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CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

STRATEGIC ANALYSIS IN COMPLEX NETWORKS WITH LOCAL EXTERNALITIES

Andrea Galeotti
Social and Information Science Laboratory, Caltech
Department of Economics, University of Essex

Fernando Vega-Redondo
Facultad de Económicas, Universidad de Alicante
Instituto Valenciano de Investigaciones Económicas
Department of Economics, University of Essex

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Abstract

In this paper, we discuss a model with local positive externalities on a complex random network that allows for wide heterogeneities among the agents. The situation can be analyzed as a game of incomplete information where each player's connectivity is her type. We focus on three paradigmatic cases in which the overall degree distribution is Poisson, exponential, and scale-free (given by a power law). For each of them, we characterize the equilibria and obtain interesting insights on the interplay between network topology and payoffs. For example, we reach the somewhat paradoxical conclusion that a broad degree distribution or/and too low a cost of effort render it difficult, if not impossible, to sustain an (efficient) high-effort configuration at equilibrium.

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1 Introduction

Local externalities are a phenomenon of great significance in a wide range of different contexts. They are important, for example, in problems of learning and search (Bramoulle and Kranton (2004), Galeotti (2004)); crime (Glaeser *et al* (1996), Calvó and Zenou (2004)); productivity and growth (Glaeser *et al* (1992), Durlauf (1993),Ciccone and Hall (1996)); technological adoption (Coleman (1988), Valente (1996), Conley and Udry (2000), Rogers (1962)); R&D collaboration (Goyal and Moraga-Gonzalez (2001)). The common approach to modelling these local effects is to posit that agents interact, rather than with the population at large, with their neighbors in some relevant socio-economic network. Often, it is assumed that agents are located along a *fixed* network, so that they can clearly identify who are (and will continue to be) their neighbors. This, in turn, allows agents to shape their behavior on the basis of what is (or is anticipated to be) the behavior of those neighbors.

As a polar opposite, we have the context where, even though interaction is "local" (i.e. restricted to a relatively small subset of the population), the particular pattern of interaction is not known beforehand. A paradigmatic example is given by the case where agents are randomly matched in pairs and interaction takes place bilaterally among those matched. *Ex ante*, agents face essentially identical matching possibilities, even if they are heterogenous in some other respects. If interaction is repeated over time, then the implicit assumption is that a fresh round of rematching occurs at every instance. This is a scenario considered by much of the theory of evolution and learning (Weibull (1995), Vega-Redondo (1996), Young (1998), Fudenberg and Levine (1998)), the literature on bargaining in population environments (Rubinstein-Wolinsky (1985), Gale(1987)), or the study of how social norms arise in large populations (Kandori (1992), Okuno-Fujiwara and Postlewaite (1995)).

We want to argue that, quite frequently in the real world, the situation is best conceived as intermediate between the former two scenarios. That is, not only is agent interaction hardly "frozen" to a given set of neighbors but, on the other hand, it is far from displaying the *ex-ante* symmetry and lack of structure displayed by random uniform matching. Patterns of interaction in the real world are, in other words, both volatile and complex. This, in turn, should of course have important implications on how local externalities shape agents' decisions.

To fix ideas, consider a simple and styled example involving academic researchers who interact with colleagues in scientific conferences. Let us suppose that researchers are heterogenous, in that some attend many conferences and others only a few, with the rest of the population found somewhere in between according to a certain nondegenerate distribution. Further assume that each conference has the same appeal to every researcher, so that the probability of meeting a certain type (i.e. a more or less "travelled" colleague) is solely determined by its respective population frequency and how many conferences that type attends. Then, the key decision each scientist has to make is how much effort to devote to research, anticipating what the others will do on their part, and realizing that the eventual payoffs attained depend on the positive externalities accruing from interaction.

In our model, agents decide independently on their own level of investment (or effort) before learning the characteristics (and therefore investment) of their partners. The optimal (equilibrium) decision so taken by each player must depend on a number of factors. First, of course, it is shaped by the precise nature of the externalities (i.e. whether they are positive or negative, and their magnitude). Second, it has to reflect as well the intensity of interaction of the player in question (e.g. the number of conferences she plans to attend, in our former example). Third, it must hinge upon the overall distribution of types prevailing in the population, which determines how agents' network characteristics mesh with each other and jointly determine the overall architecture of interaction (i.e. number of partners, partners of partners, etc. of a typical individual).

Formally, we study the problem in the framework provided by the theory of complex random networks. This theory has its precursor in the work of Erdös and Rényi (1959, 1960), who started their fruitful collaboration on this topic in the late 1950's. In recent times, this theory has been much extended to become a powerful tool in the study of large and complex networks (see

Albert and Barabási (2002), Newman (2003), and Vega-Redondo (2005) for exhaustive surveys). In essence, a random network is to be conceived as a stochastic ensemble, i.e. a probability measure (typically uniform) defined on a given family of possible networks. This family is usually characterized in terms of certain overall properties (such as a particular degree distribution, degree correlations, or clustering) that all the networks in it are taken to satisfy. Then, while all eligible networks satisfy the properties required, the specific network realized is uncertain. By way of (trivial) illustration, the traditional mechanism of bilateral random matching can be conceived as a random network where all eligible networks have each node connected to one (and only one) other neighbor, every matched pair defining a separate component of the induced network.

Here, in contrast, we shall be interested in networks with complex topologies, where the random network is solely characterized by a given degree distribution. Specifically, we shall focus on three cases that have become paradigmatic in the literature: Poisson, power-law (or scale-free), and geometric networks – i.e. random networks whose degree distribution is as respectively specified.[1] All networks consistent with a fixed such distribution is assumed formed with equal probability. In this context, we study a game (which can be regarded as one of incomplete information) where every player has to choose her costly effort, under the beliefs induced by the random network on the type of partner she may encounter. Of course, a key feature of the problem is the nature of payoffs accruing from interaction. Here, as a first step, we focus on the classical Cobb-Douglas formulation of positive externalities, the gross payoff to a player being the product of all efforts displayed by herself and her neighbors. Costs, on the other hand, are taken to be quadratic in the effort level.

