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SEPARATION OF DECISIONS IN GROUP IDENTIFICATION

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Abstract

We study a model of group identification in which individuals' opinions as to the membership of a group are aggregated to form a list of the group's members. Potential aggregation rules are studied through the axiomatic approach. We introduce two axioms, meet separability and join separability, each of which requires the list of members generated by the aggregation rule to be independent of whether the question of membership in a group is separated into questions of membership in two other groups. We use these axioms to characterize a class of "one vote" rules, in which one opinion determines whether an individual is considered to be a member of a group. We then use this characterization to provide new axiomatizations of the liberal rule, in which each individual determines for himself whether he is a member of the group, as the only non-degenerate anonymous rule satisfying the meet separability and join separability axioms.

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Separation of Decisions in Group Identification*

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1 Introduction

We study a model of group identification first introduced by Kasher and Rubinstein [6], in which individuals' opinions as to the membership of a group are aggregated to form a list of the group's members. Potential aggregation rules are studied through the axiomatic approach. Various axioms are proposed and characterizations of the classes of rules satisfying these axioms are provided.[1]

Instead of asking the question "who is a member of a group" we might pose multiple questions each asking whether a particular individual is a member of the group. The decision to study the larger problem of group identification in place of the smaller problems about the status of particular individuals requires some justification. There are two main reasons why we might prefer to study the simultaneous approach.

First, we might view certain voters as being linked to certain issues. In the group identification model there is a linkage between each voter and the issue which determines whether that individual is a member. By aggregating the opinions simultaneously we are able to preserve this linkage. Second, we might also believe that these issues are connected – that the question of whether one person is a member of a group is related to the question of whether a different person is a member of that group.

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[1]The Kasher-Rubinstein framework is applicable in studying questions that ask which individuals meet a particular standard, such as "who is an American?" or "who is an honors student?" A related but conceptually distinct problem involves ranking individuals according to a standard. For example, we might want to compare students. The latter problem has been studied axiomatically by Palacios-Huerta and Volij [7] in the context of developing a cardinal ranking of scientific publications. In these papers standards are all taken exogenously. For a very different approach which uses preferences of agents to models standards endogenously see Sobel [9].

The literature which has studied the Kasher-Rubinstein group identification model has generally studied the assumption that these issues are not connected. The stronger version of the *independence* axiom used in these papers, found in Kasher and Rubinstein [6], Samet and Schmeidler [8], Ju [5], and Çengelci and Sanver [3], requires that whether a particular individual is determined to be a member of a group is independent of the opinions regarding all of the other individuals. A weaker notion of independence is found in Kasher and Rubinstein [6], Sung and Dimitrov [10], and Dimitrov, Sung, and Xu [4], the last of which characterizes a recursive procedure for determining group membership.[2]

We depart from the independence axiom in this paper. Instead we focus our attention on a set of axioms we term *separability*. Suppose we can determine whether an individual is a member of a particular group merely from knowing whether she is a member of two other groups. Separability requires that the list of members of the former group is independent of whether the list is generated directly from aggregating the opinions regarding the membership of the former group or indirectly from aggregating the opinions regarding the membership of the two latter groups. In particular we discuss two such axioms, *meet separability* and *join separability*, which we illustrate here with two examples.

Suppose we wish to know who is an officer and a gentleman. One approach would be to gather the opinions as to who are officer-gentlemen and aggregate those opinions to form a list. An alternative approach would be to gather two sets of opinions: one with respect to the identity of the officers and one with respect to the identity of the gentlemen, aggregate them both separately to form lists of officers and gentlemen, and then take the names common to both lists. Meet separability requires that the same list of officer-gentlemen is generated regardless of which of these two approaches is used.

Alternatively, suppose we wish to know who is an Iberian.[3] One approach would be to gather the opinions as to who are Iberians and aggregate those opinions to form a list. Another approach would be to gather two sets of opinions: one with respect to the identity of the Spanish and one with respect to the identity of the Portuguese, aggregate them both separately to form lists of Spanish and Portuguese, and then take the names on either (or both) of the lists. Join separability requires that the same list of Iberians is generated regardless of which of these approaches is used.

These axioms do not require *neutrality* – that the method by which the opinions are aggregated should be independent of the subject matter of the opinions. Abstractly we might allow the use of a different method to aggregate opinions about the Spanish than we would use to aggregate opinions about the Portuguese, and we might use an even different method to aggregate opinions about Iberians. We seek to characterize the full class of non-degenerate aggregation methods such that these separability properties can be achieved. The assumption of non-degeneracy, however, implies that the aggregation

[2]The assumption of independence is not made by Billot [2] who, while motivated by Kasher and Rubinstein [6] and Samet and Schmeidler [8], studies a very different model in which group membership is determined by the individuals' preferences.

[3]That is, who is either Spanish or Portuguese.

method must be neutral. It is impossible to generate a list of Iberians which is independent of whether the question is separated into two unless the same aggregation rule is used to aggregate opinions about the Spanish, Portuguese, and Iberians, or unless the opinions are completely irrelevant in determining whether some individuals are Iberian.

Using the concepts of meet separability and join separability we characterize three classes of non-degenerate rules. The first such class is the set of non-degenerate rules satisfying the meet separability axiom, which we term *agreement* rules. In an agreement rule, for each individual there is a set of opinions such that the individual is determined to be a member of the group if and only if each of those opinions is favorable. There is no requirement that the opinions relate directly to the individual in question. Thus, if we consider a society composed of three individuals, Alice, Bob, and Charlie, one agreement rule determines Alice to be a member if and only if everyone considers Alice to be a member, while another rule determines Alice to be a member if and only if Bob and Charlie consider each other to be members. We use the term "agreement" rules because a certain set of opinions need to be in agreement (and favorable) for an individual to be qualified.[4]

The second such class is the set of non-degenerate rules satisfying the join separability axiom, which we term *nomination* rules. In a nomination rule, for each individual there is a set of opinions such that the individual is determined to be a member of the group if and only if one or more of those opinions is favorable. There is no requirement that the opinions relate directly to the individual in question. Thus, in our three person society, one nomination rule determines Alice to be a member if and only if someone considers Alice to be a member, while another rule determines Alice to be a member if and only if Bob considers Charlie to be a member, Charlie considers Bob to be a member, or both. We use the term "nomination" rules because only one opinion out of a set needs to be favorable for an individual to be qualified. This is akin to a nomination process in which any one member of a group can decide to nominate.

The third class of rules we characterize is the set of non-degenerate rules satisfying the meet separability and join separability axioms, which we term *one vote* rules. In a one vote rule, for each individual there is exactly one opinion which determines whether the individual is a member of a group. Again there is no requirement that the opinion be directly related to the individual in question. According to one rule Alice chooses whether she is a member; according to another Alice is determined to be a member if and only if Bob considers Charlie to be a member.

As a consequence, no rules in which two or more opinions are relevant in determining whether an individual is a member of a group satisfy both the meet separability and join separability axioms. These include the consent rules introduced by Samet and Schmeidler [8] (except for the cases of the liberal rule[5] and the degenerate rules), quota rules (in which an individual is qualified if a certain positive number of people consider him to be

[4]By "qualified" we mean determined to be a member of the group.

[5]Under the liberal rule each individual chooses whether that individual is qualified.

a member) and oligarchic rules (in which a set of (at least two) individuals determine who is a member).

A potentially desirable property which we might want an aggregation rule to satisfy is *self-duality*, first introduced by Aumann and Maschler [1] in the context of claims in bankruptcy. In the claims literature, two rules are *dual* if one allocates losses to the claimants in the same manner that the other rule allocates gains. A rule is *self-dual* if it allocates losses in the same way that it allocates gains.

The concept of self-duality was introduced to the group identification literature by Samet and Schmeidler [8], who defined two rules as dual if one aggregates opinions about a group's members in the same manner that the other aggregates opinions about a group's non-members. Likewise, Samet and Schmeidler defined a rule as self-dual if it aggregates opinions about a group's members in the same manner that the rule aggregates opinions about a group's non-members. Extending this notion to our context, we find that agreement rules and nomination rules are dual and, as a consequence, any self-dual rule satisfying either of the separability axioms must be a one-vote rule.

Several of the rules previously discussed in the literature do satisfy the two separability axioms. These include the *liberal rule* and the *dictatorship*, both first introduced by Kasher and Rubinstein [6]. The liberal rule has been widely studied, including a refinement of the Kasher and Rubinstein characterization by Sung and Dimitrov [10] and a separate axiomatization by Samet and Schmeidler [8]. We provide two separate axiomatizations of the liberal rule based off of our characterization of one-vote rules.

