DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

ITERATIVE ELIMINATION OF WEAKLY DOMINATED STRATEGIES IN BINARY VOTING AGENDAS WITH SEQUENTIAL VOTING

Patrick Hummel

SOCIAL SCIENCE WORKING PAPER 1236

August 2005

Iterative Elimination of Weakly Dominated Strategies in Binary Voting Agendas with Sequential Voting

Patrick Hummel

Abstract

In finite perfect information extensive (FPIE) games, backward induction (BI) gives rise to all pure-strategy subgame perfect Nash equilibria, and iterative elimination of weakly dominated strategies (IEWDS) may give different outcomes for different orders of elimination. Several conjectures were recently posed in an effort to better understand the relationship between BI and IEWDS in FPIE games. Four of these problems regard binary voting agendas with sequential voting and two alternatives. Those problems are: (1) Assuming no indifferences, is the BI strategy profile, "always vote for my preferred alternative", guaranteed to survive IEWDS using exhaustive elimination? (2) Does any order of IEWDS leave only strategy profiles that generate paths of play consistent with BI? (3) Does there exist an order of IEWDS that leaves only strategy profiles that generate paths of play consistent with BI? (4) Does any order of IEWDS leave at least one strategy profile that generates a path of play consistent with BI? This paper proves all four conjectures. Moreover, the first conjecture is generalized to agendas with indifferences, the second and third conjectures are shown to not hold for binary voting agendas with more than two alternatives, and I comment on additional results related to the last three problems.

JEL classification numbers: C72, D72

Key words: perfect information games, extensive games, backward induction, weakly dominated strategies, iterative elimination of weakly dominated strategies, binary voting agendas, sequential voting

Iterative Elimination of Weakly Dominated Strategies in Binary Voting Agendas with Sequential Voting *

Patrick Hummel

1 Introduction

Binary voting agendas are some of the most fundamental voting procedures. A finite number of alternatives are paired together to be voted on in a tree, where each node in the tree represents a majority vote between two alternatives. The vote at any point in the tree may be decided by simultaneous voting or sequential voting. In the case of simultaneous voting, the tree represents a finite imperfect information extensive game. Using sequential voting would define a finite perfect information extensive (FPIE) game ([2]-[4], [8], [9]).

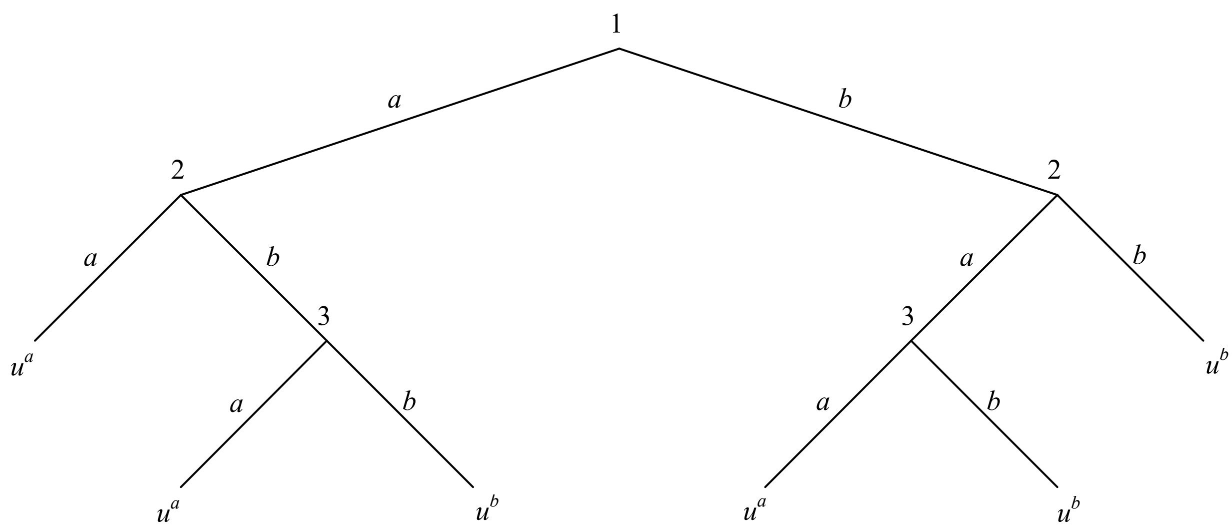
FPIE games are well-understood. Backward induction (BI) gives rise to all pure-strategy subgame perfect Nash equilibria of the game, and iterative elimination of weakly dominated strategies (IEWDS) may give different outcomes for different orders of elimination. To understand the relationship between BI and IEWDS in binary voting agendas with sequential voting, one must first understand this relationship in the case of two alternatives. Duggan ([1]) recently posed four open problems related to this question: (1) Assuming no indifferences, is the BI strategy profile, "always vote for my preferred alternative", guaranteed to survive IEWDS using exhaustive elimination? (2) Does any order of IEWDS leave only strategy profiles that generate paths of play consistent with BI? (3) Does there exist an order of IEWDS that leaves only strategy profiles that generate paths of play consistent with BI? (4) Does any order of IEWDS leave at least one strategy profile that generates a path of play consistent with BI? This paper presents solutions to all four conjectures and additional results related to these problems. I have also addressed one of Duggan's other conjectures on the relationship between BI and IEWDS in FPIE games in [5].

