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NON-EXISTENCE OF EQUILIBRIUM IN VICKREY, SECOND-PRICE, AND ENGLISH AUCTIONS

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Abstract

A simple example shows that equilibria can fail to exist in second price (Vickrey) and English auctions when there are both common and private components to bidders' valuations and private information is held on both dimensions. The example shows that equilibrium only exists in the extremes of pure private and pure common values, and that existence in standard models is not robust to a slight perturbation.

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1 Introduction

Characterizations of equilibrium bidding behavior have been a cornerstone of a large literature that includes theoretical, experimental, and empirical analyses of auctions. Having analytic characterizations of bidding functions has made the analyses both tractable and powerful. Such characterizations have largely relied, however, on modeling the private information of bidders in such a way that it can be ordered. Either private values are assumed, or information that satisfies either a monotone likelihood ratio property or an affiliation property.[1] This has provided foundations for the understanding of behavior in a wide variety of standard auctions.

While the settings studied are special, one would hope for some robustness of the results. This is especially important as almost all natural settings have some additional richness to them. For instance, it is rare to find purely private value settings, as the presence of alternative markets and/or the possibility of resale means that a bidder's effective valuation for an item is not simply a function of the bidder's preferences, but also depends on information they might have about the potential to buy or sell the good outside of the current auction. Note also that such information would generally be more than one dimensional. A bidder might have some information about the private aspects of the good and different information about the outside market conditions.[2]

The example presented here shows that there is reason for concern. The existence of non-degenerate equilibrium turns out to be sensitive to even the slightest variation in the information. The setting is the simplest possible extension to a situation where bidders' information has more than one dimension. An agent's valuation is a convex combination of a private and a common component. The agent knows his or her private value and sees a signal concerning the common value, and then bids in a sealed bid, second price auction. If preferences place weight entirely on the private component (a standard private values setting) or entirely on the common component (a standard common values setting) then symmetric equilibria as well as equilibria in undominated

*This is a revised version of Jackson (1999), where the implications of the example have been strengthened, and the discussion of its implications for the auctions literature has been updated. I thank Andy Postlewaite for encouraging me to resurrect this paper, Laurent Mathevet for helpful comments on an earlier draft, and Jeroen Swinkels for many valuable discussions of the subject.

[1]See Milgrom (1981) and Milgrom and Weber (1982) for seminal works in applying such signal ordering properties to auctions and developing implications for and characterizations of bidding behavior.

[2]See Pesendorfer and Swinkels (2000) for more discussion of the importance of such a two dimensional setting.

strategies exist. These correspond to the ones known and characterized in the literature. However, if preferences place weight on both components, *regardless of what the relative weights are*, there does not exist a symmetric equilibrium or even an equilibrium in undominated strategies.

The intuition for the non-existence is relatively straightforward. It is impossible to order bids that depend on both private and common components so that they are consistent with the incentives of the bidders given the information that they can infer from the price. Some basic equilibrium restrictions imply a certain ordering of bids under which winning at lower bids can convey more positive information about the value of the object than winning at higher bids. This produces perverse incentives in terms of best responses, and a lack of existence.

It is particularly disturbing that the example here is essentially the simplest example one can write down where there both private and common components matter in bidders' valuations. Moreover, the signals satisfy the strict monotone likelihood ratio property, and so the nonexistence problem is not an artifact of some negative interdependence. Higher signals are better news for all agents and the setting is fully symmetric.

A discussion of the relationship between this example and recent existence theorems is deferred until after the details of the example have been made clear.

2 Definitions and the Example

There is a finite number $N \geq 2$ of agents who are bidding for a single indivisible object.

Agent i has a utility for the object which is described by $at_i + (1-a)q$, where $t_i \in [0,1]$ is a private component, $q \in \{0,1\}$ is a common component, and $a \in [0,1]$ is a weighting factor.

