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QUANTILES AND MEDIANS

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SOCIAL SCIENCE WORKING PAPER 1222

April 2005

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Abstract

We provide a list of functional equations that characterize quantile functions for collections of bounded and measurable functions. Our central axiom is ordinal covariance. When a probability measure is exogeneously given, we characterize quantiles with respect to that measure through monotonicity with respect to stochastic dominance. When none is given, we characterize those functions which are simply ordinally covariant and monotonic as quantiles with respect to capacities; and we also find an additional condition for finite probability spaces that allows us to represent the capacity as a probability measure. Additionally requiring that a function be covariant under its negation results in a generalized notion of median. Finally, we show that all of our theorems continue to hold under the weaker notion of covariance under increasing, concave transformations. Applications to the theory of ranking infinite utility streams and to the theory of risk measurement are provided.

JEL classification numbers: D63,D71,D81,D90

Key words: Ordinal, quantile, median, axiom, risk measure, Value at Risk, intergenerational equity

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1 Introduction

We are concerned with the numerical measurement of certain real-valued functions. Given is some exogenous state space, and a set of functions from that space to the real numbers. A function can be understood as specifying a state-contingent monetary payoff; however, our theory is abstract.

We are interested in functionals defined on this space of functions. Of primary concern are functionals which are covariant with respect to arbitrary changes in unit of measurement. Imagine two functions, f and g, which are ranked according to the functional. If the functional ranks f as higher than g, we will require that the functional also ranks $\varphi \circ f$ as higher than $\varphi \circ g$ (where φ is some strictly increasing and continuous function). This property is referred to as *ordinal covariance*. Our other primary requirement is *monotonicity*. If one function dominates another in every state of the world, then the functional should rank the first one at least as high as the second.

For a concrete example, think of a statistical environment. As a property of a statistic, ordinal covariance simply means that the statistic should commute with respect to changes in unit of measurement. Such changes are allowed to be nonlinear. There are many environments where one would like to ignore the effects of nonlinear changes in the unit of measurement. A statistic on the distribution of incomes should work equally well for the distribution of pre-tax incomes as it does for the distribution of after-tax incomes. In a production scenario with one input and one output, a statistic on production should work equally well for distributions of inputs as it does for distributions of outputs. Units of measurement are of course even less meaningful when we speak of the utility of an agent. Here, it is well known that the only empirically meaningful statements refer to the underlying preference ordering of an agent. Thus, the cardinal structure of utility functions has no real empirical significance. Thus, a statistic of an individual's utility under different situations must be invariant under arbitrary increasing transformations.

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Monotonicity in the statistical environment means nothing more than that the statistic should aggregate data in a reasonable way.

Let us give another example. In this example, the functions that we consider are risks that map from a set of exogenous (and uncertain) states of the world to monetary payoffs. Imagine some decision maker facing a choice between two such risks. As an outside observer, what advice should we give to this decision maker as to which of the two risks to choose? One way of providing an answer to this question is by numerically ranking all such risks. This provides us with a functional. If this numerical ranking is supposed to be a list of recommendations that could be given to an arbitrary decision maker facing these risks, then the ranking should be independent of individual characteristics of the decision maker. In particular, consider an environment in which decision makers face nonlinear income taxes. An ordinally covariant ranking will recommend the same ranking, independently of whether risks are measured in before or after-tax income.

We provide a general representation theorem for all functionals which are ordinally covariant and monotonic. First we give an example, then we discuss the general result. The punchline is that any functional satisfying the two properties is a generalization of a median.

Imagine that a measurable space is endowed with a probability measure. Each function then induces a probability distribution over the real numbers, in a natural sense. Suppose that we require that our functional is monotonic with respect to stochastic dominance of the induced probability distributions. Together with ordinal covariance, this forces our functional to be a *quantile*. A quantile is a generalization of the notion of a median. The median of a distribution is defined as the smallest value for which the probability that the realized value is at least as great as this one is less than one-half. A quantile works with an arbitrary number between zero and one–thus, an α-quantile is defined exactly as a median, except for "one-half" is replaced by α.