2 The Model

2.1 The model and the notation

We extend the model introduced by Kasher and Rubinstein [6] and use the notation introduced by Samet and Schmeidler [8]. There is a set $N \equiv \{1, ..., n\}$ of individuals, $n \geq 3$. There is a given Boolean algebra of issues $\mathfrak{B}$.[6] Each element $b \in \mathfrak{B}$ is an issue pertaining to membership in a group.[7]

The individuals each mark their opinions about the groups on ballots. A ballot can be represented as an $1 \times n$ row vector $P_i \in \{0, 1\}^N$. The n ballots can be assembled into an $n \times n$ matrix $P \in \{0, 1\}^{N \times N}$, where $P_{ij} = 1$ if individual i considers individual j to be a member, and where $P_{ij} = 0$ if individual i does not consider individual j to

[6] A Boolean algebra is a set of sets closed under intersection, union, and complementation, where for any two issues $a, b \in \mathfrak{B}$, $a \wedge b$ is the *intersection* of these issues ("a and b"), $a \vee b$ is the *union* of these issues ("a and/or b"), and $\bar{a}$ and $\bar{b}$ are the *complements* of these issues ("not a" and "not b").

[7] For example, if a is the issue of being American and b is the issue of being British, then $a \wedge b$ is the issue of being American and British, and $a \vee b$ is the issue of being American or British (or both). Also, $\bar{a}$ is the issue of being non-American, and $\bar{b}$ is the issue of being non-British.

be a member. Such a matrix P is called a **profile**. A **qualification problem** is a pair $(P, b) \in \{0,1\}^{N\times N} \times \mathfrak{B}$. A **social rule** is a mapping $f : \{0,1\}^{N\times N} \times \mathfrak{B} \to \{0,1\}^N$ which maps each qualification problem into a unique vector $f(P, b) \equiv (f_1(P, b), ..., f_n(P, b))$, where $f_j(P, b) = 1$ if and only if individual j is determined to be a member of group b.

For any two matrices or vectors A and B, we define $A \wedge B$ to be the coordinatewise minimum, so that $(A \wedge B)_{ij} = \min\{(A)_{ij}, (B)_{ij}\}$, and we define $A \vee B$ to be the coordinatewise maximum, so that $(A \vee B)_{ij} = \max\{(A)_{ij}, (B)_{ij}\}$. For any two pairs (P, b) and (Q, a) we let $(P, b) \wedge (Q, a) \equiv (P \wedge Q, b \wedge a)$ and $(P, b) \vee (Q, a) \equiv (P \vee Q, b \vee a)$. For $x \in \{0, 1\}$ we define $\bar{x} \equiv 1 - x$.

We let **1** and **0** refer to the $n \times n$ matrices composed entirely of ones and zeros, respectively. For any two matrices or vectors S and T, we say that $S \geq T$ if this inequality holds coordinatewise, and $S \ngeq T$ if that inequality does not hold coordinatewise. We say that $S > T$ if $S \neq T$ and $S \geq T$.

Individuals in the model give their opinions consistently. If profiles P and Q describe the opinions about issues a and b, respectively, then $P \wedge Q$ and $P \vee Q$ describe the opinions about issues $a \wedge b$ and $a \vee b$, respectively. Similarly, $\bar{P}$ and $\bar{Q}$ would be the profiles which describe opinions about issues $\bar{a}$ and $\bar{b}$, respectively.

2.2 The axioms

Consider the following problem. Within the senior class of a small college there is a group of students who are smart, and there is a group of students who are hard-working. The board of trustees is having its annual meeting, and the administration would like to invite the smart hard-working seniors to have dinner with the trustees. While the seniors form a well defined group in this college, the administrators do not know which of the students are smart and which of the students are hard working. A severe problem of grade inflation has left the college without any reliable metric. The college president believes that only the students have this information.

A decision is made to gather this information from the students. But a debate quickly ensues as to the method. One administrator argues that every senior should be given a ballot and asked to mark off the names of the smart hard-working seniors. Another administrator argues that the seniors should be given two ballots – one of the smart seniors and another of those who are hard-working. Proponents of the one-ballot method argue that their approach is less costly while supporters of the two ballot method argue that their approach generates data that might be useful later. However, neither side can make a clear case as to why their approach will generate better results. As a result the president declares that the first method will be used (because it is cheaper), but that the final list of smart hard-working seniors must be the same regardless of which method is used.

In this case $s \in \mathfrak{B}$ is the issue of being smart, $h \in \mathfrak{B}$ is the issue of being hard-working, and $s \wedge h \in \mathfrak{B}$ is the issue of being smart and hardworking. The information from the ballots will be assembled into profiles. If the students are given a single ballot there will be one profile, which we will denote R, which contains the seniors' opinions about the smart hard-working seniors. The administrators will then aggregate the data to generate a list of the smart hard-working seniors, denoted $f(R, s \wedge h)$.

If the students are given two ballots there will be two profiles: S, which contains the students' opinions about the smart seniors, and H, which contains the students' opinions about the hard-working seniors. The administrators will then aggregate the opinions about the smart seniors to create a list, denoted $f(S, s)$. They will also aggregate the opinions about the hard-working seniors to create a list, denoted$f(H, h)$. Afterward they will generate a list of smart hard-working seniors simply by taking the names common to both lists. This is the meet of the two lists: $f(S, s) \wedge f(H, h)$.[8]

The students at the college are reasonably intelligent, and as a result the administrators are confident that the information on the ballots will be given consistently regardless of which method is used. A student would put another student's name on the ballot of smart hard-working seniors if and only if he would have put her name on both the ballot of smart seniors and on the ballot of hard-working seniors. This means that the profile of opinions about smart hard-working seniors (R) must be the same as the meet of the profile of opinions about smart seniors (S) and the profile of opinions about hard-working seniors (H), or $R = S \wedge H$.[9] Therefore the list of smart hard-working seniors generated through the one ballot method is $f(H \wedge S, h \wedge s)$, or $f((H, h) \wedge (S, s))$.

The president of the college has required that the final list of smart hard-working seniors must be the same regardless of which method is used. Therefore the social rule must satisfy the equality $f((H, h) \wedge (S, s)) = f(H, h) \wedge f(S, s)$. Our first axiom, *meet separability*, requires that this be the case.

Axiom 1 *Meet separability: A social rule f satisfies meet separability if, for every pair of issues $\{a, b\} \subset \mathfrak{B}$ and for all profiles P and Q, $f((P, a) \wedge (Q, b)) = f(P, a) \wedge f(Q, b)$.*

In the town in which the college is situated there are two gangs: the Darwins and the Bryans. A researcher wants to know who in the town is a member of a gang. Fortunately for the researcher, gang membership is not illegal and members are proud to reveal their affiliations. Unfortunately for the researcher, the members may be too proud. The gangs are fragmented into loosely affiliated "dens", and there is no clear agreement as to who all of the gang members are. The researcher plans to do an extensive survey of all the towns' residents to get the needed data on who is a member of a gang.

The researcher can design the survey in either of two methods. The survey can have two questions, asking for a list of Darwins and a list of Bryans, respectively, or it can

[8]This is true because, $f_j(S, s) \wedge f_j(H, h) = 1$ if and only if $h_j(S, s) = 1$ and $f_j(H, h) = 1$.
[9]This is true because $(H \wedge S)_{ij} = 1$ if and only if $H_{ij} = 1$ and $S_{ij} = 1$.

have one question, asking for a list of gang members. The researcher needs to justify her methodology but cannot elucidate a clear reason as to one method should be preferred over the other. Furthermore, she wants to make sure that her study will be comparable to work done by later researchers who may not have the same choice. As a result she chooses the single question survey (to save on costs) but decides that the ultimate list of gang members should be independent of which method is used.

In this case, $d \in \mathfrak{B}$ is the issue of being a Darwin, $b \in \mathfrak{B}$ is the issue of being a Bryan, and $d \vee b \in \mathfrak{B}$ is the issue of being a gang member. The two question survey will generate two profiles of opinions: D, which contains the views of the townspeople about the Darwins, and B, which contains the views of the townspeople about the Bryans. Each of these profiles will be aggregated to generate a list: $f(D, d)$, a list of Darwins, and $f(B, b)$, a list of Bryans. The list of gang members is the join of these lists, $f(D, d) \vee f(B, b)$.[10]

The one question survey, on the other hand, will generate a single profile of opinions, $D \vee B$, which contains the views of the townspeople regarding the gang members.[11] The opinions in the profile will be aggregated to form a list of gang members, $f(D \vee B, d \vee b)$, or $f((D, d) \vee (B, b))$.

The constrain imposed by the researcher is that the ultimate list of gang members should be independent of which method is used. Therefore the social rule must satisfy the equality $f((D, d) \vee f(B, b)) = f(D, d) \vee f(B, b)$. Our second axiom, *join separability*, requires that this be the case.

Axiom 2 *Join separability: A social rule f satisfies join separability if for every pair of issues $\{a, b\} \subset \mathfrak{B}$ and for all profiles P and Q, $f((P, a) \vee (Q, b)) = f(P, a) \vee f(Q, b)$.*

Our third axiom is adapted from Samet and Schmeidler [8]. This axiom excludes constant rules; rules for which there exists an individual who is, or is not, a member of the group regardless of which names are on the ballots.