When $a = 0$ the setting is a standard common values setting. When $a = 1$ the setting is a standard private values setting. When $1 > a > 0$ there are both private and common components to an agent's valuation.

An individual's private component, t_i, is described by the random variable T_i. T_i takes on values $\{0, \varepsilon, 2\varepsilon, \ldots, 1\}$ with equal probability, where 1 is a multiple of ε. The T_i's are independently distributed across individuals. For convenience, assume that $1-a$ is not a multiple of $a\varepsilon$, which is clearly true generically.

The value of the common parameter, q, is described by the random variable Q. Q is independent of the T_i's. Q takes on the value 0 with probability $\frac{1}{2}$ and value 1 with probability $\frac{1}{2}$.

Bidder i observes his or her private type T_i and also the realization of a random signal S_i which provides information about the value of Q. S_i takes on values in $\{0, \frac{1}{2}, 1\}$. The S_i's are independently distributed conditional on Q, and satisfy $P(S_i = Q) = 1 - m$ and $P(S_i = \frac{1}{2}) = m$.[3]

The above signal structure leads to easy updating by Bayes' rule. Conditional on $S_i = 0$ or $S_i = 1$, agent i knows the state; while conditional on $S_i = \frac{1}{2}$, i places equal probability on the two states.

The auction is a standard sealed-bid second-price auction. Each bidder i submits a bid $b_i \in I\!R_+$. The highest bidder is awarded the object and pays the second highest bid. In the event that there is a tie at the highest bid, the object is awarded with equal probability to one of the highest bidders.

[3]Note that signals satisfy the strict monotone likelihood ratio property so that higher signals provide more favorable news concerning the value of Q.

2.1 Equilibrium

A bidder's strategy is a profile of probability distribution functions on $I\!R_+$, $(F^i_{t_i,s_i})_{t_i,s_i}$, that describe how i bids conditional on each possible combination of realizations of t_i and s_i. So, $F^i_{t_i,s_i}(b)$ is the probability that i bids no more than b conditional on observing t_i, s_i. Let F^i denote the profile $(F^i_{t_i,s_i})_{t_i,s_i}$ and F^{-i} the profile $(F^j)_{j\neq i}$.

An equilibrium is a Bayesian equilibrium of the game where a bidder i's payoff is $aT_i+(1-a)Q-p$ if i is awarded the object and p is the second highest bid, and where i's utility is 0 if i is not awarded an object.

It is well-known that for the second price auction there always exists an asymmetric equilibrium. For example, let bidder 1 unconditionally bid X, where $X \geq 1$, and have all other bidders bid 0. This is an equilibrium regardless of the weighting factor a.

There is more than one way to rule out this sort of degenerate equilibrium. Let us consider two different refinements that do so: (i) symmetric equilibrium and (ii) equilibrium that only uses strategies that are in the closure of the set of undominated strategies.[4] Here, the refinement to undominated strategies is more general than a refinement to symmetric equilibrium and thus provides for a stronger non-existence result. However, as symmetric equilibria are the main ones studied in the literature, I include results concerning both.

An equilibrium is *symmetric* if $F^i_{t,s} = F^j_{t,s}$ for each t, s and i, j.

A bid $b'_i \in [0,1]$ is *weakly dominated* by a bid b_i for i at t_i, s_i if for every F^{-i}, the expected payoff to b_i conditional on t_i, s_i and F^{-i} is greater than the expected payoff to b'_i conditional on t_i, s_i and F^{-i}, with strict inequality holding for some F^{-i}.[5]

A strategy b_i is *undominated* for i at t_i, s_i if it is not weakly dominated by any other strategy.