The intuition for this result actually comes from several important theorems from the social choice literature. Theorems found in [3, 7, 8, 13, 14] use concepts similar to ordinal covariance to characterize leximin rankings and rank-order dictatorships (see also Sprumont [17] for a related condition in the cost-sharing literature). In these models, states are agents, and a real-valued function on the set of states specifies a utility profile. The ordinal covariance axiom in this framework simply means that utility units are comparable across individuals, but have no cardinal significance. These authors postulate a Pareto condition (similar to our monotonicity), and a symmetry condition, which together imply monotonicity with respect to stochastic dominance where the probability measure in consideration is the uniform probability over states. The aforementioned result only slightly generalizes these by working on arbitrary spaces and removing the uniform prior assumption.

Denneberg [5] also studies the ordinal covariance concept. His Proposition 4.1 shows that the quantiles are ordinally covariant; in fact, they are covariant under any weakly monotonic transformation.

In the general case, a probability measure over states is not specified. Any ordinally invariant and monotonic functional is characterized by a particular collection of sets of states. In the case of quantiles, these sets are those sets which have probability weakly less than α. In the general case, it turns out that it can be any arbitrary family of sets satisfying two conditions. The first of these is that the empty set is contained in the collection, and that the set consisting of all states is never contained in the collection. The second condition is that if a set is contained in the collection, then so is any measurable subset. For any such collection of sets, the functional applied to an arbitrary function finds the smallest value such that the upper contour set according to that value is contained in the collection.

It turns out that all such functionals can be equivalently characterized as quantiles. They are not necessarily quantiles with respect to probability measures, but with respect to (possibly non-additive) monotonic set functions. A functional satisfies ordinal covariance and monotonicity if and only if there exists some (possibly non-additive) monotonic set function ν and some $\alpha < 1$ for which the function finds the smallest value for which the set function applied to the upper contour set of this value is weakly less than α. An alternative representation for this class of functionals is given by the supremum and infimum operators. Fix any nonempty collection of nonempty, measurable subsets of the underlying state space. For each set in the collection and any given function, find the supremum over this set. Then, take the infimum of supremums over all sets. Such a functional is ordinally covariant and monotonic.

When does there exist a quantile representation with a probability measure? Generally, such a representation need not exist. We provide another condition, which we call betting consistency, that allows us to write the functional as a quantile with respect to some probability measure. Betting consistency states that for any two sequences of indicator functions of equal length, if the sum of the first sequence is weakly greater than the sum of the second sequence, then there must be an element of the first sequence which is ranked at least as high according to the functional as the corresponding element of the second sequence. It is interpreted as stating that it is impossible to recommend that an agent choose a sequence of bets (strictly) that will make him weakly worse off in the aggregate.

We find a condition that is characteristic of a general notion of median. This condition states that the negative of a functional applied to a function is the functional applied to the negative of the function.

Lastly, we show that our ordinal covariance axiom can be significantly weakened without altering any of the results. We show that requiring covariance with respect to arbitrary concave and strictly monotonic functions produces the same results.

There are applications of the theorems to the theory of social choice and to the theory of risk measurement. In social choice, functions are utility profiles. The functional provides a social utility. We provide (using one of our general representation theorems) a Rawlsian-style [12] representation for any functional that is ordinally covariant and monotonic. Such a functional fixes a set of coalitions, finds the most well-off agent in each such coalition, and then finds the least well-off agent among all of these.

Moreover; we show how our representation applies to the theory of intergenerational equity. There are a countably infinite set of agents, so speak of utility streams. We use a consistency-like notion to axiomatize exactly four methods of ranking utility streams: the supremum, the infimum, the limit superior, and the limit inferior. The consistency condition requires that, when combining a finite list of utility streams whose social utility is the same, the resulting social utility of the combination should be the same as the social utilities of the original utility streams.

We also study the theory of risk measurement, which is also a normative theory. When functions are risks, a risk measure is a *recommendation* of which risks to choose in any given situation. In a sense, the notion is very similar in motivation to the concept of a social welfare function; the only primitive distinction here being that the set of decision makers and their preferences remain unspecified. Our primary representation theorem provides a foundation for a commonly used risk measure–the Value at Risk. We show that one of the primary criticisms of the Value at Risk can be bypassed by working with our more general representations.