Axiom 3 *Non-degeneracy: For every individual j and every issue $b \in \mathfrak{B}$ there exist profiles P and P' such that $f_j(P, b) = 1$ and $f_j(P', b) = 0$.*

The separability axioms do not require that the social rule must use the same method to aggregate opinions about different issues. We might use one method to aggregate opinions as to the group of officers, a different method to aggregate opinions as to gentlemen, and a third method to aggregate opinions as to officer-gentlemen. If the social rule is non-degenerate, however, then the separability axioms imply that the method by which opinions are aggregated must be independent of the issue. We prove this in the following theorem.

[10] This is true because, $f_j(D, d) \vee f_j(B, b) = 1$ if and only if $f_j(D, d) = 1$, $f_j(B, b) = 1$, or both.

[11] This is true because $(D \vee B)_{ij} = 1$ if and only if $D_{ij} = 1$, $B_{ij} = 1$, or both.

Theorem 2.1 *(i) If a social rule f satisfies non-degeneracy and meet separability then, for all issues $a, b \in \mathfrak{B}$ and for every profile P, $f(P,a) = f(P,b)$.*

(ii) If a social rule f satisfies non-degeneracy and join separability then, for all issues $a, b \in \mathfrak{B}$ and for every profile P, $f(P,a) = f(P,b)$.

Proof: We prove (i). The proof of (ii) is similar. Let $\{a, b\} \in \mathfrak{B}$. By meet separability, for all profiles P and Q, $f((P,a) \wedge (Q,b)) = f(P,a) \wedge f(Q,b)$. We know that $f((P,a) \wedge (Q,b)) = f(P \wedge Q, a \wedge b)$, and that $f(P \wedge Q, a \wedge b) = f(Q \wedge P, a \wedge b)$, and therefore $f((P,a) \wedge (Q,b)) = f((Q,a) \wedge (P,b))$. It follows that $f(P,a) \wedge f(Q,b) = f(Q,a) \wedge f(P,b)$.

Let $j \in N$. By non-degeneracy there must exist profiles R^j and S^j such that $f_j(R^j, a) = f_j(S^j, b) = 1$. Let $Q^j \equiv R^j \wedge S^j$. It follows that $f_j((R^j, a) \wedge (S^j, b)) = 1$ and consequently $f_j(R^j \wedge S^j, a \wedge b) = f_j(Q^j, a \wedge b) = f_j(Q^j \wedge Q^j, a \wedge b) = f_j((Q^j, a) \wedge (Q^j, b)) = 1$. Therefore it must be that $f_j(Q^j, a) = f_j(Q^j, b) = 1$.

Let $P \in \{0,1\}^{N \times N}$. Because $f_j(Q^j, a) = 1$ it follows that $f_j(Q^j, a) \wedge f_j(P, b) = f_j(P, b)$. Likewise, because $f_j(Q^j, b) = 1$ it follows that $f_j(P, a) \wedge f_j(Q^j, b) = f_j(P, a)$. We know that $f_j(Q^j, a) \wedge f_j(P, b) = f_j(P, a) \wedge f_j(Q^j, b)$ and therefore, $f_j(P, b) = f_j(P, a)$.

It follows that for all profiles P, $f_j(P, a \wedge b) = f_j(P \wedge P, a \wedge b) = f_j((P,a) \wedge (P,b)) = f_j(P,a) \wedge f_j(P,b) = f_j(P,a) \wedge f_j(P,a) = f_j(P,a)$, and therefore $f_j(P,a) = f_j(P,b) = f_j(P, a \wedge b)$. Because this is true for an arbitrary $j \in N$ and an arbitrary pair $\{a, b\} \subset \mathfrak{B}$, $f(P,a) = f(P,b) = f(P, a \wedge b)$. It follows that for all $a, b \in \mathfrak{B}$, $f(P,a) = f(P,b)$. ∎

Our last axiom, also adapted from Samet and Schmeidler [8], requires that as additional names are added to the ballots, no names are removed from the list of qualified persons.

Axiom 4 *Monotonicity: For all profiles P and P' such that $P \geq P'$, $f(P,b) \geq f(P',b)$.*

The monotonicity axiom is implied by either of the meet separability and join separability axioms, as we demonstrate in the following lemma.

Lemma 2.2 *If a social rule f satisfies either of the meet separability or join separability axioms then it satisfies the monotonicity axiom.*

2.3 Agreement Rules

The first class of social rules we characterize are **agreement** rules, in which for every individual there is a set of votes which "matter" such that the individual is qualified if and only if each and every one of those votes is in the affirmative. Then the minimal profile under which the individual is qualified is the profile such that all of the votes which matter are in the affirmative and the others are against.

Social Rule 1 *Agreement rules: For all individuals j there exists a profile $P^{j-} > \mathbf{0}$ such that, for all issues $b \in \mathfrak{B}$, $f_j(P, b) = 1$ if and only if $P \geq P^{j-}$.*

We characterize these rules in the following theorem:

Theorem 2.3 *A social rule f satisfies the meet separability and non-degeneracy axioms if and only if it is an agreement rule.*

Proof: Let $\{a, b\} \subset \mathfrak{B}$ and $P, Q \in \{0,1\}^{N \times N}$. By the definition of meet separability, $f((P, a) \wedge (Q, b)) = f(P \wedge Q, a \wedge b) = f(P, a) \wedge f(Q, b)$. It follows from Theorem 2.1 that $f(P, a) = f(P, b) = f(P, a \wedge b)$ and therefore $f(P \wedge Q, b) = f(P, b) \wedge f(Q, b)$.

Let $j \in N$. Define $\mathcal{P}_j \equiv \{P \in \{0,1\}^{N \times N} : f_j(P, b) = 1\}$. We know that $\mathcal{P}_j \neq \varnothing$ because if $\mathcal{P}_j = \varnothing$ then $f_j(P, b) = 0$ for all profiles P, and this would contradict the non-degeneracy axiom.

Define $P^{j-} = \bigwedge_{P \in \mathcal{P}_j} P$. For all profiles $P', P'' \in \mathcal{P}_j$, $f_j(P', b) = f_j(P'', b) = 1$. By the meet separability axiom, $f_j(P' \wedge P'', b) = 1$. It follows by an induction argument that $f_j(\bigwedge_{P \in \mathcal{P}_j} P, b) = f_j(P^{j-}, b) = 1$. Therefore, $P^{j-} \in \mathcal{P}_j$.

Clearly, for all profiles $P \in \mathcal{P}_j$, $P \geq \bigwedge_{P \in \mathcal{P}_j} P = P^{j-}$. Furthermore, $P^{j-} \neq \mathbf{0}$, otherwise $f_j(P, b) = 1$ for all profiles P, which would violate the non-degeneracy axiom.

Lastly, we show that for all profiles P such that $P \geq P^{j-}$, $P \in \mathcal{P}_j$. This follows from Lemma 2.2: $P \geq P^{j-}$ implies that $f_j(P, b) \geq f_j(P^{j-}, b) = 1$, which implies that $f_j(P, b) = 1$. Hence $P \in \mathcal{P}_j$ if and only if $P \geq P^{j-}$. Therefore, for all issues $b \in \mathfrak{B}$, $f_j(P, b) = 1$ if and only if $P \geq P^{j-}$. ∎

2.4 Nomination Rules

The second class of social rules we characterize are **nomination** rules, in which for every individual there is a set of votes which matter such that the individual is qualified if and only if any one (or more) of those votes is in the affirmative. Then the maximal profile under which the individual is not qualified is the profile such that all of the votes which matter are against and the others are in the affirmative.

Social Rule 2 *Nomination rules: For all individuals j there exists a profile $P^{j+} < \mathbf{1}$ such that, for all issues $b \in \mathfrak{B}$, $f_j(P, b) = 0$ if and only if $P \leq P^{j+}$.*

We characterize these rules in the following theorem:

Theorem 2.4 *A social rule f satisfies the join separability and non-degeneracy axioms if and only if it is an nomination rule on $\mathfrak{B}$.*

Proof: Let $\{a, b\} \subset \mathfrak{B}$ and $P, Q \in \{0,1\}^{N\times N}$. By the definition of join separability, $f((P,a) \vee (Q,b)) = f(P \vee Q, a \vee b) = f(P,a) \vee f(Q,b)$. It follows from Theorem 2.1 that $f(P,a) = f(P,b) = f(P, a\vee b)$ and therefore $f(P \vee Q, b) = f(P,b) \vee f(Q,b)$.

Let $j \in N$ be arbitrary. Define $\mathcal{P}_j \equiv \{P \in \{0,1\}^{N\times N} : f_j(P,b) = 0\}$. We know that $\mathcal{P}_j \neq \varnothing$ because if $\mathcal{P}_j = \varnothing$ then $f_j(P,b) = 1$ for all profiles P, and this would contradict the non-degeneracy axiom.

Define $P^{j+} = \bigvee_{P\in\mathcal{P}_j} P$. For all profiles $P', P'' \in \mathcal{P}_j$, $f_j(P',b) = f_j(P'',b) = 0$. By the join separability axiom, $f_j(P' \vee P'', b) = 0$. It follows by an induction argument that $f_j(\bigvee_{P\in\mathcal{P}_j} P, b) = f_j(P^{j+}, b) = 0$. Therefore, $P^{j+} \in \mathcal{P}_j$.