An equilibrium is in *undominated** strategies, if for every i the support of $F^i_{t_i,s_i}$ lies in the closure of the set of undominated strategies for i at t_i, s_i.[6]

Vickrey (1961) showed that when $a = 1$ the auction has a unique equilibrium in dominant strategies which is also symmetric: i puts probability 1 on the bid t_i. Milgrom (1981) has shown that when $a = 0$ the auction has a symmetric equilibrium (which is in undominated strategies). In the example here when $N = 2$, it is a symmetric equilibrium where each bidder bids 0 when observing $S_i = 0$, bids $\frac{1}{2}$ when observing $S_i = \frac{1}{2}$, and bids 1 when observing $S_i = 1$. Thus, for the extremes of $a = 0$ and $a = 1$, equilibria exist.

Proposition 1 *For any $N \geq 2$ and any $1 > a > 0$ there exists $\overline{m} > 0$ and $\overline{\varepsilon} > 0$ such that if $m < \overline{m}$ and $\varepsilon < \overline{\varepsilon}$, then the second price auction does not have either a symmetric equilibrium or an equilibrium in undominated * strategies.*

To get some intuition for Proposition 1, let me provide the proof for a situation where $N = 2$. The full proof appears in the next section.

[4] Generally, equilibria in undominated strategies are a superset of trembling hand perfect equilibria, except for $N = 2$ where they coincide. As this is a larger class of equilibrium, it provides for a stronger conclusion of non-existence.

[5] Note that this definition can also be required to hold relative to the realization of final utility Q without changing any of the implications of the paper.

[6] The importance of using the closure of the set of undominated strategies in games with continuum action spaces is discussed in Jackson and Swinkels (2004).

First, note that if bidders use strategies in the closure of the set of undominated strategies, then conditional on any T_i and $S_i \in \{0,1\}$, a bidder must bid $aT_i + (1-a)S_i$, and conditional on any T_i and $S_i = \frac{1}{2}$, a bidder must bid in $[aT_i, aT_i + (1-a)]$. Under a symmetric equilibrium these strategies must also be followed. For example, suppose that instead there was some T_i and $S_i \in \{0,1\}$ such that each i bid above $aT_i + (1-a)$ with positive probability (the argument ruling out bids below is similar). In that case some i would win with positive probability when the price was above $aT_i + (1-a)$ and their value was at most $aT_i + (1-a)$. They could improve their payoff by lowering their bid to $aT_i + (1-a)$.

Next, consider the strategy followed by a bidder i conditional on observing $T_i = 0, S_i = \frac{1}{2}$. From above, we know that his or her bids are confined to the interval $[0, 1-a]$. Let $\varepsilon < \frac{1-a}{2a}$, so that $a\varepsilon < \frac{1-a}{2}$. Let us argue that if m is small enough, then neither bidder places positive probability in the range $[a\varepsilon, 1-a]$. Conditional on winning with a bid $b_i \in [a\varepsilon, 1-a]$, if m is small enough, then conditional on the price being below the maximal possible value of $1-a$ the overwhelming probability is that the other bidder observed a signal regarding the common component of 0 (all bidders having observed signals of 1 bid at least $1-a$). Thus, for small enough m, the overall conditional expected value when winning in this interval is below $a\varepsilon$, and so the bidder has a negative expected value conditional on winning in this interval and could improve by instead bidding below $a\varepsilon$. So, both bidders conditional on observing $T_i = 0, S_i = \frac{1}{2}$, bid in the range $[0, a\varepsilon)$. A bidder i faces two possibilities in this range. First that the other bidder observed $T_j = 0, S_j = 0$ and bid 0 in which case the price is 0. This is inconsequential to i's incentives. Second, the other bidder observed $T_j = 0, S_j = \frac{1}{2}$. In this case, i's conditional valuation is $\frac{1-a}{2} > a\varepsilon$ and i would strictly prefer to win against the other bidder at any price in $[0, a\varepsilon)$. This contradicts the fact that both bidders place probability one in this interval. We are left with no possible bids conditional on having observed $T_i = 0, S_i = \frac{1}{2}$ and so equilibrium does not exist.

3 Discussion

The example presented here shows that with even a slight perturbation to multidimensional types, inference from prices may be non-monotone and can lead to nonexistence.