Depending on the assumptions made on the form of the FPIE game, one can demonstrate various correspondences between the results given by BI and IEWDS. While some BI strategy profiles may always be eliminated by IEWDS, and each BI strategy profile may be eliminated by some order of IEWDS ([10]), for a large class of

* I thank the Caltech SURF program for funding and Federico Echenique for advice.

games satisfying transference of decision-maker indifference (TDI), all orders of IEWDS leave only strategy profiles that give rise to the unique BI payoff vector ([6], [7]). A game satisfies TDI if when some player is indifferent between two strategy profiles that differ only in that player's choice of strategy, all other players are indifferent as well.

When there are no indifferences in binary voting agendas with two alternatives, TDI is satisfied, and IEWDS gives rise to the unique majority-preferred alternative. Though there need not exist a BI strategy profile that survives every order of IEWDS for arbitrary FPIE games, here we might wonder if the BI strategy profile where each player always votes for his or her preferred alternative survives IEWDS for all orders of elimination. The answer is negative ([1]) as the following example illustrates:



(I have omitted the decision nodes for player *3* that always lead to the same payoff because they are irrelevant). Suppose all players strictly prefer *a* to *b*. Then one order of IEWDS is to first eliminate strategies *ab* and *bb* for players *2* and *3* and then eliminate strategy *a* for player *1*. Thus always voting for one's preferred alternative need not survive IEWDS, though in this example, such a strategy profile would survive IEWDS if all weakly dominated strategies were eliminated at every round (exhaustive elimination).

But not all BI strategy profiles will survive exhaustive elimination either. In the above example, (*a, ba, aa*) is a BI strategy profile, but strategy *ba* for player *2* is eliminated in the first round of exhaustive elimination. These results motivate the following question ([1]): Assuming no indifferences, is the BI strategy profile, "always vote for my preferred alternative", guaranteed to survive IEWDS using exhaustive elimination?

If some voters are indifferent between the two alternatives, TDI no longer holds, and the BI outcome might not be unique. The outcome selected by BI is unique if and only if an absolute majority of players strictly prefer one alternative to the other. Since TDI may not be satisfied, we no longer know if an arbitrary order of IEWDS leaves only strategy profiles that generate paths of play which select an outcome consistent with BI. In fact, unlike the case with no indifferences, here some BI strategy profiles might not survive IEWDS for any order of elimination. For example, if we consider the above *3*-player voting agenda where players *1* and *3* are indifferent and player *2* strictly prefers *a* to *b*, in every order of IEWDS, the only strategies eliminated are *ab*, *ba*, and *bb* for player *2*. Thus the paths of play consistent with the BI strategy profiles, (*a*, *ba*, *aa*), (*b*, *ab*, *ab*), and (*b*, *bb*, *ab*) are never realized after IEWDS. This leads to the following questions: Does any order of IEWDS leave only strategy profiles that generate paths of play consistent with BI? If not, will at least one such strategy profile survive any order of IEWDS? Will some order of IEWDS leave only strategy profiles that generate paths of play consistent with BI ([1])?