In this context, we are able to fully characterize symmetric equilibria for each of the three scenarios considered: Poisson, scale-free, and geometric random networks. Table 1 summarizes the main results of the paper. We find a rich interplay between payoffs and network topology that determines the extent to which players can sustain high effort levels in equilibrium. For example, one general, and somewhat paradoxical, insight that transpires, in different forms, from all the three cases considered can be succinctly described as follows. If the network connectivity is too broadly distributed and/or effort is not costly enough, the "snowball forces" bearing on the optimal effort

[1] As explained below (cf. Section 4), they correspond to three benchmark scenarios.

of highly connected individuals makes the overall situation “explosive” and, in some cases, inconsistent with equilibrium. But even if an equilibrium can be constructed under these circumstances, this equilibrium is only the grossly inefficient one were an individual exerts a lower effort the more intense her exposure to the externalities! That is, under those circumstances, we find that the higher the degree of the node, the lower the effort. Overall, this points to the fact that low cost or/and a wide range of possible connectivity may be counterproductive, in that it curtails the ability to support, at equilibrium, high levels of effort.

The rest of the paper is organized as follows. First, in Section 2, we present a general framework to study local externalities in random networks. This framework is then specialized in Section 3 to the case where externalities are positive and of a multiplicative (Cobb-Douglas) type. The formal analysis is undertaken in Section 4, divided into three scenarios: Poisson, scale-free, and geometric networks. Section 5 concludes. To facilitate the discussion, the detailed proof of the results is relegated to the Appendix.

2 General framework

There is an infinite population of agents, N, who meet randomly. Specifically, each agent $i \in N$ meets a number of other agents, as determined by her degree κ_i. We assume that the *degree distribution* is fixed, as given by a probability density

$$\mathbf{p} = \{p_\kappa\}_{\kappa=0}^{\infty} \tag{1}$$

where each p_κ denotes the fraction of individuals who have κ neighbors.

Using expression (1), we can define the corresponding probability density for the degree distribution of a neighboring node (i.e. one that is chosen as the neighbor of some randomly selected node). This distribution

$$\tilde{\mathbf{p}} = \{\tilde{p}_\kappa\}_{\kappa=1}^{\infty}. \tag{2}$$

is of the form

$$\tilde{p}_\kappa = \frac{p_\kappa \kappa}{\sum_{x'=0}^{\infty} p_{x'} x'}. \tag{3}$$

since the frequency with which a node is encountered is proportional to the product κp_κ. For example, in terms of our example of travelling researchers, the probability of meeting a colleague who attends κ conferences is proportional to the frequency of those of this type, p_κ, multiplied by the number of occasions/conferences, κ, in which they can be met.

Players interact with each other as determined by the prevailing social network. This network is chosen equiprobably from all networks that display the given degree distribution $\mathbf{p}$. Ex ante, therefore, we are in the presence of a random network characterized by that degree distribution. Each player i knows her own degree κ_i but ignores the degree of other players she will meet. The overall degree distribution, however, is assumed common knowledge.

Prior to interaction, each individual i has to choose an effort (or investment) level $e_i \in \mathbb{R}_+$. In general, of course, this choice can be tailored to her degree κ_i (which she knows) but cannot depend on the identity, characteristics, or behavior (i.e. index, degree, or effort level) of each of her future κ_i partners. Given the profile of effort levels $[e_i, (e_j)_{j \in N_i}]$ chosen by player i and each of the κ_i agents in her neighborhood N_i, the payoffs earned by player i are given by:

$$\pi_i[e_i, (e_j)_{j \in N_i}] = f[e_i, (e_j)_{j \in N_i}] - c(e_i), \tag{4}$$

where, assuming ex-ante symmetry across players,

$$f : \mathbb{R}_+ \times \left[\bigcup_{m=0}^{\infty} \mathbb{R}_+^m \right] \to \mathbb{R}_+$$

stands for the (symmetric)[2] gross payoff function of each player, depending on her own effort and that of her m neighbors, and

$$c : \mathbb{R}_+ \to \mathbb{R}_+$$

is the cost function for individual effort.

Prior to interaction, consider any given agent with degree κ who has to choose an effort level before knowing her future partners' characteristics. We posit that every such agent chooses an effort level e so as to maximize the expected value of (4) induced by the probability density (2) and some *predicted* degree-contingent (symmetric) strategy

$$\hat{\mathbf{e}} = \{\hat{e}(\kappa')\}_{\kappa'=0}^{\infty}$$

[2]Naturally, the function f must be symmetric, in the sense of being independent to any permutation in its arguments.

that specifies how every other individual, depending on her degree κ', is anticipated to choose her effort level. We denote by $\psi_\kappa(e, \hat{\mathbf{e}})$ the expected payoff function embodying the aforementioned considerations for an agent of degree κ.

To provide a precise specification of $\psi_\kappa(e, \hat{\mathbf{e}})$, we need to introduce some additional notation. First, for any degree $\kappa = 0, 1, 2, ...$ of any given player, let

$$S_\kappa = \left\{ r = (r_1, r_2, ...) \in (N \cup \{0\})^\infty : \sum_{l=1}^{\infty} r_l = \kappa \right\}$$

with the following interpretation. Each sequence $r = (r_1, r_2, ...)$ specifies, for each $l = 1, 2, ...$, the number of neighbors with degree l. Naturally, only those sequences for which $\sum_{l=1}^{\infty} r_l = \kappa$ are valid. Now, if each of the κ neighbors of the player in question are randomly chosen from the overall population, any given one of them has degree κ' with probability $\tilde{p}_{\kappa'}$ given by (3) since, as explained before, the suitable degree distribution in this case is that of a *neighboring* node. Therefore, the distribution induced on S_k follows a multinomial distribution, with probability

$$P_\kappa (r_1, r_2, ...) = \frac{\kappa!}{r_1! r_2! ...} (\tilde{p}_1)^{r_1} \times (\tilde{p}_2)^{r_2} \times ... \tag{5}$$