While the results are stated for second-price or Vickrey auctions, these are easily extended to cover English auctions. As it holds for Vickrey auctions, it also holds for some classes of Groves mechanisms (at least respecting some symmetry conditions). However, for the case of first-price auctions and Dutch auctions, it is not clear whether existence holds or not. The strategic considerations are a bit different in those auctions and the type of dominance and symmetry arguments used here do not have the same sort of implications.

Let me discuss three possible changes to the setting that could potentially restore existence of equilibrium. These are to change the setting so that there are:

(1) finite bidding grids,

(2) type-dependent tie-breaking rules in the auction, or

(3) atomless type and information distributions.

Let me first discuss (1). By changing to a finite grid of potential bids, the game would become finite and so an equilibrium (in mixed strategies) would clearly exist. In fact, Reny (2005) proves existence of pure strategy monotone equilibria with multidimensional types in a class of Bayesian games.[7] As his theorem requires continuity of payoffs in actions, it applies if the bids are required to fall on a finite grid, but does not apply to standard models of auctions with continuum bidding sets, where there are discontinuities at points of ties in bids. The example here makes it clear that a general result of that form that would apply to games with continuum action spaces cannot hold in cases where there may be discontinuities in payoffs, such as at tied bids.

As auctions in practice involve finite bidding grids, why should we be disturbed if equilibria only fail to exist for continuum bidding spaces? The answer is twofold. First, models where interesting and nontrivial properties of bidding behavior have been obtained almost without exception have continuum bidding spaces, where differentiation can be used to deduce necessary conditions for equilibrium. In fact, there are models where some important properties of bidding behavior have been deduced conditional on equilibrium existence, but without knowing whether equilibria exist (e.g., Pesendorfer and Swinkels (2000)). Second, the fact that equilibria can fail to exist in the limit should cause us to wonder why the properties of equilibrium are sensitive in this way, and this should give us pause to wonder whether the equilibria of the finite auctions are robust.

Next, let me discuss the related point (2). Jackson, Simon, Swinkels and Zame (2002) prove an existence theorem for a general class of Bayesian games that may have discontinuities. Their theorem covers the example presented here, provided one is allowed to alter the tie-breaking rule to depend on the private information of the bidders (in an incentive compatible manner) and not just their bids. In the setting analyzed here, existence is obtained if one uses a tie-breaking rule that gives the object to one of the bidders who has a highest private value. This ties back to (1), as the tie-breaking rule is found by looking at the limiting behavior of auctions where bids are restricted to finite grids, as the grids become finer.

That sort of result is useful in two ways. First, it gives us some understanding of where the non-existence issue with fixed tie-breaking in games with a continuum of actions comes from, and provides a reasonable solution to the problem. Second, it provides a tool that can be used, at least in some cases, to prove existence even with a fixed tie-breaking rule. For instance, Jackson and Swinkels (2004) show that in a wide class of private value auctions, ties occur with 0 probability and are immaterial, and thus are able to build from the results of Jackson, Simon, Swinkels and Zame (as well as results of Reny (1999)) to show that there exists an equilibrium with any tie-breaking rule.[8]

Although the use of such endogenous and type-dependent tie-breaking rules is useful, this does not mean that we should be happy to ignore the non-existence with standard tie-breaking. When equilibria only exist under type-dependent tie breaking, we should expect to face substantial difficulties in deducing properties of equilibria, as the points of discontinuity are being hit with non-trivial probability and are mattering. Moreover, we also know that all equilibria in such situations are non-robust to small perturbations in the game, as they only exist for some particular specifications of tie-breaking.

Finally, let me discuss point (3). One aspect of the example here that greatly simplifies the argument and allows one to prove nonexistence is the finite type space. There is a possibility that

[7] His results generalize earlier results of Athey (2001) and McAdams (2003).