2 Notation and Definitions

This paper deals with games of the form

$$G = (X, x^0, \alpha, \iota, u),$$

where X is a finite set of nodes, $x^0 \in X$ is the initial node, $\alpha: X \backslash \{x^0\} \to X$ is the anterior node function, $T = X \backslash \alpha(X)$ is the set of terminal nodes, $\iota: X \backslash T \to \{1, \ldots, n\}$ is the player function, and $u: T \to \mathbb{R}^n$ is the payoff function. For $x \in X \backslash T$, $A(x) = \{y \in X | \alpha(y) = x\}$ denotes the actions available at x. A strategy for player i is a cross product of actions at all nodes x where $\iota(x) = i$. For any strategy profile, s, $u(s)$ denotes the payoff function at the terminal node reached when s is employed. The depth of x, $d(x)$, is the $d \in \mathbb{Z}$ satisfying $\alpha^d(x) = x^0$, and the depth of G, $d(G)$, is the maximum depth of any of its nodes. For any node, x, $SG(x)$ denotes the subgame rooted at x.

For binary voting agendas with two alternatives, $\iota(x) = d(x) + 1$, and the actions available at any non-terminal node are to vote for either a or b. We assume the number of players, n, is odd to eliminate ties. The payoff vector at a terminal node is u^a if the majority of voters voted for a, and u^b otherwise. These u^c are the n-tuple of payoffs to each player, u_i^c, resulting from the selection of candidate c. If $u_i^a > u_i^b$, player i strictly prefers a to b. If $u_i^a = u_i^b$, player i is indifferent. Otherwise player i strictly prefers b to a. If player i has strict preferences, let i^+ denote player i's preferred candidate and i^- player i's less preferred candidate.

A strategy for player i, s_i, is weakly dominated in a set of strategies, S, if there exists another strategy for player i in S, s_i', such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for some opposing strategies, $s_{-i} \in S$, and $u_i(s_i^{'}, s_{-i}) \succeq u_i(s_i, s_{-i})$ $\forall s_{-i} \in S$. If S denotes the set of strategies in G, IEWDS is any order of removal of strategies in S, $S \backslash Z^1, \ldots, Z^m$, where each strategy in Z^k is weakly dominated in $S \backslash Z^1, \ldots, Z^{k-1}$ $\forall k$, and no strategies are weakly dominated in $S \backslash Z^1, \ldots, Z^m$. An order of elimination is exhaustive if each Z^k contains all strategies weakly dominated in $S \backslash Z^1, \ldots, Z^{k-1}$.

3 Results

Duggan ([1]) asked if the BI strategy profile, "always vote for my preferred alternative", is guaranteed to survive IEWDS using exhaustive elimination when there are no indifferences. Here I demonstrate that this strategy profile is not eliminated even if some voters are indifferent.

Theorem 1. *In binary voting agendas with two alternatives, any strategy profile that requires each player with strict preferences to always vote for his or her preferred alternative is guaranteed to survive IEWDS using exhaustive elimination.*

Proof. Suppose player i is to make a decision at an arbitrary node, x, in the game. There are three possibilities: (1) Regardless of the decision made at x, all possible strategies for the players yet to act result in the same candidate being chosen. (2) Regardless of the decision made at x, there are some strategies for the players yet to act that result in a being chosen, and other strategies that result in b being chosen. (3) If player i makes one decision at x, all strategies for the players yet to act result in one candidate being chosen, but if player i makes the other decision at that node, there are strategies for the players yet to act that result in the other candidate being selected.

For each node, let A denote the number of votes received by candidate a thus far and B the number of votes received by candidate b. Consider the first round of IEWDS using exhaustive elimination. Note that the tally (A, B) at x uniquely determines the category in which x falls. If either A or B is greater than $(n-1)/2$ at x, x falls under category (1). If both A and B are less than $(n-1)/2$ at x, x falls under category (2). Otherwise x falls under category (3).