Clearly, for all profiles $P \in \mathcal{P}_j$, $P \leq \bigvee_{P\in\mathcal{P}_j} P = P^{j+}$. Furthermore, $P^{j+} \neq \mathbf{1}$, otherwise $f_j(P,b) = 0$ for all profiles P, which would violate the non-degeneracy axiom.

Lastly, we show that for all profiles P such that $P \leq P^{j+}$, $P \in \mathcal{P}_j$. This follows from Lemma 2.2: $P \leq P^{j+}$ implies that $f_j(P,b) \leq f_j(P^{j+},b) = 0$, which implies that $f_j(P,b) = 0$. Hence $P \in \mathcal{P}_j$ if and only if $P \leq P^{j+}$. Therefore, for all issues $b \in \mathfrak{B}$, $f_j(P,b) = 0$ if and only if $P \leq P^{j-}$. ∎

2.5 One-Vote Rules

We now characterize the set of social rules which are both agreement rules and nomination rules. These are the rules in which for every individual there is one vote which matters such that the individual is qualified if and only if that vote is in the affirmative.

Social Rule 3 *One-vote rules: For all individuals j there exists (i,k) in $N \times N$ such that, for all issues $b \in \mathfrak{B}$, $f_j(P,b) = P_{ik}$.*

From this follows our main result:

Theorem 2.5 *A social rule f satisfies the meet separability, join separability, and non-degeneracy axioms if and only if it is a one-vote rule.*

Proof: That the one-vote rules satisfy the three axioms is trivial. We show that any social rule that satisfies the three axioms is necessarily a one-vote rule. Suppose a social rule f satisfies the meet separability, join separability, and non-degeneracy axioms. Let $j \in N$ and let $b \in \mathfrak{B}$. Because f satisfies meet separability and non-degeneracy it must be an agreement rule (by Theorem 2.3). Therefore, there must exist a profile $P^{j-} > \mathbf{0}$ such that $f_j(P, b) = 1$ if and only if $P \geq P^{j-}$. This implies that there exists (i, k) in $N \times N$ such that $f_j(P, b) = 0$ if $P_{ik} = 0$ and therefore $f_j(P, b) \leq P_{ik}$. Because f satisfies join separability and non-degeneracy it must be an agreement rule (by Theorem 2.4). Therefore, there must exist a profile $P^{j+} < \mathbf{1}$ such that $f_j(P, b) = 0$ if and only if $P \leq P^{j+}$. This implies that $P^{j+}_{ik} = 0$, which implies that $f_j(P, b) = 1$ if $P_{ik} = 1$ and therefore $f_j(P, b) \geq P_{ik}$. It follows that $f_j(P, b) = P_{ik}$. By Theorem 2.1 it follows that $f(P, b) = P_{ik}$ for every issue $b \in \mathfrak{B}$. ∎

2.6 Parition and Duality

Individuals in the model vote consistently: if P is the profile describing opinions about group b, then $\bar{P}$ is the profile describing opinions about the group $\bar{b}$. Then $f(P, b)$ is the list of individuals determined to be members of group b and $f(\bar{P}, \bar{b})$ is the list of individuals determined to be members of group $\bar{b}$. A logical requirement for the social rule is that these lists form a partition of N; that is, $f(P, a) \wedge f(\bar{P}, \bar{a}) = \mathbf{0}$ and $f(P, a) \vee f(\bar{P}, \bar{a}) = \mathbf{1}$.[12] For every individual $j \in N$, $f_j(P, b) \neq f_j(\bar{P}, \bar{b})$, and therefore $f(P, a) = \overline{f(\bar{P}, \bar{a})}$. Our next axiom, *partition*, requires that this be the case.

Axiom 5 *Partition: A social rule f satisfies partition if $f(P, b) = \overline{f(\bar{P}, \bar{b})}$ for all profiles P and all issues $b \in \mathfrak{B}$.*

A concept related to partition is duality, first introduced in the context of group identification by Samet and Schmeidler [8]. Suppose we wish to know who is a non-member of b. The profile describing the opinions about the non-members of b is $\bar{P}$. Thus $f(\bar{P}, b)$ gives us the list of people considered to be non-members of b by the aggregation rule used to determine membership in b, and $\overline{f(\bar{P}, b)}$ gives us a list of people who are not non-members of b. We denote this rule by $\bar{f}(P, b) = \overline{f(\bar{P}, b)}$ and call $\bar{f}$ the **dual** of f. If $f = \bar{f}$ we say that f is **self-dual**.

Axiom 6 *Self-duality (Samet-Schmeidler): A social rule f satisfies self-duality if $f = \bar{f}$.*

Partition and self-duality are very different concepts. However, in the presence of non-degeneracy and either separability axiom, the partition and self-duality axioms are equivalent.

Proposition 2.6 *If a social rule f satisfies non-degeneracy and either meet separability or join separability then the following two statements are equivalent:*
(i) f satisfies partition,
(ii) f satisfies self-duality.

[12] Here $\mathbf{0}$ and $\mathbf{1}$ refer to $1 \times n$ vectors of zeros and ones, respectively, and not to matrices.

Proof: By Theorem 2.1, for all $a, b \in \mathfrak{B}$, $f(P, a) = f(P, b)$. Because $\mathfrak{B}$ is closed under complementation it follows that $f(P, b) = f(P, \bar{b})$, and therefore $\overline{f(\bar{P}, \bar{b})} = \overline{f(\bar{P}, b)}$. ∎

We also use a notion of duality of axioms which we take from Thomson [11], who discusses related issues in a claims context. We say that two axioms are the **dual of each other** if whenever a social rule f satisfies one axiom, $\bar{f}$ satisfies the other. We use this notion to establish the following proposition.

Proposition 2.7 *The meet separability and join separability axioms are dual of each other.*

Proof: Let $a, b \in \mathfrak{B}$ and let $P, Q \in \{0,1\}^{N \times N}$. Let f satisfy the meet separability axiom. Then $f((\bar{P}, a) \wedge (\bar{Q}, b)) = f(\bar{P} \wedge \bar{Q}, a \wedge b) = f(\bar{P}, a) \wedge f(\bar{Q}, b)$. It follows from Theorem 2.1 that $f(\bar{P} \wedge \bar{Q}, b) = f(\bar{P}, b) \wedge f(\bar{Q}, b)$, and therefore $\overline{f(\bar{P} \wedge \bar{Q}, b)} = \overline{f(\bar{P}, b) \wedge f(\bar{Q}, b)}$. By DeMorgan's laws, $\overline{f(\bar{P} \wedge \bar{Q}, b)} = \overline{f(\overline{P \vee Q}, b)} = \bar{f}(P \vee Q, b)$. Also by DeMorgan's laws, $\overline{f(\bar{P}, b) \wedge f(\bar{Q}, b)} = \overline{f(\bar{P}, b)} \vee \overline{f(\bar{Q}, b)} = \bar{f}(P, b) \vee \bar{f}(Q, b)$. Therefore, $\bar{f}(P \vee Q, b) = \bar{f}(P, b) \vee \bar{f}(Q, b)$. By Theorem 2.1 it follows that $\bar{f}(P \vee Q, a \vee b) = \bar{f}((P, a) \vee (Q, b)) = \bar{f}(P, b) \vee \bar{f}(Q, b)$. This shows that if a social rule f satisfies the meet separability axiom, its dual $\bar{f}$ satisfies the join separability axiom. The second half of the proof is symmetric to the first. ∎

The following corollary to the proposition follows directly from Theorems 2.3 and 2.4.

Corollary 2.8 *The dual of an agreement rule is a nomination rule.*

We note that self-duality implies non-degeneracy.

Lemma 2.9 *If a social rule f satisfies the self-duality axiom then it satisfies the non-degeneracy axiom.*

Second, we establish the following relationships between the meet separability, join separability, and self-duality axioms.

Proposition 2.10 *If a social rule f satisfies the meet separability and self-duality axioms then it satisfies join separability and non-degeneracy. If a social rule f satisfies the join separability and self-duality axioms then satisfies meet separability and self-duality.*

Proof: By Lemma 2.9, any social rule that satisfies the self-duality axiom also satisfies non-degeneracy. By Proposition 2.7, meet separability and join separability are the dual of each other. Therefore any social rule that satisfies self-duality and meet separability must also satisfy join separability and non-degeneracy. Similarly, any social rule that satisfies self-duality and join separability must also satisfy meet separability and non-degeneracy. ∎

Proposition 2.11 *If a social rule f satisfies the meet separability, join separability and non-degeneracy axioms then it satisfies self-duality axiom.*

Proof: Let a social rule f satisfy meet separability, join separability and non-degeneracy. Let $j \in N$, $P \in \{0,1\}^{N \times N}$, and $b \in \mathfrak{B}$. By Theorem 2.5 f must be a one-vote rule , and therefore there exists a pair $(i,k) \in N \times N$ such that $f_j(P,b) = P_{ik}$. It follows that $f_j(\bar{P},b) = \bar{P}_{ik}$ and thus $\overline{f_j(\bar{P},b)} = \bar{\bar{P}}_{ik} = P_{ik}$. Therefore, for every issue $b \in \mathfrak{B}$ and every profile P, $f(P,b) = \overline{f(\bar{P},b)}$. ∎

3 Discussion

3.1 The Liberal Rule and Dictatorship

Kasher and Rubinstein [6] provided axiomatizations for two types of social rules: the **liberal rule** and the **dictatorship**.[13]

Under the liberal rule, each individual decides for herself whether she is qualified.