In terms of these probabilities, the expected-payoff function $\psi_\kappa(e, \hat{\mathbf{e}})$ can be formally defined as follows:

$$\psi_\kappa(e, \hat{\mathbf{e}}) = \sum_{r \in S_\kappa}^{\infty} P_\kappa (r) \;\; f[e; \underbrace{\hat{e}(1), ..., \hat{e}(1)}_{r_1 \text{ times}}, \underbrace{\hat{e}(2), ..., \hat{e}(2)}_{r_2 \text{ times}}, ...] - c(e) \tag{6}$$

Then, as customary, we say that a profile $\mathbf{e}^* = \{e^*(\kappa)\}_{\kappa=0}^{\infty}$ is a (symmetric) *Nash equilibrium* if it satisfies:[3]

$$e^*(\kappa) \in \arg\max_{e \in \mathbb{R}_+} \psi_\kappa(e, \mathbf{e}^*) \qquad\qquad (\kappa = 0, 1, 2, ...). \tag{7}$$

Note that this equilibrium can also be regarded as a Bayes-Nash equilibrium of a (Bayesian) incomplete-information game where the type space of every agent coincides with the set of possible degrees.

[3]For simplicity, we speak of $\mathbf{e}^*$ as an "equilibrium," although it is only the identical strategy played by every player in a symmetric equilibrium.

3 Strategic Complementarities

In this paper we restrict attention to the case in which individuals' efforts are strategic complements. More specifically, we posit that the *gross* payoff of a player is the product of her own efforts and the efforts exerted by each of her neighbors. On the other hand, we suppose that the agent's investment cost is quadratic, the magnitude of these costs being parametrized by some $\alpha > 0$. Combining both components (gross payoffs and costs), and relying on (5), the expected *net* payoffs for an agent with κ neighbors can be written as follows:

$$\psi_\kappa(e, \hat{\mathbf{e}}) = \sum_{r \in S_\kappa}^{\infty} \left\{ \frac{\kappa!}{r_1! r_2! \ldots} \; e \; \left[\prod_{\kappa'=1}^{\infty} \left[\tilde{p}_{\kappa'} \cdot e\left(\kappa'\right) \right]^{r_{\kappa'}} \right] \right\} - \frac{\alpha}{2} e^2. \tag{8}$$

The functions $\psi_\kappa(e, \hat{\mathbf{e}})$ are obviously differentiable with respect to e. Thus, the conditions (7) that define a symmetric Nash equilibrium $\mathbf{e}^* = \{e^*(\kappa)\}_{\kappa=0}^{\infty}$ can be formulated as follows:

$$\left. \frac{\partial \psi_\kappa}{\partial e}(e, \mathbf{e}^*) \right|_{e=e^*(\kappa)} = 0 \qquad (\kappa = 0, 1, 2, \ldots), \tag{9}$$

which, using (8), yield:

$$\alpha e^*(0) = 1 \tag{10}$$

$$\alpha e^*(1) = \sum_{\kappa'=1}^{\infty} \tilde{p}_{\kappa'} e^*\left(\kappa'\right) \tag{11}$$

$$\alpha e^*(\kappa) = \left(\sum_{\kappa'=1}^{\infty} \tilde{p}_{\kappa'} e^*\left(\kappa'\right) \right)^{\kappa} \qquad (\kappa = 2, 3, \ldots). \tag{12}$$

We first observe that the equilibrium effort level for $\kappa = 0$ (i.e. an isolated player) is simply $e^*(0) = 1/\alpha$. On the other hand, by introducing (11) in (12), we find that for $\kappa \geq 2$ (a player having at least two neighbors), the optimal effort level can be written in terms of $e^*(1)$ as follows:

$$e^*(\kappa) = \frac{1}{\alpha} \left[\alpha e^*(1) \right]^{\kappa}, \qquad (\kappa = 2, 3, \ldots), \tag{13}$$

while the value of $e^*(1)$ can then be solved from the equation

$$\frac{1}{\alpha} \sum_{\kappa'=1}^{\infty} \tilde{p}_{\kappa'} \left[\alpha e^*(1) \right]^{\kappa'-1} = 1. \tag{14}$$

4 Analysis

In general, of course, Nash equilibria depend on the degree distribution of the network. It is clear, however, that, regardless of this distribution, there always exists a symmetric Nash equilibrium where players with zero degree invest $1/\alpha$ while players with positive degree do not exert any effort at all. Our main concern, therefore, is to explore whether other non-trivial equilibria exist where connected players exert positive effort and there is an interesting dependence on network characteristics.

We shall address this issue in three different scenarios that can be conceived as benchmark cases in the network literature (cf. Vega-Redondo (2005)). The first one is the original setup studied by Erdös and Rényi (1959, 1960), where the degree distribution is assumed Poisson. This is generated by a mechanism where connectivity is set at random (every possible link is formed with a fixed independent probability) and the framework is stationary (the set of nodes is large but given). The second scenario posits a degree distribution that is scale-free (i.e. it is given by a power law). This has been shown by Barabási and Albert (1999) to arise in a growing environment where new links are again set at random, but with a (linear) bias in favor of those nodes that are more highly connected. Finally, the third scenario, where the degree distribution is geometric, embodies random connectivity as well but in a growing setup where the set of nodes increase unboundedly over time.

For each of these three cases, we first characterize the non-trivial equilibria where connected players do invest. Then, we analyze how the profile of equilibrium efforts and the induced utilities depend on the key underlying parameters – specifically, the moments of the distribution and the cost of investment.

4.1 Poisson degree distributions

Let the network degree be Poisson distributed. Thus, the probability that a randomly selected agent has degree $\kappa = 0, ...$ is:

$$p(\kappa) = \exp(-z)\frac{z^{\kappa}}{\kappa!} \tag{15}$$

where z is the average network degree. Then, recall that the probability a neighboring agent (i.e. an agent who is the neighbor of some randomly selected node) has degree κ is:

$$\tilde{p}(\kappa) = \frac{p(\kappa)\kappa}{\sum_{\kappa'=0}^{\infty} p'_{\kappa'}\kappa'} = \exp(-z)\frac{z^{\kappa-1}}{(\kappa-1)!} \tag{16}$$

The next result provides a full characterization of non-trivial equilibria. Let us define $\underline{\alpha}(z) = 1/\exp(z)$.