If player i is indifferent between a and b, no strategies for that player will ever be eliminated in IEWDS. But if player i has strict preferences, some of player i's strategies may be eliminated by IEWDS. In particular, if x is a type (3) node, some strategy s_i for player i that specifies to vote for one candidate at x, say candidate c, will be weakly dominated by a strategy $s_i^{'}$ that is identical at all other nodes but specifies to vote for the other candidate at x: By making the best decision at x, player i can either assure that i^+ will be selected or eliminate the possibility that i^+ will definitely not be selected. So $u_i(s_i^{'}, s_{-i}) \succeq u_i(s_i, s_{-i})$ $\forall s_{-i}$, and $u_i(s_i^{'}, s_{-i}) > u_i(s_i, s_{-i})$ when (s_i, s_{-i}) selects a different candidate than $(s_i^{'}, s_{-i})$.

Thus applying a round of IEWDS using exhaustive elimination will eliminate all strategies that specify that player *i* vote for *c* at nodes corresponding to a tally where it has been determined that this decision is less favorable.

The above result indicates that after one round of IEWDS, the tally at any node, *y*, still uniquely determines the category in which *y* falls. Thus for the second round of elimination, again all nodes with certain tallies will fall under category (3), and by the same reasoning, IEWDS using exhaustive elimination will eliminate all strategies that specify that a player with strict preferences vote for a fixed candidate at nodes corresponding to this set of tallies. Applying this reasoning over and over again, we see that in every round of IEWDS using exhaustive elimination, the strategies that are eliminated are those which require a player with strict preferences to vote for a fixed candidate whenever a certain tally is reached.

Thus determining whether always voting for one's strictly preferred alternative can ever be weakly dominated amounts to determining if there is ever a tally at which voting for one's strictly preferred alternative is weakly dominated, as the above analysis indicates that under IEWDS with exhaustive elimination, all nodes with the same tally are treated the same way despite the different paths of play through which they are reached. Suppose (A, B) is such a tally with player *i's* turn to vote, and assume that player *i* strictly prefers *a* to *b*. Now it need be shown that if (A, B) is a tally that falls under category (3), then the tally ($A+1$, B) cannot necessarily lead to the selection of candidate *b*, while the tally (A, $B+1$) might lead to the selection of *a*. Likewise, it must be show that the tally (A, $B+1$) cannot necessarily lead to the selection of candidate *a*, while the tally ($A+1$, B) might lead to the selection of *b*.

Suppose ($A+1$, B) necessarily leads to the selection of *b* but (A, $B+1$) might lead to the selection of *a*. Then there must be some path of play from the tally (A, $B+1$) where candidate *a* receives more votes than candidate *a* receives in a path of play necessarily leading to the selection of *b* starting from the tally ($A+1$, B). Candidate *a* starts with fewer votes at the tally (A, $B+1$) than at the tally ($A+1$, B). So if a path of play starting from (A, $B+1$) is to ultimately lead to more votes for *a* than a path of play starting from ($A+1$, B), there must be some player for which after the player has voted, the tallies on both paths are identical. But this identical tally can be reached from the tally ($A+1$, B), a tally which necessarily leads to the selection of *b*. Thus the tally (A, $B+1$) necessarily leads to the selection of *b*, contradicting the original statement.

A symmetrical argument shows that the tally (A, $B+1$) cannot necessarily lead to the selection of candidate *a*, while the tally ($A+1$, B) might lead to the selection of *b*. Thus we see that at every round of IEWDS using exhaustive elimination, we never eliminate a strategy that requires a player to select his or her strictly preferred candidate at every node. So any strategy profile that requires each player with strict preferences to always vote for his or her preferred alternative is guaranteed to survive IEWDS using exhaustive elimination when there are just two alternatives. ■

Duggan ([1]) also wondered if, in binary voting agendas with two alternatives, IEWDS leaves at least one strategy profile that generates a path of play consistent with BI regardless of the order of elimination. Here I demonstrate that a similar result holds under much more general conditions:

Theorem 2. *In FPIE games where the player function is never the same at two nodes of different depths, IEWDS always leaves at least one strategy profile that generates a path of play consistent with BI regardless of the order of elimination used.*

Proof. Suppose by means of contradiction that there exists some such FPIE game for which some order of IEWDS does not leave at least one strategy profile that generates a path of play consistent with BI. Let d denote the minimum depth of the games satisfying these properties, G one such game with depth d, and i the player who moves at the initial node, x^0. Since the player function is never the same at two nodes with different depths, player i does not move at any nodes besides x^0, and the strategies for player i are the set of actions available at x^0. Let $x^1, \ldots, x^m$ denote this set of actions.