Social Rule 4 *Liberal rule: For every $j \in N$ and for every issue $b \in \mathfrak{B}$, $f_j(P,b) = P_{jj}$.*

Under a dictatorship, a pre-designated individual decides who is qualified.

Social Rule 5 *Dictatorship: There exists an $i \in N$ such that for every $j \in N$ and every $b \in \mathfrak{B}$, $f_j(P,b) = P_{ij}$.*

Each of these rules is a one-vote rule and consequently satisfies the two separability axioms as well as the non-degeneracy axiom.

Of these rules, the liberal rule has received the more extensive treatment in the literature, including a refinement of the Kasher-Rubinstein axiomatization by Sung and Dimitrov [10] and a separate axiomatization by Samet and Schmeidler [8]. We provide a separate axiomatization of the liberal rule as the only social rule which satisfies the two separability axioms as well as particular concepts of symmetry and non-degeneracy.

Kasher and Rubinstein [6] and Samet and Schmeidler [8] provided two different concepts of symmetry. The **symmetry** condition used by Kasher and Rubinstein requires that if any two individuals are symmetric with respect to their views about others and others' views toward them, then either both or neither are qualified.

[13] Kasher and Rubinstein also provide an axiomatization for a third class of social rules, the *oligarchic* rules; however, these rules rely on a model substantially different from that discussed in this paper. While the Kasher and Rubinstein axiomatization of the dictatorship uses a slightly different model as well, it is nonetheless similar enough to be understood in our framework. The term "liberal rule" is taken from Samet and Schmeidler. Kasher and Rubinstein call this rule the "strong liberal collective identity function".

Axiom 7 *Symmetry: Let* $j, k \in N$*. If (a)* $P_{jj} = P_{kk}$*, (b)* $P_{kj} = P_{jk}$*, and, for all* $i \in N \setminus \{j, k\}$*, (c)* $P_{ij} = P_{ik}$*, and (d)* $P_{ji} = P_{ki}$*, then, for every issue* $b \in \mathfrak{B}$*,* $f_j(P, b) = f_k(P, b)$*.*

The symmetry condition used by Samet and Schmeidler [8], which we term **anonymity** (to minimize confusion), requires that the list of the qualified individuals does not depend on their names. We switch names through a permutation π of N. Thus, for a given permutation π, i is the new name of the individual formerly known as $\pi(i)$. For a given profile $P \in \{0, 1\}$ we let πP be the profile in which the names are switched. Then $(\pi P)_{ij} = P_{\pi(i)\pi(j)}$. We denote $\pi f(P, b) \equiv (f_{\pi(1)}(P, b), f_{\pi(2)}(P, b), ..., f_{\pi(n)}(P, b))$. Anonymity requires that if individual i is qualified in profile πP, then individual $\pi(i)$ is qualified in profile P.

Axiom 8 *Anonymity (Samet-Schmeidler): For every permutation* π *of* N *and every issue* $b \in \mathfrak{B}$*,* $f(\pi P, b) = \pi f(P, b)$*.*

We show that the liberal rule is the only one-vote rule which satisfies the anonymity axiom.

Theorem 3.1 *The liberal rule is the only rule that satisfies the meet separability, join separability, non-degeneracy, and anonymity axioms.*

Proof: That the liberal rule satisfies the four axioms is trivial. We show that any rule that satisfies the four axioms must necessarily be a one-vote rule. Let $j \in N$ and let $b \in \mathfrak{B}$. Let f satisfy separability, join separability, non-degeneracy, and anonymity. By Theorem 2.5 f must be a one vote rule, and therefore there must be a pair of individuals i and k such that $f_j(P, b) = P_{ik}$. Because the pair of individuals may differ for every individual j, we denote these individuals $i(j)$ and $k(j)$. Therefore, $f_j(P, b) = P_{i(j)k(j)}$. Let π be a permutation of N. Then, $f_j(\pi P, b) = (\pi P)_{i(j)k(j)} = P_{\pi(i(j))\pi(k(j))}$, and $f_{\pi((j))}(P, b) = P_{i(\pi(j))k(\pi(j))}$. By the anonymity axiom, it follows that $P_{\pi(i(j))\pi(k(j))} = P_{i(\pi(j))k(\pi(j))}$, which implies that $\pi(i(j)) = i(\pi(j))$ and $\pi(k(j)) = k(\pi(j))$, which hold if and only if $i(j) = j$ and $k(j) = j$. Thus, for every individual $j \in N$ and every issue $b \in \mathfrak{B}$, $f_j(P, b) = P_{jj}$. ■

From Propositions 2.10 and 2.10 we can establish the following two corollaries.

Corollary 3.2 *The liberal rule is the only social rule that satisfies the meet separability, self-duality, and anonymity axioms.*

Corollary 3.3 *The liberal rule is the only social rule that satisfies the join separability, self-duality, and anonymity axioms.*

If we replace the anonymity axiom with the symmetry axiom, however, this result no longer holds. Consider the rule in which, for every $j \in N$ and every issue $b \in \mathfrak{B}$, $f_j(P, b) = P_{11}$. This is a one-vote rule and clearly satisfies the join separability, meet separability, and non-degeneracy axioms. Furthermore, it trivially satisfies the symmetry axiom, as for all $i, j \in N$, $f_i(P,b) = f_j(P,b)$. But this is not the liberal rule. To characterize the liberal rule using the symmetry axiom we need an additional axiom, which we term **subgroup non-degeneracy**. The subgroup non-degeneracy axiom requires that for every potential subgroup of the larger population there is a profile such the members of that subgroup, and only the members of that subgroup, are qualified.

Axiom 9 *Subgroup non-degeneracy: For all $S \subset N$ and all $b \in \mathfrak{B}$, there exists a profile P such that $\{j : f_j(P,b) = 1\} = S$.*

Subgroup non-degeneracy implies non-degeneracy, as we show in the following lemma.

Lemma 3.4 *If a social rule f satisfies the subgroup non-degeneracy axiom then it satisfies the non-degeneracy axiom.*

The symmetry and subgroup non-degeneracy axioms provide us with a separate characterization of the liberal rule.

Theorem 3.5 *The liberal rule is the only social rule that satisfies the meet separability, join separability, subgroup non-degeneracy, and symmetry axioms.*

Proof: That the liberal rule satisfies the axioms is trivial. We show that any rule that satisfies the four axioms is necessarily the liberal rule. Let f satisfy the meet separability, join separability, subgroup non-degeneracy, and symmetry axioms. Because f satisfies the subgroup non-degeneracy axiom it must satisfy the non-degeneracy axiom by Lemma 3.4. Let $j \in N$ and $b \in \mathfrak{B}$. By Theorem 2.5, all rules which satisfy the meet separability, join separability, and non-degeneracy axioms are one-vote rules. Therefore $f_j(P,b) = P_{ik}$ for some pair $(i,k) \in N \times N$.

The subgroup non-degeneracy axiom implies there must be a different such pair for every individual. To show this, assume that a one-vote rule satisfies the subgroup non-degeneracy axiom but that there are two individuals whose qualification depends on the same vote. Then there exists $g \in N \setminus \{j\}$ such that $f_g(P,b) = P_{ik}$. This implies that $f_j(P,b) = f_g(P,b)$ for all profiles P. Because $\{j\} \subset N$, the subgroup non-degeneracy axiom implies that there is some profile P such that $f_j(P,b) = 1$ and $f_g(P,b) = 0$. This contradiction proves that the subgroup non-degeneracy axiom implies that for each $j \in N$ there exists a distinct pair of individuals $(i,k) \in N \times N$ such that $f_j(P,b) = P_{ik}$.

Let P' be the $n \times n$ matrix such that all elements of this matrix are zero except that $P'_{ik} = P'_{jk} = 1$. Because i and j satisfy the conditions of the symmetry axiom, $f_i(P',b) = f_j(P',b)$. Because $f_j(P,b) = P_{ik}$, it follows that $f_j(P',b) = P'_{ik} = 1$. This implies that $f_i(P',b) = 1$ and therefore that $f_i(P,b) = P_{jk}$. From this we learn that:

$$\text{for all } i,j,k \in N,\ f_j(P,b) = P_{ik} \text{ if and only if } f_i(P,b) = P_{jk}. \tag{1}$$

From statement (1) it follows that $f_j(P, b) = P_{ik}$ if and only if $f_k(P, b) = P_{ij}$ if and only if $f_i(P, b) = P_{kj}$ if and only if $f_j(P, b) = P_{ki}$ if and only if $f_k(P, b) = P_{ji}$. Therefore $i = j = k$. It follows that $f_j(P, b) = P_{jj}$. By Theorem 2.1 it follows that $f_j(P, b) = P_{jj}$ for all $b \in \mathfrak{B}$. ∎

From Propositions 2.10 and 2.11 we can establish the following two corollaries.