Theorem 1 *Let the degree distribution be given by (15). There exists at most a unique non-trivial equilibrium* $\mathbf{e}^* = \{e^*(\kappa)\}_{\kappa=0}^{\infty}$. *This equilibrium exists if and only if* $\alpha > \underline{\alpha}(z)$ *and takes the following form:*

$$e^*(\kappa) = \frac{1}{\alpha}\left(\frac{\ln\alpha + z}{z}\right)^{\kappa}, \qquad (\kappa = 0, 1, ...).$$

The previous result establishes the existence of a unique non-trivial equilibrium when costs are sufficient high, i.e. if $\alpha > \underline{\alpha}(z)$. It is easy to see that $\underline{\alpha}(z)$ decreases in z and that $\underline{\alpha}(z) \leq 1$ for any $z \geq 1$. Thus, the existence region for the non-trivial equilibrium widens as the average degree of the network increases.

Having characterized the conditions under which a non-trivial equilibrium exists, our next concern is to understand how efforts and payoffs change with connectivity. As an immediate consequence of Theorem 1, we have the following result.

Corollary 1 *Let the degree distribution be given by (15) and assume* $\alpha > \underline{\alpha}(z)$ *so that a non-trivial equilibrium* $\mathbf{e}^* = \{e^*(\kappa)\}_{\kappa=0}^{\infty}$ *exists. Then, equilibrium efforts* $e^*(\kappa)$ *and expected payoffs* $\psi_\kappa(e^*(\kappa), \mathbf{e}^*)$ *satisfy:*

- *if* $\alpha < 1$, *both are decreasing in the degree* κ;
- *if* $\alpha = 1$, *both are constant in the degree* κ;
- *if* $\alpha > 1$, *both are increasing in the degree* κ.

Figure 1 illustrates the dependence of equilibrium effort level on κ established by Corollary 1. When cost are high enough ($\alpha > 1$), the equilibrium profile has the seemingly natural property that the individual whose externalities are stronger (i.e. has more neighbors) finds it worthwhile to invest a higher level of effort. This contrasts with the situation when the costs are low

($\alpha < 1$). In this case, there is no equilibrium where the effort levels increase with κ. Intuitively, the reason for this somewhat paradoxical state of affairs can be explained as follows. If effort is too cheap, an increasing equilibrium profile would have the expected payoff per link grow so fast that it would generate a snow-ball effect incompatible with equilibrium.

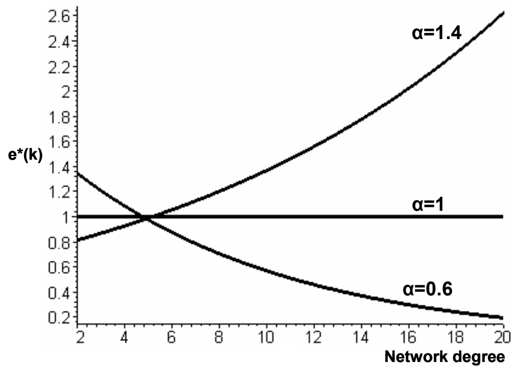

Figure 1. Equilibrium effort profile in a Poisson network.
Three different values of α are considered: $\alpha = 0.6, 1, 1.4$ for an average degree $z = 5$.

To clarify the previous statement, the following heuristic reasoning may be helpful. First, note that the equilibrium conditions (11)-(12) imply that the key factor shaping the equilibrium profile is the expected marginal gross payoff of a single link:

$$\varpi \equiv \sum_{\kappa'=1}^{\infty} \tilde{p}_{\kappa'} e^* (\kappa') , \tag{17}$$

so that the equilibrium conditions can be rewritten in terms of ϖ as follows:

$$e^* (\kappa) = \varpi^{\kappa} / \alpha \qquad (\kappa = 1, 2, ...). \tag{18}$$

Clearly, only if $\varpi > 1$ can the equilibrium effort grow with κ. Thus let assume this is the case and suppose that, given any such ϖ, agents start from the putative equilibrium given by (18) and reconsider their choices. If, in particular, an agent with just one link were to compute afresh her marginal gross payoff $\hat{\varpi}$, she would arrive at the expression

$$\begin{aligned} \hat{\varpi} &= \frac{1}{\alpha} \sum_{\kappa'=1}^{\infty} \tilde{p}_{\kappa'} \varpi^{\kappa'} = \frac{1}{\alpha} \exp(-z) \sum_{\kappa'=1}^{\infty} \frac{z^{\kappa'-1}}{(\kappa'-1)!} \varpi^{\kappa'} = \frac{\varpi}{\alpha} \exp(-z) \sum_{\kappa'=0}^{\infty} \frac{(z\varpi)^{\kappa'}}{\kappa'!} \\ &= \frac{1}{\alpha z} \exp(-z) \exp(z\varpi) = \frac{1}{\alpha z} \exp(z(\varpi - 1)) \end{aligned}$$

Note that, under the maintained assumption that $\varpi > 1$, we have that $\hat{\varpi} > \varpi$ if $\alpha < 1$. Thus, in this case, if the agent in question were to continue with her putative effort level $e^* (1) = \varpi/\alpha$, she would be facing a marginal benefit $\hat{\varpi}$ higher than the marginal cost, $\alpha e^* (1)$. This should induce her to exert an effort higher than $e^* (1)$. Analogous considerations, of course, could be made by agents with any degree κ. Again, they would be led to increasing their effort levels over what is prescribed by (18). In turn, this would increase the marginal gross payoff of a single link over the level specified in (17), still leading to a further subsequent increase and generating an unbounded process of revisions inconsistent with equilibrium. If $\alpha < 1$, such snow-ball effect can only be checked if $\varpi < 1$ (i.e. if the effort profile is decreasing), as stated in Corollary 1. Of course, it could be that it is not possible at all to check it if α is very small. This is indeed what underlies the lower bound on α contemplated in Theorem 1 for the existence of a non-trivial equilibrium.

Now we turn to analyzing how the equilibrium effort profile varies with the average degree of the network and the magnitude of costs. This is addressed in turn by the following two additional corollaries of Theorem 1.