Each $SG(x^k)$ for $k=1, \ldots, m$ is itself a FPIE game where the player function is never the same at two nodes of different depths. No $SG(x^k)$ has depth greater than $d-1$, so by definition of d, any order of IEWDS applied to each $SG(x^k)$ leaves at least one strategy profile which follows a path of play consistent with BI in the subgame. Thus in applying IEWDS to G, there will always remain a strategy profile for the players opposing i, s_{-i}, identical to a strategy profile which follows a path of play consistent with BI in each $SG(x^k)$ for $k=1, \ldots, m$. Let u^k denote the payoff received by player i at each outcome reached by s_{-i} in $SG(x^k)$, and u^{max} the maximum of these u^k. At least one strategy for player i, s_i, where $u_i(s_i, s_{-i}) = u^{max}$ must survive IEWDS.

This (s_i, s_{-i}) must lead to a path of play consistent with BI in G. To see this, note that a path of play is consistent with BI in G if two things happen:

(1) The path of play is consistent with BI in $SG(x^k)$, where x^k is the node of depth 1 reached by the path of play.

(2) For each x^k there is a minimum payoff for player i amongst the paths of play consistent with BI in $SG(x^k)$. The path of play must also give a payoff to player i that is at least as high as the maximum of these minimum payoffs.

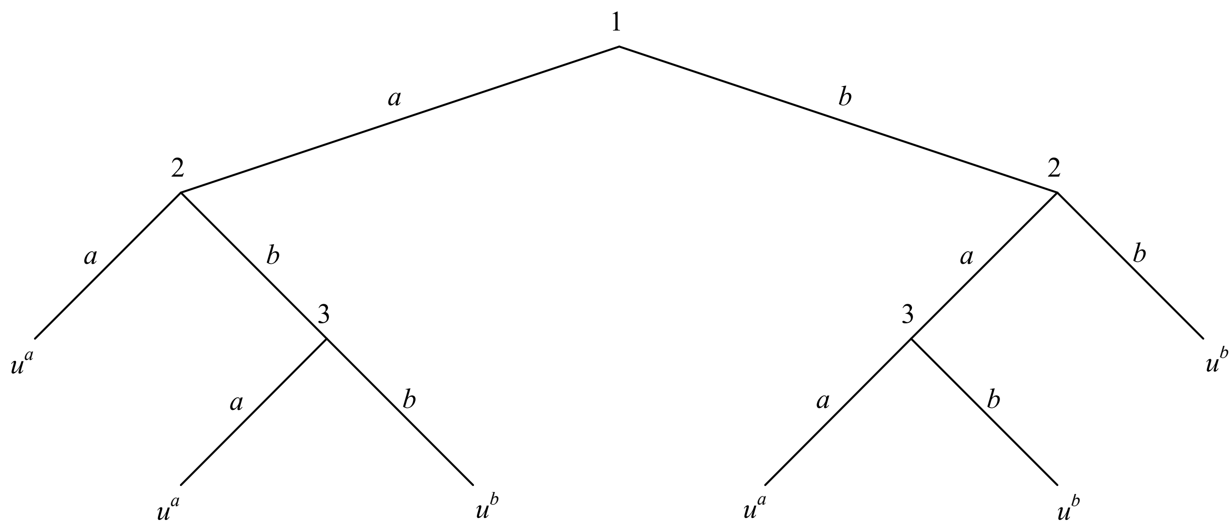
(s_i, s_{-i}) follows a path of play consistent with BI in each $SG(x^k)$ for $k=1, \ldots, m$, and u^{max} is greater or equal to the maximum payoff in (2). Thus both of the above conditions are met, and any order of IEWDS applied to G leaves at least one strategy profile that generates a path of play consistent with BI, a contradiction that proves the desired result. ■