Corollary 3.6 *The liberal rule is the only social rule that satisfies the meet separability, self-duality, subgroup non-degeneracy and symmetry axioms.*

Corollary 3.7 *The liberal rule is the only social rule that satisfies the join separability, self-duality, subgroup non-degeneracy and symmetry axioms.*

3.2 The Generalized Model

This paper has focused on the question of group identification, in which the binary opinions of n persons on n issues are simultaneously aggregated.[14] Alternatively, we might consider a more general model involving the simultaneous aggregation of the binary opinions of n persons on m issues, where $n \neq m$. All of the results in section 2 are applicable to the more general case of simultaneous aggregation of binary opinions on multiple issues. We provide here a simple example of how the results might be applied in the case in which the set of issues is distinct from the set of individuals whose opinions are considered.

Consider an economics department which needs to decide which of several potential visitors should be invited to give a seminar. The chair of the department can choose to hold one-vote in which all of the invitees for the semester will be chosen. Alternatively, the chair can separate this decision into several votes: one in which the invitees for the microeconomics seminars are chosen, one for the econometrics seminars, one for the political economy seminars, and so on. Fearful that the chair will manipulate the result, the faculty wish to choose a voting rule under which the ultimate list of invited seminars is irrespective of how the chair separates the votes.

The requirement that the chair must not be able to manipulate the result is equivalent to the requirement that the voting rule must satisfy the join separability axiom. Presumably the faculty also wish to choose a rule that is non-degenerate – otherwise they would not bother to vote.

From Theorem 2.4, it follows that the faculty must use a nomination rule to select the invitees – for each potential visitor there must be a group of faculty such that the support of one member of that group is necessary and sufficient for the visitor to be invited. The composition of this group must be exogenously determined. We could

[14]Each of the n issues is the issue of whether a particular individual is a member of the group. These issues are distinct from the set $\mathfrak{B}$ of issues pertaining to specific groups.

envision a rule under which any member of the faculty, or of the senior faculty, is able to invite a seminar speaker. Or we could have a rule under which econometricians invite econometricians, political economists invite political economists, and microeconomists can invite microeconomists.

3.3 Weakening of the Axioms

The meet separability and join separability axioms are both formulated with respect to every pair of issues $\{a, b\} \subset \mathfrak{B}$. It is possible to weaken these axioms by having them apply only to pairs of issues in a subset of $\mathfrak{B}$. For example, we might want a rule to aggregate opinions about British, Americans, and British-Americans consistently but not care about how the rule aggregates opinions about people who are either British and/or American. As a result we could choose a subset of $\mathfrak{B}$ such that the axioms only place restrictions on relationships between those issues in which we are interested.

With respect to these issues, the results of Theorems 2.1, 2.3, 2.4, and 2.5 would be entirely the same if the weakened forms of the axioms were used. The only difference would be with respect to the issues in which we are not interested. Proposition 2.10 would hold with respect to the issues in which we are interested, as well.

4 Appendix

4.1 Independence of the Axioms

We make seven claims about the independence of the axioms used in this paper.

Claim 1 *The meet separability, join separability, non-degeneracy, and anonymity axioms are independent.*

Claim 2 *The meet separability, join separability, subgroup non-degeneracy, and anonymity axioms are independent.*

Claim 3 *The meet separability, join separability, subgroup non-degeneracy, and symmetry axioms are independent.*

Proof: We present four rules. The first rule satisfies all of the above axioms except for meet separability. The second rule satisfies all of the above axioms except for join separability. The third rule satisfies all of the above axioms except for non-degeneracy and subgroup non-degeneracy. The fourth rule satisfies all of the above axioms except for symmetry and anonymity. This is sufficient to prove all three claims.

Rule 1: Consider the social rule f in which, for every $j \in N$ and $b \in \mathfrak{B}$, $f_j(P, b) = 1$ if and only if $P_{ij} = 1$ for some $i \in N$. This is a nomination rule and therefore satisfies join separability and non-degeneracy (by Theorem 2.4).

To show that it satisfies subgroup non-degeneracy, let $S \subset N$ and let $P^S \in \{0,1\}^{N\times N}$ such that $P^S_{ij} = 1$ if and only if $j \in S$. Then $\{j : f_j(P^S, b) = 1\} = S$.

To show that it satisfies anonymity, consider an arbitrary $j \in N$ and let π be a permutation of N. According to this rule, $f_j(P, b) = 1$ if and only if there exists an $i \in N$ such that $P_{ij} = 1$. Then $f_j(\pi P) = 1$ if and only if there exists an $i \in N$ such that $(\pi P)_{ij} = P_{\pi(i)\pi(j)} = 1$. Because this is true for any $i \in N$, $f_j(\pi P, b) = 1$ if and only if there exists an $i \in N$ such that $P_{i\pi(j)} = 1$. Furthermore, $\pi f_j(P, b) = f_{\pi(j)}(P, b) = 1$ if and only if there exists an $i \in N$ such that $P_{i\pi(j)} = 1$. Therefore, $\pi f_j(P, b) = f_j(\pi P)$. Because this is true for an arbitrary $j \in N$ it follows that $\pi f(P, b) = f(\pi P)$.

To show that this rule satisfies symmetry, assume that there exist $j, k \in N$ such that (a) $P_{jj} = P_{kk}$, (b) $P_{kj} = P_{jk}$, and, for all $i \in N \setminus \{j, k\}$, (c) $P_{ij} = P_{ik}$, and (d) $P_{ji} = P_{ki}$. Conditions (a), (b), and (c) imply that $|\{i \in N : P_{ij} = 1\}| = |\{i \in N : P_{ik} = 1\}|$, and therefore $f_j(P, b) = f_k(P, b)$.

Lastly, to show that the rule does not satisfy the meet separability axiom, let $P \in \{0,1\}^{N\times N}$ such that, for all $j \in N$, $P_{ij} = 1$ if and only if $i = 1$, and let $Q \in \{0,1\}^{N\times N}$ such that, for all $j \in N$, $Q_{ij} = 1$ if and only if $i = 2$. Then, for all $j \in N$, $f_j(P, b) = f_j(Q, b) = f_j(P, b) \wedge f_j(Q, b) = 1$ but $f_j(P \wedge Q, b) = f_j(\mathbf{0}, b) = 0$. Therefore $f(P \wedge Q, b) \neq f(P, b) \wedge f(Q, b)$. Because f satisfies non-degeneracy it follows from Theorem 2.1 that it fails meet separability.

Rule 2: Consider the social rule f in which, for every $j \in N$ and $b \in \mathfrak{B}$, $f_j(P, b) = 1$ if and only if $P_{ij} = 1$ for all $i \in N$. This is an agreement rule and therefore satisfies meet separability and non-degeneracy (by Theorem 2.3).

To show that it satisfies subgroup non-degeneracy, let $S \subset N$ and let $P^S \in \{0,1\}^{N\times N}$ such that $P^S_{ij} = 1$ if and only if $j \in S$. Then $\{j : f_j(P^S, b) = 1\} = S$.

To show that it satisfies anonymity, consider an arbitrary $j \in N$ and let π be a permutation of N. According to this rule, $f_j(P, b) = 1$ if and only if $P_{ij} = 1$ for all $i \in N$. Then $f_j(\pi P, b) = 1$ if and only if $(\pi P)_{ij} = P_{\pi(i)\pi(j)} = 1$ for all $i \in N$. Because this must be true for all $i \in N$, $f_j(\pi P, b) = 1$ if and only if $P_{i\pi(j)} = 1$ for all $i \in N$. Furthermore, $\pi f_j(P, b) = f_{\pi(j)}(P, b) = 1$ if and only if $P_{i\pi(j)} = 1$ for all $i \in N$. Therefore, $\pi f_j(P, b) = f_j(\pi P)$. Because this is true for an arbitrary $j \in N$ it follows that $\pi f(P, b) = f(\pi P)$.

To show that this rule satisfies symmetry, assume that there exist $j, k \in N$ such that (a) $P_{jj} = P_{kk}$, (b) $P_{kj} = P_{jk}$, and, for all $i \in N \setminus \{j, k\}$, (c) $P_{ij} = P_{ik}$, and (d) $P_{ji} = P_{ki}$. Conditions (a), (b), and (c) imply that $|\{i \in N : P_{ij} = 1\}| = |\{i \in N : P_{ik} = 1\}|$, and therefore $f_j(P, b) = f_k(P, b)$.