Corollary 2 *Let the degree distribution be given by (15) and assume* $\alpha > \underline{a}(z)$. *Consider the non-trivial equilibrium* $\mathbf{e}^* = \{e^* (\kappa)\}_{\kappa=0}^{\infty}$ *established in Theorem (1). Then, for any given* $\kappa \geq 1$, *we have:*

- *if* $\alpha < 1$, *then* $e^* (\kappa)$ *is increasing in the average degree* z;
- *if* $\alpha = 1$, *then* $e^* (\kappa)$ *is constant in the average degree* z;
- *if* $\alpha > 1$, *then* $e^* (\kappa)$ *is decreasing in the average degree* z.

Corollary 3 *Let the degree distribution be given by (15) and assume* $\alpha > \underline{a}(z)$. *Consider the non-trivial equilibrium* $\mathbf{e}^* = \{e^* (\kappa)\}_{\kappa=0}^{\infty}$ *established in Theorem (1). Then,*

- *if* $\kappa > \ln \alpha + z$, *then* $e^* (\kappa)$ *is increasing in* α;
- *if* $\kappa < \ln \alpha + z$, *then* $e^* (\kappa)$ *is decreasing in* α;

The above corollaries point to interesting dependencies of the effort levels on the parameters of the model. Corollary 2 establishes that changes in the average degree have a qualitatively different effect on equilibrium effort levels depending on the value of α (i.e. on whether it is lower or higher than the threshold value of one). In essence, what it indicates is that individual efforts uniformly adjust upward or downward so as to offset the effect on externalities of any changes in average connectivity. Specifically, if $\alpha < 1$, a rise in z strengthens the *negative* externality imposed on players by their neighbors and, consequently, the effort levels rise to mitigate the effect. Instead, when $\alpha > 1$, an increase in z strengthens the *positive* externality enjoyed by agents and, correspondingly, the effort levels uniformly fall at equilibrium in order to restrain that effect. These conclusions are illustrated in Figure 2 for some representative cases.

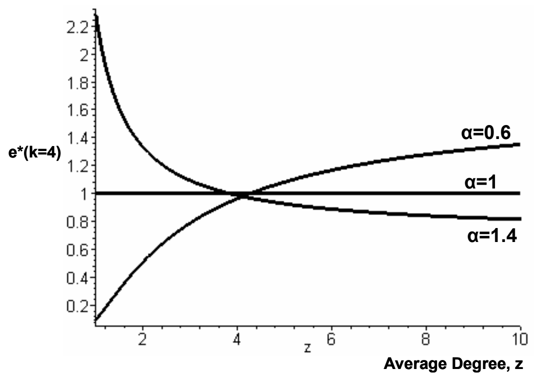

Figure 2. Equilibrium effort of a player with degree 4 in a Poisson network. Three different values of α are considered, $\alpha = 0.6, 1, 1.4$

Corollary 3, on the other hand, focuses on the effect of changes in costs, as parametrized by α. Here, quite interestingly, the effect is no longer uniform for all agents. Naturally, if α changes, say rises, the effort profile cannot remain unchanged. One might have expected that the effect on efforts were uniform, as it was for z. Interestingly, we find that it is heterogenous, inducing always an upward adjustment on those agents who generate a higher externality (i.e. have a higher degree) and a downward adjustment for the rest. The boundary of separation between the two groups is marked by the average degree z, but depends on α as well. In particular, it is higher or lower than z depending on whether α is above or below 1. The situation is illustrated in Figure 3 for a number of different cases.

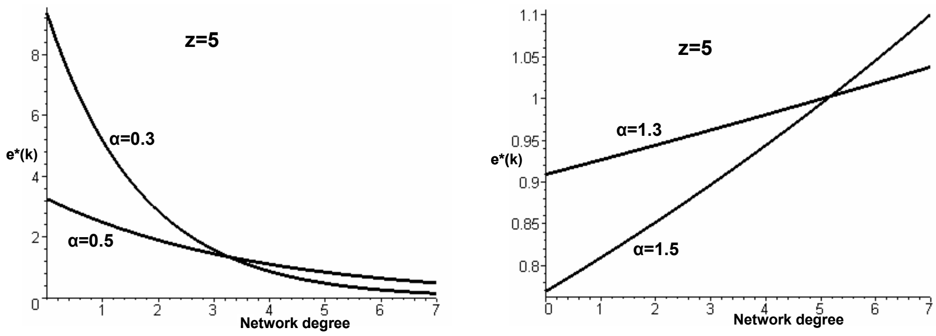

Figure 3. Equilibrium effort profile in a Poisson network.
Four different values of α are considered, $\alpha = 0.3, 0.5, 1.3, 1.5$, for an average degree $z = 5$.

4.2 Scale-free degree distributions

Next we consider a case that, in a sense, can be conceived as polar to the previous one. The degree distribution is now assumed to be scale-free, in the sense of being governed by a power law of the form:

$$p_\kappa = \begin{cases} 0 \text{ if } \kappa = 0 \\ \frac{\kappa^{-\gamma}}{\mathcal{R}(\gamma)} \text{ if } \kappa = 1, ..., \infty \end{cases} \tag{19}$$

where $\gamma \geq 2$ is a parameter that determines the decay of the distribution and $\mathcal{R}(\gamma) \equiv \sum_{\kappa=1}^{\infty} \kappa^{-\gamma}$ is the Riemann zeta function. The average degree is simply given by

$$z(\gamma) = \frac{\mathcal{R}(\gamma - 1)}{\mathcal{R}(\gamma)},$$

that diverges when $\gamma \downarrow 2$ and converges to 1 when $\gamma \to \infty$.

In contrast with the Poisson distribution, a distribution given by (19) lacks a characteristic scale, i.e. for any $r \in \mathbb{N}$ we have that the ratio

$$\frac{p_{r\kappa}}{p_\kappa} = r^{-\gamma} > 0$$

is the same at "all scales," i.e. for *all* κ. Intuitively, this represents a distribution with "fat tails," where the decay in probability for higher degrees is only relatively slow (in particular, "infinitely slower" than the exponential decay displayed by Poisson networks).

Let us denote $\underline{\alpha}(\gamma) = 1/\mathcal{R}(\gamma - 1)$. Then we can establish the following result.