From this theorem the following result follows immediately:

Corollary 1. *In binary voting agendas with two alternatives, IEWDS always leaves at least one strategy profile that generates a path of play consistent with BI regardless of the order of elimination used.*

Proof. In binary voting agendas with two alternatives, the player function is never the same at two nodes of different depths. Applying Theorem 2 gives the desired result. ■

Finally I turn to Duggan's question ([1]) of whether IEWDS must only leave strategy profiles that generate paths of play consistent with BI in binary voting agendas with two alternatives. Before examining this problem, first note that there may be no order of IEWDS that leaves only strategy profiles consistent with BI:



Assume player 1 strictly prefers a to b and the other players are indifferent. In this case IEWDS never eliminates any strategies for any players, as players 2 and 3 are indifferent between all strategies, and neither of player 1's strategies dominates the other, since $u_1(b, ba, ba) > u_1(a, ba, ba)$ and $u_1(a, ab, ab) > u_1(b, ab, ab)$. So IEWDS gives the original game for any order of elimination. But (a, ba, ba) is not a BI strategy profile, so not all strategy profiles in the original game are consistent with BI. Thus there need not exist an order of IEWDS that leaves only BI strategy profiles.

However, all paths of play in the above example are consistent with BI. For example, the path of play reached by (a, ba, ba) is the same as that reached by (a, bb, bb), a BI strategy profile. In fact, we can prove the following general result:

Theorem 3. *In binary voting agendas with two alternatives, IEWDS only leaves strategy profiles that generate paths of play consistent with BI regardless of the order of elimination used.*

Proof. Suppose by means of contradiction that there exists some binary voting agenda with two alternatives, G, where some order of IEWDS leaves a strategy profile that generates a path of play inconsistent with BI, and suppose there are n players in G. Let t denote a terminal node reached by one of these paths of play inconsistent with BI, m the smallest positive integer such that t is not consistent with BI in $SG(\alpha^m(t))$, $y^0 = \alpha^m(t)$, $y^1 = \alpha^{m-1}(t)$, and y^2 the node reached when $\iota(y^0) = i$ does not move to y^1 at y^0.

Since t is not consistent with BI in $SG(y^0)$, but is consistent with BI in $SG(y^1)$, all paths of play consistent with BI in $SG(y^2)$ must lead to the selection of i^+, and i^- must be selected at t. Recall that in binary voting agendas with indifferences, BI selects a unique candidate if and only if an absolute majority of players strictly prefer one alternative to the other. By analogy, we see that in $SG(y^2)$, BI selects i^+ as the unique candidate if and only if the number of voters that act in $SG(y^2)$ and strictly prefer i^+ plus the number of voters that have already voted for i^+ at y^2 is at least $(n+1)/2$.