Lastly, to show that the rule does not satisfy the join separability axiom, let $P \in \{0,1\}^{N\times N}$ such that, for all $j \in N$, $P_{ij} = 1$ if and only if $i = 1$, and let $Q \in \{0,1\}^{N\times N}$ such that, for all $j \in N$, $Q_{ij} = 1$ if and only if $i \neq 1$. Then, for all $j \in N$, $f_j(P,b) = f_j(Q,b) = f_j(P,b) \vee f_j(Q,b) = 0$ but $f_j(P \vee Q, b) = f_j(\mathbf{1}) = 1$. Therefore $f(P \vee Q, b) \neq f(P,b) \vee f(Q,b)$. Because f satisfies non-degeneracy it follows from Theorem 2.1 that it fails join separability.

Rule 3: Consider the degenerate social rule f in which $f_j(P,b) = 1$ for every $j \in N$, $b \in \mathfrak{B}$, and all profiles $P \in \{0,1\}^{N\times N}$. This trivially satisfies the meet separability, join separability, anonymity, and symmetry axioms, but violates non-degeneracy and subgroup non-degeneracy.

Rule 4: Consider the social rule f in which, for every $j \in N$ and $b \in \mathfrak{B}$, $f_j(P,b) = 1$ if and only if $P_{1j} = 1$. This is a one-vote rule and therefore satisfies the meet separability, join separability, and non-degeneracy axioms (by Theorem 2.5). To show that it satisfies subgroup non-degeneracy, let $S \subset N$ and let $P^S \in \{0,1\}^{N\times N}$ such that $P^S_{ij} = 1$ if and only if $j \in S$. Then $\{j : f_j(P^S, b) = 1\} = S$. However, f is not the liberal rule and therefore clearly violates the anonymity and symmetry axioms (by Theorems 3.1 and 3.5). ∎

Claim 4 *The meet separability, self-duality, and anonymity axioms are independent.*

Claim 5 *The meet separability, self-duality, subgroup non-degeneracy, and symmetry axioms are independent.*

Proof: We present four rules. The first rule satisfies all of the above axioms except for meet separability. The second rule satisfies all of the above axioms except for self-duality. The third rule satisfies meet separability, self-duality, and symmetry, but does not satisfy subgroup non-degeneracy. The fourth rule satisfies all of the above axioms except for symmetry and anonymity. This is sufficient to prove both claims.

Rule 1: Consider the rule in which, for every $j \in N$ and $b \in \mathfrak{B}$, $f_j(P,b) = P_{jj}$ if and only if there exists an $i \in N$, $i \neq j$, such that $P_{jj} = P_{ij}$. To show that this rule satisfies self-duality, there are two cases. First, suppose there exists an $i \in N$, $i \neq j$, such that $P_{jj} = P_{ij}$. Then there is an $i \in N$, $i \neq j$, such that $\bar{P}_{jj} = \bar{P}_{ij}$. This implies that $f_j(P,b) = P_{jj}$ and $f_j(\bar{P},b) = \bar{P}_{jj}$, which implies that $\bar{f}_j(P,b) = \overline{f_j(\bar{P},b)} = \bar{\bar{P}}_{jj} = P_{jj}$. Therefore $\bar{f}_j(P,b) = f_j(P,b)$. Next, suppose that there does not exist an $i \in N$, $i \neq j$, such that $P_{jj} = P_{ij}$. Then there does not exist an $i \in N$, $i \neq j$, such that $\bar{P}_{jj} = \bar{P}_{ij}$. This implies that $f_j(P,b) = \bar{P}_{jj}$ and $f_j(\bar{P},b) = P_{jj}$, which implies that $\bar{f}_j(P,b) = \overline{f_j(\bar{P},b)} = \bar{P}_{jj}$. Therefore $\overline{f}_j(P,b) = f_j(P,b)$.

To show that it satisfies subgroup non-degeneracy, let $S \subset N$ and let $P^S \in \{0,1\}^{N\times N}$ such that $P^S_{ij} = 1$ if and only if $j \in S$. Then $\{j : f_j(P^S, b) = 1\} = S$.

To show that this rule satisfies the anonymity axiom, consider an arbitrary $j \in N$ and let π be a permutation of N. According to this rule, $f_j(P, b) = P_{jj}$ if and only if there exists an $i \in N$, $i \neq j$, such that $P_{jj} = P_{ij}$. Then $f_j(\pi P, b) = (\pi P)_{jj}$ if and only if there exists an $i \in N$, $i \neq j$, such that $P_{\pi(j)\pi(j)} = P_{\pi(i)\pi(j)}$. Furthermore, $\pi f_j(P, b) = f_{\pi(j)}(P, b) = P_{\pi(j)\pi(j)}$ if and only if there exists an $\pi(i) \in N$, $\pi(i) \neq \pi(j)$, such that $P_{\pi(j)\pi(j)} = P_{\pi(i)\pi(j)}$. Therefore, $\pi f_j(P, b) = f_j(\pi P)$. Because this is true for an arbitrary $j \in N$ it follows that $\pi f(P, b) = f(\pi P)$.

To show that this rule satisfies symmetry, assume that there exist $j, k \in N$ such that (a) $P_{jj} = P_{kk}$, (b) $P_{kj} = P_{jk}$, and, for all $i \in N \setminus \{j, k\}$, (c) $P_{ij} = P_{ik}$, and (d) $P_{ji} = P_{ki}$. Conditions (b) and (c) imply that $|\{i \in N \setminus \{j\} : P_{ij} = 1\}| = |\{i \in N \setminus \{k\} : P_{ik} = 1\}|$. From this and from condition (a) it follows that $f_j(P, b) = f_k(P, b)$.

Lastly, to show that the rule does not satisfy the meet separability axiom, let $P \in \{0, 1\}^{N \times N}$ such that, for all $j \in N$, $P_{ij} = 1$ if and only if $i = j$ or $i = j - 1$, and let $Q \in \{0, 1\}^{N \times N}$ such that, for all $j \in N$, $Q_{ij} = 1$ if and only if $i = j$ or $i = j + 1$. Then $f_2(P, b) = f_2(Q, b) = f_2(P, b) \wedge f_2(Q, b) = 1$ and $f_2(P \wedge Q, b) = 0$. Therefore $f(P \wedge Q, b) \neq f(P, b) \wedge f(Q, b)$. Because f satisfies self-duality (and therefore non-degeneracy) it follows from Theorem 2.1 that it fails meet separability.

Rule 2: Consider the rule in which, for all $j \in N$ and $b \in \mathfrak{B}$, $f_j(P, b) = 1$ if and only if $P_{ij} = 1$ for all $i \in N$. This is an agreement rule and therefore satisfies meet separability (by Theorem 2.1).

To show that it satisfies subgroup non-degeneracy, let $S \subset N$ and let $P^S \in \{0, 1\}^{N \times N}$ such that $P^S_{ij} = 1$ if and only if $j \in S$. Then $\{j : f_j(P^S, b) = 1\} = S$.

To show that it satisfies anonymity, consider an arbitrary $j \in N$ and let π be a permutation of N. According to this rule, $f_j(P, b) = 1$ if and only if $P_{ij} = 1$ for all $i \in N$. Then $f_j(\pi P, b) = 1$ if and only if $(\pi P)_{ij} = P_{\pi(i)\pi(j)} = 1$ for all $i \in N$. Because this must be true for all $i \in N$, $f_j(\pi P, b) = 1$ if and only if $P_{i\pi(j)} = 1$ for all $i \in N$. Furthermore, $\pi f_j(P, b) = f_{\pi(j)}(P, b) = 1$ if and only if $P_{i\pi(j)} = 1$ for all $i \in N$. Therefore, $\pi f_j(P, b) = f_j(\pi P, b)$. Because this is true for an arbitrary $j \in N$ it follows that $\pi f(P, b) = f(\pi P, b)$.

To show that this rule satisfies symmetry, assume that there exist $j, k \in N$ such that (a) $P_{jj} = P_{kk}$, (b) $P_{kj} = P_{jk}$, and, for all $i \in N \setminus \{j, k\}$, (c) $P_{ij} = P_{ik}$, and (d) $P_{ji} = P_{ki}$. Conditions (a), (b), and (c) imply that $|\{i \in N : P_{ij} = 1\}| = |\{i \in N : P_{ik} = 1\}|$, and therefore $f_j(P, b) = f_k(P, b)$.

Lastly, to show that this rule does not satisfy self-duality, construct a profile P' such that $P'_{ij} = 1$ if and only if $i = 1$. Then for all $j \in N$, $f_j(P', b) = 0$ and $f_j(\bar{P}', b) = 0$. This implies that $\overline{f_j(\bar{P}', b)} = \bar{f}_j(P', b) = 1$. Because $\bar{f}_j(P', b) \neq f_j(P', b)$, f is not self-dual.

Rule 3: Consider the rule in which, for all $j \in N$ and $b \in \mathfrak{B}$, $f_j(P, b) = 1$ if and only if $P_{11} = 1$. This rule is a one-vote rule and therefore satisfies the meet separability and self-duality axioms. Furthermore it satisfies symmetry as for all $i, j \in N$ and all $P \in \{0,1\}^{N \times N}$, $f_i(P, b) = f_j(P, b)$. However it does not satisfy subgroup non-degeneracy as for all $S \subset N$, $S \neq N$, $S \neq \varnothing$, there is no profile P such that $\{j : f_j(P, b) = 1\} = S$.