Theorem 2 *Let the degree distribution be given by (19). There exists at most a unique non-trivial equilibrium* $\mathbf{e}^* = \{e^*(\kappa)\}_{\kappa=0}^{\infty}$*. This equilibrium exists if and only if* $\alpha \in (\underline{\alpha}(\gamma), 1]$ *and* $\gamma > 2$*. Furthermore if* $\alpha \in (\underline{\alpha}(\gamma), 1)$ *the equilibrium effort levels as well as the equilibrium expected payoffs are decreasing in the degree, while they are constant if* $\alpha = 1$*.*

Thus, if the degree distribution is scale-free, we find that the existence of a non-trivial equilibrium requires that costs be low ($\alpha < 1$). And, if an equilibrium does exist (which requires that costs not be too low either – $\alpha > \underline{\alpha}(\gamma)$), then it has the "paradoxical features" found in the Poisson case for the analogous situation. Namely, the equilibrium induces an effort profile that is decreasing in the degree, so that those agents subject to stronger externalities exert lower effort. Along the lines explained for Poisson distributions, this can be intuitively understood as the only way of preventing the unfolding of snow-ball effects that are obviously incompatible with equilibrium. When the degree distribution has a fat upper tail, such considerations are preeminent and lead to a situation where the only possible equilibrium controls those effects by prescribing a decreasing effort profile.[4]

4.3 Geometric degree distributions

Finally, we consider the case of geometric distributions, which can be interpreted as sharing some features with each of the previous two. It is again described by a one-parameter family with probabilities

$$p_\kappa = (1-\gamma)\,\gamma^\kappa \qquad (\kappa = 0, 1, ...) \tag{20}$$

where $\gamma \in (0,1)$. The average network degree is $z(\gamma) = \gamma/(1-\gamma)$, which is increasing in γ.

This distribution shares with Poisson networks the fact that it has a finite characteristic scale. That is, for any $r \in \mathbb{N}$ ($r \geq 2$), we have:

$$\frac{p_{r\kappa}}{p_\kappa} \to 0$$

[4] The dependence of the equilibrium efforts on the parameters is much less clearcut than in the Poisson case. For example, the effect of small changes in α is qualitatively different for low and high values of it. If we focus, specifically, on $e^*(1)$, the effect of increasing α can be shown to be always positive when α is small (i.e. close to $1/\mathcal{R}(\gamma-1)$) but depends on γ when α is close to 1. In this latter case, it is negative if γ is close to its lower bound of 2 but it is positive if γ is large.

as $\kappa \to \infty$, which means that large scales have no more than an infinitesimal weight. In this sense, therefore, we may say that the distributions do not exhibit fat tails, as scale-free distributions do. But, on the other hand, it shares with scale-free distributions that the probability decay is infinitely slower than for the Poisson case. Based on this mixed comparison with Poisson and scale-free distributions, one might expect that the equilibrium in the present case should display a "mixture" of the properties found in each of the two previous scenarios. A partial confirmation of this idea is established by the following result.

Let us denote $\underline{\alpha}(\gamma) = (1-\gamma)^2$. Then we can show:

Theorem 3 *Let the degree distribution be given by (20). There are two non-trivial equilibria. First, for any* $\alpha > 0$ *and* γ, *there exists a high-effort equilibrium which takes the following form:* $e^H(\kappa) = \alpha^{\kappa-1}\left(\frac{\alpha+\sqrt{\alpha}(1-\gamma)}{\gamma\alpha^2}\right)^{\kappa}$, $\kappa = 0, 1, \ldots.$ *Second, for any* $\alpha > \underline{\alpha}(\gamma)$, *there also exists a low-effort equilibrium which takes the following form:* $e^L(\kappa) = \alpha^{\kappa-1}\left(\frac{\alpha-\sqrt{\alpha}(1-\gamma)}{\gamma\alpha^2}\right)^{\kappa}$, $\kappa = 0, 1, \ldots.$

The previous theorem tells us that, provided that α is sufficiently high, there exists two non-trivial equilibria: one with uniformly higher efforts than the other. Instead, when α is low, the high-effort equilibrium is the only non-trivial equilibrium. In the region where the two equilibria coexist (when $\alpha > \underline{\alpha}(\gamma)$), they can display substantially different features, as stated by the following straightforward Corollary.

Corollary 4 *The high-effort equilibrium induces efforts and payoffs that are increasing in the degree. Whenever the low-effort equilibrium exists, the same applies to its induced effort profile if* $\alpha > 1$; *otherwise, the induced efforts and payoffs are decreasing in the degree. In every case (i.e. for all relevant* α *and all* γ*), the high-effort equilibrium Pareto-dominates the low-effort equilibrium.*

The above Corollary shows, as in the case of Poisson networks, that the existence of an equilibrium with a decreasing effort profile depends on the cost parameter α being less than one. In the present case, however, there always exists an equilibrium with an increasing effort profile, even if α is arbitrarily small. The multiplicity of equilibria arising in this case leads to an equilibrium selection problem, akin to that common in coordination problems. It is interesting to observe that this coordination aspect of the problem (an inherent consequence of the strategic complementarities involved) materializes in quite different ways depending on the characteristics of the social network. Note that for Poisson and scale-free networks, the coordination dilemma only involves a trivial and a unique non-trivial equilibrium. In contrast, the present case opens up a new possibility, namely, an additional non-trivial equilibrium that is also possible for wide parameter range.[5]

5 Summary and conclusions

In this paper, we analyze the role of local externalities in shaping the behavior of agents who interact in a complex and volatile network. The model is found to display a rich and subtle interplay between the network topology and strategic individuals' behavior. Specifically, we show that, somewhat surprisingly, low costs of investment or/and a wide distribution in the connectivity of agents may render it impossible to sustain, at equilibrium, high levels of effort. Table 1 provides a schematic description of our results for each of the three paradigmatic scenarios considered: Poisson, scale-free, and geometric random networks.

Our approach to the study of strategic interaction in network setups attempts to make some progress concerning two important limitations of existing network literature.