[8] See also Lebrun (1999) Maskin and Riley (2000), and Araujo, de Castro, and Moreira (2004), who also work with non-standard tie-breaking techniques and show that in some cases they are inconsequential.

a change to a setting with atomless type distributions might restore existence in standard auctions. However, there is not much insight on this question to be gained from the previous literature, as it has either relied on pure private values, or else on types that are unidimensional together with some sort of monotone ordering. As the example here shows, the nature of multi-dimensional types and non-private values introduces issues of inference and ordering that are not easily overcome. The temptation is to guess that ties will occur with probability 0 in the face of atomless distributions, and so ties won't matter and existence can be established. (That is the route that Jackson and Swinkels (2004) follow to prove existence in the case of private values.) However, ruling out ties appears to be hard once one moves beyond private values, even with atomless distributions. The difficulty comes from the fact that bidders make inferences from winning, and might not prefer to win against some lower types, who might be bidding a high amount because of a high private signal. That is, only some dimensions of other bidders' information might be important to me as a bidder, and yet all of their information affects their bids. This can introduce interesting non-monotonicities in the best response correspondence, even in atomless settings with nice affiliation properties, and that is the principal hurdle in obtaining existence.

The extent to which equilibria exist in standard auctions (i.e., continuum bidding sets and standard tie-breaking) with multi-dimensional types remains an open and apparently difficult question.

4 Proof of Proposition 1

Consider a symmetric equilibrium or an equilibrium in undominated* strategies.

Claim 1 For any realization of T_i, a bidder i bids $aT_i + (1-a)S_i$ with probability one if $S_i \neq \frac{1}{2}$, and bids in the interval $[aT_i, aT_i + 1 - a]$ with probability one if $S_i = \frac{1}{2}$.

This is a straightforward extension of the argument given for the case of $N = 2$.

Claim 2 At most one bidder i has an atom at $a\varepsilon + 1 - a$ conditional upon $T_i = \varepsilon, S_i = \frac{1}{2}$.

Suppose to the contrary that both i and j have atoms at $a\varepsilon + 1 - a$ when observing $T_i = \varepsilon, S_i = \frac{1}{2}$. This implies, given Claim 1, that conditional on winning at a price of $a\varepsilon + 1 - a$, i has an expected value of Q that is less than 1. So, i could gain by slightly lowering the atom at $a\varepsilon + 1 - a$. This bidding could thus not have been part of an equilibrium, which is a contradiction.

Claim 3 If $\varepsilon < \frac{1-a}{2a}$, then there exists at most one i who with positive probability bids above $1 - a$ conditional on $T_i = \varepsilon, S_i = \frac{1}{2}$.

Let W_i denote the highest bid among the bidders other than i.

If there were more than one bidder with probability in this interval, then by the previous claims there is a positive probability that the price ends up in the open interval between $1 - a$ and $a\varepsilon + 1 - a$, and that a bidder observing $S = \frac{1}{2}$ wins. Given Claim 1, conditional on the price being in the open interval between $1 - a$ and $a\varepsilon + 1 - a$, a winning bidder who sees $S_i = \frac{1}{2}$ has a conditional expectation $E[Q|W_i \in ((1-a), a\varepsilon + (1-a)), S_i = \frac{1}{2}] \leq \frac{1}{2}$. [To see this note that conditional on $W_i \in ((1-a), a\varepsilon + (1-a))$ i must infer that any $b_j \leq W_i$ where $j \neq i$ and $S_j = 0$ or $S_j = 1$ are more likely to have come from bidders with $S_j = 0$, as they are more likely to have bid below W_i (as more than one realization of T_j conditional on $S_j = 0$, bids below $1 - a$, while only the realization of $T_j = 0$ leads to a bid of $1 - a$ conditional on $S_j = 1$).] Given that $\varepsilon < \frac{1-a}{2a}$, it follows that

$$0 > E[(a\varepsilon + (1-a)Q - (1-a))|W_i \in ((1-a), a\varepsilon + (1-a)), S_i = \frac{1}{2}].$$

So, a bidder i who has observed $T_i = a\varepsilon, S_i = \frac{1}{2}$ and who wins an object when the price falls in $((1-a), a\varepsilon + (1-a))$ would strictly prefer not to and could gain by lowering his or her bid (and this could not hurt the bidder if the price was at $a\varepsilon + (1-a)$.