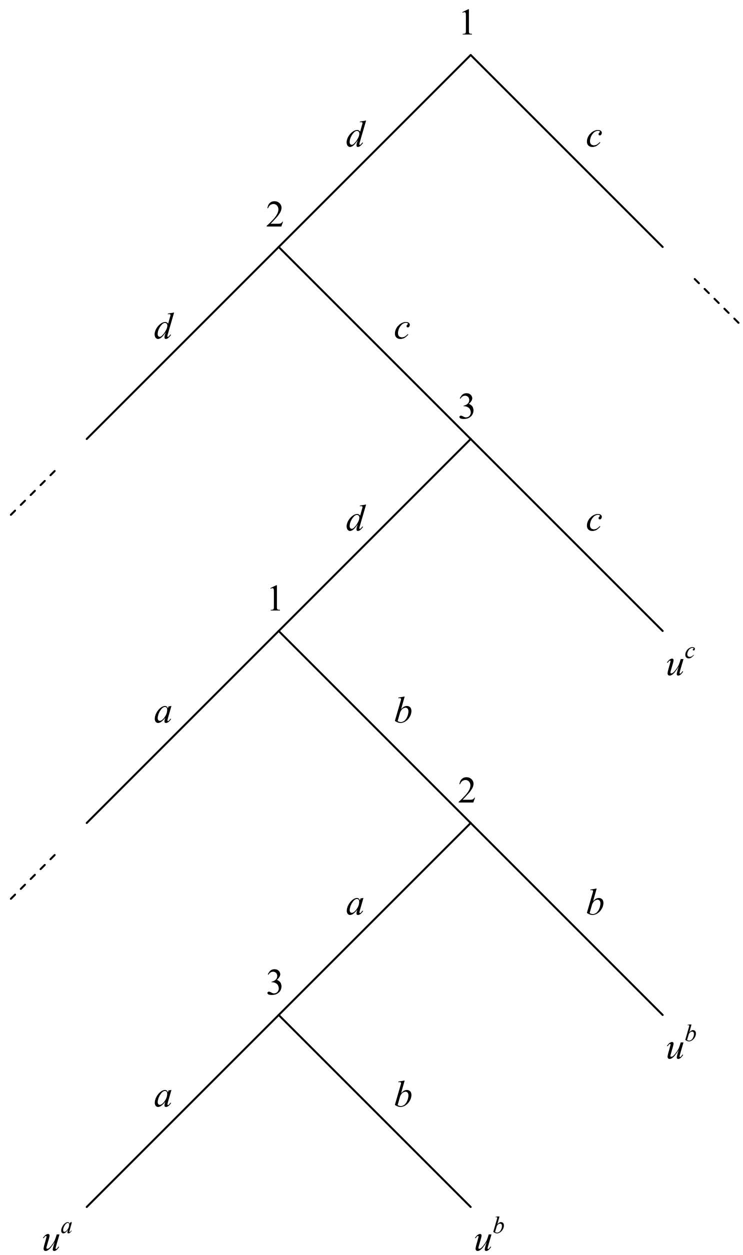
If at least $(n+1)/2$ voters have already voted for i^+ at y^2, any path of play which reaches y^2 leads to the selection of i^+. Otherwise, assume exactly $(n+1)/2 - r$ voters have voted for i^+ at y^2, where r is a positive integer. Then there are at least r players to act in $SG(y^2)$ that strictly prefer i^+ to i^-. Consider the last such player to act, player h. At nodes where exactly $(n-1)/2$ players have already voted for i^+, player h will be able to assure the selection of i^+ by voting for i^+. Thus any strategy profile that survives IEWDS and follows a path of play which reaches a node where player h has yet to act and at least $(n-1)/2$ players have already voted for i^+ automatically leads to the selection of i^+.

Now suppose it has been demonstrated that any strategy profile that survives IEWDS and follows a path of play which reaches a node where k such players have not yet acted and at least $(n+1)/2 - k$ players have already voted for i^+ automatically leads to the selection of i^+. Consider the $k+1^{\text{th}}$ to last such player to act, player j. At nodes where exactly $(n-1)/2 - k$ players have already voted for i^+, player j will be able to assure the selection of i^+ by voting for i^+. Thus any strategy profile that survives IEWDS and follows a path of play which reaches a node where player j has yet to act and at least $(n-1)/2 - k$ players have already voted for i^+ automatically leads to the selection of i^+. By induction it follows that any strategy profile that survives IEWDS and reaches y^2 automatically leads to the selection of i^+.

Thus for any order of IEWDS applied to G there are two possibilities: (1) All strategy profiles that follow a path of play which reaches y^2 are eliminated or (2) Some such strategy profiles are not eliminated, but all strategy profiles that follow a path of play which reaches y^2 and fails to select i^+ are eliminated. Since we assume there is a strategy profile that follows a path of play which reaches y^1 and leads to the selection of i^- at t and there are strategy profiles that follow paths of play which reach y^2 and lead to the selection of i^+, we do not eliminate all strategy profiles for player i which specify to move to y^2 at y^0. Thus the first possibility can only hold if all strategy profiles that follow a path of play which reaches y^0 are eliminated. Since this is inconsistent with the existence of paths of play that reach t, only the second possibility is valid.