First, it does not shun contexts where the underlying social network displays significant interagent heterogeneity and substantial topological complexity. These two features – a mark of many interesting social networks in the real world – are accommodated by modelling the system as a large stochastic system that, despite its intrinsic complexity, displays *given* overall *statistical* regularities. The analysis may then rely on the versatile tools afforded by the modern theory of complex systems.

Second, players' behavior is made to reflect the presumed environmental complexity by positing that they have only imprecise information on their individual circumstances (i.e. the type of their neighbors), although they

[5] By way of comparative statics, sharp results can be obtained for the high-effort equilibrium on the dependence of the underlying parameters. Specifically, it can be shown that $e^H(\kappa)$ is always decreasing (for any degree κ) in both γ and α. For the low-effort equilibrium, however, matters are less clear-cut. For example, it is easy to see that, for every κ, the effort level $e^L(\kappa)$ is increasing in γ (respectively, decreasing) if $\alpha > 1$ (respectively, if $\alpha < 1$).

all share the same global information (accurate but “anonymous”) on the whole network. Interestingly, these natural informational constraints allows us to reduce the huge multiplicity of equilibria that are so prevalent in network models, allowing instead for definite theoretical predictions and clearcut comparative-statics analysis.

The present paper just represents a first step into a new terrain (that could be labelled “strategic complex-network analysis”) and of course should be extended and enriched in a number of important directions. For example, a key feature in understanding how network effects shape equilibrium behavior is the nature of the payoffs accruing from interaction. In this paper we have focused on the classical case of strategic complementarity embodied by a Cobb-Douglas formulation. In work in progress by Galeotti, Goyal and Vega-Redondo (2005) we turn attention to a class of aggregative functions (i.e. the dependence on what others do is through the sum of their actions) and analyze the case in which individuals’ efforts are strategic substitutes as well as strategic complements. One of our main concerns there is to understand how, for general degree distributions, changes in players’ knowledge about the underlying random network affects equilibrium behavior.

Degree Distribution	Existence	Equilibrium Effort
Poisson	$1/\exp(z)<\alpha<1$	$e(0)>e(1)>e(2)....$
	$\alpha=1$	$e(0)=e(1)=e(2)....$
	$\alpha>1$	$e(0)<e(1)<e(1)....$
Scale free	$1/R(\gamma-1)<\alpha<1$ and $\gamma>2$	$e(0)>e(1)>e(2)....$
Geometric	$0<\alpha<(1-\gamma)^2$	$e^H(0)<e^H(1)<e^H(2)....$
	$(1-\gamma)^2<\alpha<1$	$e^H(0)<e^H(1)<e^H(2)....$ $e^L(0)>e^L(1)>e^L(2)....$
	$\alpha=1$	$e^H(0)<e^H(1)<e^H(2)....$ $e^L(0)=e^L(1)=e^L(2)....$
	$\alpha>1$	$e^H(0)<e^H(1)<e^H(2)....$ $e^L(0)<e^L(1)<e^L(2)....$

Table 1. Summary of the main results

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Appendix

Proof of Theorem 1:. We can rewrite (14) as follow:

$$\frac{1}{\alpha^2 e^*(1)} \sum_{\kappa'=1}^{\infty} \tilde{p}_{\kappa'} (\alpha e^*(1))^{\kappa'} = \frac{1}{\alpha^2 e^*(1)} G_1(\alpha e^*(1)) = 1, \tag{21}$$

where $G_1(\cdot)$ is the generating function of the degree distribution of a neighboring node. Note that

$$G_1(x) = \frac{x G_0'(x)}{G_0'(1)}$$

and $G_0(x) = \sum_{\kappa'=0}^{\infty} p_{\kappa'} x^{\kappa'}$ is the generating function of the original degree distribution. Then:

$$G_1(\alpha e^*(1)) = \frac{\alpha e^*(1)\, G_0'(\alpha e^*(1))}{z}.$$

Since in the case of a Poisson distribution we have:

$$G_0'(x) = z \exp(z(x-1))$$

we can write expression (21) as follows:

$$\frac{1}{\alpha} \exp(z(\alpha e^*(1) - 1)) = \frac{\exp(z \alpha e^*(1))}{\alpha \exp(z)} = 1,$$

and solving for $e^*(1)$ we obtain:

$$e^*(1) = \frac{\ln \alpha + z}{\alpha z}$$

Therefore:

$$e^*(\kappa) = \frac{1}{\alpha} \left(\frac{\ln \alpha + z}{z} \right)^{\kappa}, \ \kappa = 0, 1, ... \tag{22}$$

We now must require that for any possible degree κ, $e^*(\kappa) > 0$. This holds if, and only if, $\ln \alpha + z > 0$, which is satisfied if and only if $\alpha > \underline{\alpha}(z)$. Finally note that the same condition assures that the expected utility of each player is positive. Indeed, for any i with degree κ, we have that:

$$\psi_\kappa(e, \mathbf{e}^*) = e^*(\kappa) \left(\sum_{\kappa'=1}^{\infty} \tilde{p}_{\kappa'} e^*(\kappa') \right)^\kappa - \alpha \frac{e^*(\kappa)^2}{2}$$

and using the equilibrium conditions it follows that:

$$\psi_\kappa(e, \mathbf{e}^*) = \frac{\alpha}{2} e^*(\kappa)^2 > 0. \tag{23}$$

This completes the proof of the Theorem. ■

Proof of Corollary 1:. In view of (22), note that $e^*(\kappa)$ is increasing in κ if, and only if,

$$\frac{\ln \alpha + z}{z} > 1$$

or

$$\ln \alpha > 1,$$

which yields the desired conclusion and proves the Corollary. ■

Proof Corollary 2. Note that:

$$\frac{\partial e^*(\kappa)}{\partial z} = -\frac{\kappa}{\alpha}(1 + \frac{\ln \alpha}{z})^{\kappa-1} \frac{\ln \alpha}{z^2}$$

As long as $\alpha > \underline{\alpha}(z)$, the sign of $\partial e^*(\kappa) / \partial z$ is opposite to that of $\ln \alpha$. This proves the Corollary. ■

Proof Corollary 3. Note that

$$\frac{\partial e^*(\kappa)}{\partial \alpha} = \left(\frac{\ln \alpha + z}{\alpha z} \right)^\kappa \frac{\alpha^{\kappa-2}}{(\ln \alpha + z)} (\kappa - \ln \alpha - z)$$