Rule 4: Consider the social rule f in which, for all $j \in N$ and $b \in \mathfrak{B}$, $f_j(P, b) = 1$ if and only if $P_{1j} = 1$. This is a one-vote rule and therefore satisfies the meet separability and self-duality axioms. To show that it satisfies subgroup non-degeneracy, let $S \subset N$ and let $P^S \in \{0,1\}^{N \times N}$ such that $P^S_{ij} = 1$ if and only if $j \in S$. Then $\{j : f_j(P^S, b) = 1\} = S$. However, f is not the liberal rule and therefore clearly violates the anonymity and symmetry axioms (by Corollaries 3.2 and 3.6). ∎

Claim 6 *The join separability, self-duality, and anonymity axioms are independent.*

Claim 7 *The join separability, self-duality, subgroup non-degeneracy, and symmetry axioms are independent.*

Proof: We present four rules. The first rule satisfies all of the above axioms except for join separability. The second rule satisfies all of the above axioms except for self-duality. The third rule satisfies join separability, self-duality, and symmetry, but does not satisfy subgroup non-degeneracy. The fourth rule satisfies all of the above axioms except for symmetry and anonymity. This is sufficient to prove both claims.

Rule 1: Consider the rule in which, for every $j \in N$ and $b \in \mathfrak{B}$, $f_j(P, b) = P_{jj}$ if and only if there exists an $i \in N$, $i \neq j$, such that $P_{jj} = P_{ij}$. To show that this rule satisfies self-duality, there are two cases. First, suppose there exists an $i \in N$, $i \neq j$, such that $P_{jj} = P_{ij}$. Then there is an $i \in N$, $i \neq j$, such that $\bar{P}_{jj} = \bar{P}_{ij}$. This implies that $f_j(P, b) = P_{jj}$ and $f_j(\bar{P}, b) = \bar{P}_{jj}$, which implies that $\bar{f}_j(P, b) = \overline{f_j(\bar{P}, b)} = \bar{\bar{P}}_{jj} = P_{jj}$. Therefore $\bar{f}_j(P, b) = f_j(P, b)$. Next, suppose that there does not exist an $i \in N$, $i \neq j$, such that $P_{jj} = P_{ij}$. Then there does not exist an $i \in N$, $i \neq j$, such that $\bar{P}_{jj} = \bar{P}_{ij}$. This implies that $f_j(P, b) = \bar{P}_{jj}$ and $f_j(\bar{P}, b) = P_{jj}$, which implies that $\bar{f}_j(P, b) = \overline{f_j(\bar{P}, b)} = \bar{P}_{jj}$. Therefore $\bar{f}_j(P, b) = f_j(P, b)$.

To show that it satisfies subgroup non-degeneracy, let $S \subset N$ and let $P^S \in \{0,1\}^{N \times N}$ such that $P^S_{ij} = 1$ if and only if $j \in S$. Then $\{j : f_j(P^S, b) = 1\} = S$.

To show that this rule satisfies the anonymity axiom, consider an arbitrary $j \in N$ and let π be a permutation of N. According to this rule, $f_j(P, b) = P_{jj}$ if and only if there exists an $i \in N$, $i \neq j$, such that $P_{jj} = P_{ij}$. Then $f_j(\pi P, b) = (\pi P)_{jj}$ if and only if there exists an $i \in N$, $i \neq j$, such that $P_{\pi(j)\pi(j)} = P_{\pi(i)\pi(j)}$. Furthermore, $\pi f_j(P, b) = f_{\pi(j)}(P, b) = P_{\pi(j)\pi(j)}$ if and only if there exists an $\pi(i) \in N$, $\pi(i) \neq \pi(j)$, such that $P_{\pi(j)\pi(j)} = P_{\pi(i)\pi(j)}$. Therefore, $\pi f_j(P, b) = f_j(\pi P)$. Because this is true for an arbitrary $j \in N$ it follows that $\pi f(P, b) = f(\pi P)$.

To show that this rule satisfies symmetry, assume that there exist $j, k \in N$ such that (a) $P_{jj} = P_{kk}$, (b) $P_{kj} = P_{jk}$, and, for all $i \in N \setminus \{j, k\}$, (c) $P_{ij} = P_{ik}$, and (d) $P_{ji} = P_{ki}$. Conditions (b) and (c) imply that $|\{i \in N \setminus \{j\} : P_{ij} = 1\}| = |\{i \in N \setminus \{k\} : P_{ik} = 1\}|$. From this and from condition (a) it follows that $f_j(P, b) = f_k(P, b)$.

Lastly, to show that the rule does not satisfy the join separability axiom, let $P \in \{0,1\}^{N \times N}$ such that, for all $j \in N$, $P_{ij} = 1$ if and only if $i = j$, and let $Q \in \{0,1\}^{N \times N}$ such that, for all $j \in N$, $Q_{ij} = 1$ if and only if $i = j + 1$. Then $f_2(P, b) = f_2(Q, b) = f_2(P, b) \vee f_2(Q, b) = 0$ but $f_2(P \vee Q, b) = 1$. Therefore $f(P \vee Q, b) \neq f(P, b) \vee f(Q, b)$. Because f satisfies self-duality (and therefore non-degeneracy) it follows from Theorem 2.1 that it fails join separability.

Rule 2: Consider the rule in which, for every $j \in N$ and $b \in \mathfrak{B}$, $f_j(P, b) = 1$ if and only if $P_{ij} = 1$ for some $i \in N$. This is a nomination rule and therefore satisfies join separability (by Theorem 2.1).

To show that it satisfies subgroup non-degeneracy, let $S \subset N$ and let $P^S \in \{0,1\}^{N \times N}$ such that $P^S_{ij} = 1$ if and only if $j \in S$. Then $\{j : f_j(P^S, b) = 1\} = S$.

To show that it satisfies anonymity, consider an arbitrary $j \in N$ and let π be a permutation of N. According to this rule, $f_j(P, b) = 1$ if and only if there exists an $i \in N$ such that $P_{ij} = 1$. Then $f_j(\pi P, b) = 1$ if and only if there exists an $i \in N$ such that $(\pi P)_{ij} = P_{\pi(i)\pi(j)} = 1$. Because this is true for any $i \in N$, $f_j(\pi P, b) = 1$ if and only if there exists an $i \in N$ such that $P_{i\pi(j)} = 1$. Furthermore, $\pi f_j(P, b) = f_{\pi(j)}(P, b) = 1$ if and only if there exists an $i \in N$ such that $P_{i\pi(j)} = 1$. Therefore, $\pi f_j(P, b) = f_j(\pi P, b)$. Because this is true for an arbitrary $j \in N$ it follows that $\pi f(P, b) = f(\pi P, b)$.

To show that this rule satisfies symmetry, assume that there exist $j, k \in N$ such that (a) $P_{jj} = P_{kk}$, (b) $P_{kj} = P_{jk}$, and, for all $i \in N \setminus \{j, k\}$, (c) $P_{ij} = P_{ik}$, and (d) $P_{ji} = P_{ki}$. Conditions (a), (b), and (c) imply that $|\{i \in N : P_{ij} = 1\}| = |\{i \in N : P_{ik} = 1\}|$, and therefore $f_j(P, b) = f_k(P, b)$.

Lastly, to show that this rule does not satisfy self-duality, construct a profile P' such that $P'_{ij} = 1$ if and only if $i = 1$. Then for all $j \in N$, $f_j(P', b) = 1$ and $f_j(\bar{P}', b) = 1$. This implies that $\overline{f_j(\bar{P}', b)} = \bar{f}_j(P', b) = 0$. Because $\bar{f}_j(P', b) \neq f_j(P', b)$, f is not self-dual.

Rule 3: Consider the rule in which, for all $j \in N$ and $b \in \mathfrak{B}$, $f_j(P, b) = 1$ if and only if $P_{11} = 1$. This rule is a one-vote rule and therefore satisfies the join separability and self-duality axioms. Furthermore it satisfies symmetry as for all $i, j \in N$ and all $P \in \{0,1\}^{N \times N}$, $f_i(P, b) = f_j(P, b)$. However it does not satisfy subgroup non-degeneracy as for all $S \subset N$, $S \neq N$, $S \neq \varnothing$, there is no profile P such that $\{j : f_j(P, b) = 1\} = S$.

Rule 4: Consider the social rule f in which, for all $j \in N$ and $b \in \mathfrak{B}$, $f_j(P, b) = 1$ if and only if $P_{1j} = 1$. This is a one-vote rule and therefore satisfies the join separability and self-duality axioms. To show that it satisfies subgroup non-degeneracy, let $S \subset N$ and let $P^S \in \{0,1\}^{N \times N}$ such that $P^S_{ij} = 1$ if and only if $j \in S$. Then $\{j : f_j(P^S, b) = 1\} = S$. However, f is not the liberal rule and therefore clearly violates the anonymity and symmetry axioms (by Corollaries 3.3 and 3.7). ∎

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