The claims above imply that when observing $(\varepsilon, \frac{1}{2})$ there is at most one bidder who bids above $(1-a)$ with positive probability, and so the remaining bidders must bid in entirely in the interval $[a\varepsilon, 1-a]$. So consider a bidder who when observing $T_i = \varepsilon, S_i = \frac{1}{2}$ places positive probability in the interval $[a\varepsilon, 1-a]$. Consider i moving the probability under $F^i_{\varepsilon,\frac{1}{2}}$ from $[a\varepsilon, 1-a]$ to slightly above $1-a$. There would be two potential changes. First, i would win when $W_i = 1-a$, whereas before he or she either tied or lost. Note that for small enough m, $E[Q|W_i = 1-a]$ can be made arbitrarily close to 1, and so for small enough m, i would strictly gain from winning in this case and the case occurs with a probability that is bounded below as m is made small. Second, i wins whenever $W_i \in [a\varepsilon, 1-a)$. For small enough m there is a vanishing probability that W_i falls in $(a\varepsilon, 1-a)$, and i is at least indifferent when $W_i = a\varepsilon$. Thus, for small enough m i will strictly gain from this deviation, which contradicts equilibrium. ■

Araujo, A. L.I. de Castro, H. Moreira (2004), " Pure Strategy Equilibria of Multidimensional and Non-Monotonic Auctions," mimeo: IMPA.

Athey, S. (2001), "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica,* Vol. 69, pp. 861-890.

Jackson, M.O. (1999), "Non-Existence of Equilibrium in Second Price Auctions with Two Dimensional Types," Caltech.

Jackson, M.O., L.K. Simon, J.M. Swinkels, and W.R. Zame (2002), "Communication and Equilibrium in Discontinuous Games of Incomplete Information," *Econometrica,* Vol. 70, pp. 1711 - 1740.

Jackson, M.O., and J.M. Swinkels (2004), "Existence of Equilibria in Single and Double Private Value Auctions," *Econometrica,* Vol. 73, pp. 93-139.

Lebrun, B. (1999), "First Price Auctions in the Asymmetric N Player Case," *International Economic Review,* Vol. 40, pp. 125-142.

Maskin, E. and J. Riley (2000), "Equilibrium in Sealed High Bid Auctions," *Review of Economic Studies,* Vol. 67, pp. 439-452.

McAdams, D. (2003), "Isotone Equilibrium in Games of Incomplete Information," *Econometrica,* Vol. 71, pp. 1191-1214.

Milgrom, P. (1981), "Rational Expectations, Information Acquisition, and Competitive Bidding," *Econometrica,* Vol. 49, pp. 921–944.

Milgrom, P. and R Weber (1982), "A Theory of Auctions and Competitive Bidding," *Econometrica,* Vol. 50, pp. 1089-1122.

Pesendorfer, W. and J. Swinkels (2000), "Efficiency and Information Aggregation in Auctions," *American Economic Review*, Vol. 90, pp. 499-525.

Reny, P.J. (1999), "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games," *Econometrica,* Vol. 67, pp. 1029-1056.

Reny, P.J. (2005), "On the Existence of Monotone Pure Strategy Equilibria in Bayesian Games," mimeo: University of Chicago.

Simon, L. and W. Zame (1990), "Discontinuous Games and Endogenous Sharing Rules," *Econometrica,* Vol. 58, pp. 861-872.

Vickrey, W. (1961), "Counterspeculation, Auctions, and Competitive Sealed-Tenders," *Journal of Finance*, Vol. 16, pp. 8-37.