Since the first two terms are positive, the sign of $\partial e^*(\kappa) / \partial \alpha$ depends on the sign of $(\kappa - \ln \alpha - z)$. This proves the Corollary. ■

Proof Theorem 2. Consider the equilibrium condition

$$\frac{1}{\alpha^2 e^*(1)} G_1(\alpha e^*(1)) = 1$$

We can rewrite it as

$$G_1(x) = \alpha x$$

where $x = \alpha e^* (1)$. Note that:

$$G_1(x) = \frac{xG_0'(x)}{G_0'(1)} = \frac{1}{\mathcal{R}(\gamma - 1)} \sum_{k=1}^{\infty} \kappa^{1-\gamma} x^{\kappa}$$

and therefore the following holds:

$$G_1(0) = 0 \text{ and } G_1(1) = 1$$

We now show that $G_1(x)$ is increasing and convex in $x \in [0, 1]$, while it is undefined for any $x > 1$. Note that

$$\frac{dG_1(x)}{dx} = \frac{1}{\mathcal{R}(\gamma - 1)} \sum_{k=1}^{\infty} \kappa^{1-\gamma} \kappa x^{\kappa-1}$$

The function $\mathcal{R}(\gamma - 1)$ is well define and positive for any $\gamma > 2$. Furthermore, $\lim_{\gamma \to 2} \mathcal{R}(\gamma - 1) = \infty$ and $\lim_{\gamma \to \infty} \mathcal{R}(\gamma - 1) = 1$. Therefore for any $\gamma > 2$, $G_1(x)$ is increasing in x. To show convexity note that:

$$\frac{d^2G_1(x)}{dx^2} = \frac{1}{\mathcal{R}(\gamma - 1)} \sum_{k=1}^{\infty} \kappa^{1-\gamma} \kappa (\kappa - 1) x^{\kappa-2} > 0 \text{ for any } x > 0$$

and that:

$$\left.\frac{d^2G_1(x)}{dx^2}\right|_{x=0} = \frac{2^{2-\gamma}}{\mathcal{R}(\gamma - 1)} > 0$$

These facts imply that there exists a unique solution $x^* = \alpha e^* (1) \leq 1$ if, and only if, $\alpha \in (\frac{1}{\mathcal{R}(\gamma-1)}, 1]$ and $\gamma > 2$. In particular $x^* < 1$ if $\alpha < 1$, and $x^* = 1$ if $\alpha = 1$. Note that this range is never empty and it becomes wider as γ is close to 2. Finally, since $\alpha e^* (1) < 1$, it is clear that the equilibrium effort levels are decreasing in the players' degree. This also implies that, at equilibrium, expected payoffs are decreasing in the degree of a player, which completes the proof of the Theorem. ■

Proof Theorem 3. As before, we start by rewriting (14) as follows:

$$\frac{1}{\alpha^2 e^* (1)} G_1(\alpha e^* (1)) = 1$$

Since
$$G_1(\alpha e^*(1)) = \frac{\alpha e^*(1)\, G_0'(\alpha e^*(1))}{G_0'(1)} = \frac{\alpha e^*(1)(1-\gamma)^2}{(1-\alpha e^*(1)\gamma)^2},$$
it follows that $e^*(1)$ must satisfy:
$$\frac{(1-\gamma)^2}{\alpha(1-\gamma\alpha e^*(1))^2} = 1$$
Solving this equilibrium condition we obtain the following solutions:
$$e^*(1) = \begin{cases} \frac{\alpha+\sqrt{\alpha}(1-\gamma)}{\gamma\alpha^2} \\ \frac{\alpha-\sqrt{\alpha}(1-\gamma)}{\gamma\alpha^2} \end{cases}$$
Therefore, there are two solutions of the system (10)-(12), which are:
$$e^H(\kappa) = \alpha^{\kappa-1}\left(\frac{\alpha+\sqrt{\alpha}(1-\gamma)}{\gamma\alpha^2}\right)^{\kappa},\ \ \kappa = 0,1,2,... \tag{24}$$
and
$$e^L(\kappa) = \alpha^{\kappa-1}\left(\frac{\alpha-\sqrt{\alpha}(1-\gamma)}{\gamma\alpha^2}\right)^{\kappa},\ \ \kappa = 0,1,2,... \tag{25}$$
We now must require that for any possible degree κ, $e^H(\kappa), e^L(\kappa) \geq 0$. It is easy to see that $e^H(\kappa)$ always satisfies this condition. Consider now $e^L(\kappa)$. In this case we have must have that $\alpha - \sqrt{\alpha}(1-\gamma) > 0$, which is satisfied if and only if $\alpha > (1-\gamma)^2$. Furthermore, by investigation of expression (23), it follows that in both equilibria players' payoffs are non negative. This completes the proof. ■

Proof Corollary 4. By rewriting (24) as
$$e^H(\kappa) = \frac{1}{\alpha}\left(\frac{\alpha+\sqrt{\alpha}(1-\gamma)}{\gamma\alpha}\right)^{\kappa},\ \ \kappa = 0,1,2,...,$$
we can readily see that $e^H(\kappa)$ is increasing in the degree since
$$\frac{\alpha+\sqrt{\alpha}(1-\gamma)}{\gamma\alpha} > \frac{1}{\gamma} > 1$$
for all parameter values. Instead, from (25), it follows that $e^L(\kappa)$ is increasing when
$$\frac{\alpha-\sqrt{\alpha}(1-\gamma)}{\gamma\alpha} > 1,$$

or

$$\sqrt{\alpha}\,(1-\gamma) > \alpha(1-\gamma),$$

which requires that $\alpha > 1$. Instead, if $\alpha < 1$, the effort levels $e^L(\kappa)$ are decreasing with κ.

We finally show that the high intensity equilibrium Pareto dominates the low intensity equilibrium. This follows because $e^H(0) = e^L(0)$ and $e^H(\kappa) > e^L(\kappa)$ for any $\kappa = 1, ..., \infty$ and the fact that the expected equilibrium payoffs to a player is increasing in his own effort. ■