Thus after some round of IEWDS, all remaining strategy profiles that follow a path of play which reaches y^2 select i^+. After this round, any strategy for player i which specifies to go from y^0 to y^1 is weakly dominated by both the strategy identical at all other nodes which specifies to go from y^0 to y^2 and any other strategies that weakly dominate this second strategy. Therefore, regardless of what eliminations have taken place, all strategies that specify to move from y^0 to y^1 will be weakly dominated and ultimately eliminated. But in this case no strategy profile which follows a path of play which reaches t survives IEWDS. This contradicts our assumption that such a strategy profile exists, proving the desired result. ■

While this result holds in the case of binary voting agendas with two alternatives, for binary voting agendas with more than two alternatives, there may not be any order of IEWDS that leaves only paths of play which select an alternative consistent with BI. Consider, for example, a *3*-player binary voting agenda where the players first vote to accept or reject *c*. If *c* is accepted, the game finishes. Otherwise the players vote between *a* or *b*. Since the entire game is quite large, I depict only the parts of the game relevant to my example:



Suppose the preferences follow the ordering $u_1^a = u_1^b = u_1^c$, $u_2^c > u_2^a = u_2^b$, and $u_3^a > u_3^c > u_3^b$. Note that selecting candidate *b* is not consistent with BI since an absolute majority of players strictly prefer *c* to *b*. Likewise the strategy profile (*db*, *cb*, *da*) does not follow a path of play consistent with BI; *da* is not a best-response for player *3* when players *1* and *2* employ the strategies *db* and *cb* respectively because player *3* would receive a higher payoff by selecting strategy *ca*. But although (*db*, *cb*, *da*) selects a candidate that is not selected by any BI strategy profile, this strategy profile is never eliminated by any order of IEWDS: No strategies are ever eliminated for player *1* since player *1* is completely indifferent between all outcomes. For player *2*, *ca* always affords the same payoff as *cb*, and *cb* weakly dominates both *da* and *db* since *da* and *db* automatically lead to the selection of either *a* or *b*. For player *3*, *da* weakly dominates *db*, and neither *ca* nor *cb* ever weakly dominate *da* since *da* leads to the selection of candidate *a* against some opposing strategies, but *ca* and *cb* automatically select *c*. Thus the strategy profile (*db*, *cb*, *da*) always survives IEWDS, and in arbitrary binary voting agendas, IEWDS may always leave at least one strategy profile that generates a path of play that selects a candidate that would never be selected by BI.

Note that this example depends on the use of player indifferences. For arbitrary binary voting agendas with no indifferences, TDI holds, and as noted by Duggan ([1]), IEWDS leaves only strategy profiles that select the unique candidate consistent with BI.

References

[1] J. Duggan (2003) "A Note on Backward Induction, Iterative Elimination of Weakly Dominated Strategies, and Voting in Binary Agendas", http://www.rochester.edu/college/PSC/duggan/notes/IEWDS2.PDF.

[2] R. Farquharson (1969) *The Theory of Voting*, New Haven: Yale University Press.

[3] G. Kramer (1972) "Sophisticated Voting over Multidimensional Choice Spaces", *Journal of Mathematical Sociology*, 2: 165-180.

[4] R.J. Gretlein (1982) "Dominance Solvable Voting Schemes: A Comment", *Econometrica*, 50: 527-528.

[5] P. Hummel (2005) "Finite Perfect Information Extensive Games with Generic Payoffs", unpublished.

[6] L. Marx and J. Swinkels (1997) "Order Independence for Iterated Weak Dominance", *Games and Economic Behavior*, 18: 219-245.

[7] L. Marx and J. Swinkels (2000) "Corrigendum", *Games and Economic Behavior*, 31: 324-329.

[8] R. McKelvey and R. Niemi (1978) "A Multistage Game Representation of Sophisticated Voting for Binary Procedures", *Journal of Economic Theory*, 18: 1-22.

[9] H. Moulin (1979) "Dominance Solvable Voting Schemes", *Econometrica*, 47: 1337-1351.

[10] M. Osborne and A. Rubinstein (1994) *A Course in Game Theory*, Cambridge: MIT Press.