Our final result provides conditions for an general order structure on a set of functions that are necessary and sufficient for it to be represented by a functional satisfying ordinal covariance and monotonicity.

Section 2 provides our general representation theorems. Section 3 studies the problem with an exogenously specified probability measure. Section 4 goes further, and endogenizes the probability. Section 5 gives our result on medians, and Section 6 is devoted to the study of concave covariance. Section 7 offers an application of our general theory to the theory of intergenerational equity, and Section 8 discusses an application to the theory of risk measurement. Section 9 discusses a theory based on orders. Section 10 concludes.

2 General representations for ordinally covariant and monotonic functionals

Let (Ω, Σ) be a measurable space, and let $\mathcal{F}$ be the vector space of real-valued, bounded, Σ measurable functions. The set $\mathcal{F}$ might represent a set of state-contingent monetary payoffs, or "risks." It may also represent a set of possible utility profiles. We will study functionals $T : \mathcal{F} \to \mathbb{R}$.

We wish to study those functionals that are covariant under arbitrary strictly increasing and continuous transformations.

Ordinal covariance: For all $f \in \mathcal{F}$ and all strictly increasing and continuous $\varphi : \mathbb{R} \to \mathbb{R}$, $T(\varphi \circ f) = \varphi(T(F))$.

Here, we study a general environment in which a probability measure on (Ω, Σ) is not explicitly given. We will therefore use the following notion of monotonicity:

Monotonicity: For all $f, g \in \mathcal{F}$, if $f \leq g$, then $Tf \leq Tg$.

We characterize all functionals that satisfy ordinal covariance and monotonicity.

Say a collection of sets $\mathcal{E}$ is an **ideal** if $A \in \mathcal{E}$ and $B \subset A$ implies that $B \in \mathcal{E}$.

First, we provide the following lemma:

Lemma: Suppose that T is ordinally covariant and monotonic. Then for all $f \in \mathcal{F}$, $Tf \in \overline{f(\Omega)}$.

Proof. Suppose that the statement of the Lemma is false. Then there exists $f \in \mathcal{F}$ such that $Tf \in \mathbb{R} \backslash \overline{f(\Omega)}$. In particular, there exists a neighborhood U of Tf so that $U \cap f(\Omega) = \varnothing$. Without loss of generality, we may assume that U is an open interval, say $(Tf - \varepsilon, Tf + \varepsilon)$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined as

$$\varphi(x) \equiv \begin{cases} x \text{ for } x \notin (Tf - \varepsilon, Tf + \varepsilon) \\ \frac{(x-(Tf-\varepsilon))^2}{2\varepsilon} + (Tf - \varepsilon) \text{ for } x \in (Tf - \varepsilon, Tf + \varepsilon) \end{cases}.$$

For all $x \in (Tf - \varepsilon, Tf + \varepsilon)$, $\varphi(x) < x$. In particular, as $(Tf - \varepsilon, Tf + \varepsilon) \cap f(\Omega) = \varnothing$, $\varphi \circ f = f$. As $Tf \in (Tf - \varepsilon, Tf + \varepsilon)$, $\varphi(Tf) < Tf$. Hence $Tf = T(\varphi \circ f) = \varphi(Tf) < Tf$. Here, the second equality follows from ordinal covariance. This is a contradiction. ■

The main representation result follows.

Theorem 1: A functional T satisfies ordinal covariance and monotonicity if and only if there exists an ideal $\mathcal{E} \subset \Sigma$ such that $\varnothing \in \mathcal{E}$ and $\Omega \notin \mathcal{E}$ for which

$$Tf = \inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}.$$

Proof. Suppose first that the functional T can be represented as $Tf = \inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}$ for some ideal $\mathcal{E}$. To see that it is ordinally covariant, let $f \in \mathcal{F}$ and let φ be strictly monotonic and continuous. Then

$$\begin{aligned} \varphi(Tf) &= \varphi(\inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}) \\ &= \inf \{\varphi(x) : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\} \\ &= \inf \left\{x : \left\{\omega : f(\omega) \geq \varphi^{-1}(x)\right\} \in \mathcal{E}\right\} \\ &= \inf \{x : \{\omega : \varphi(f(\omega)) \geq x\} \in \mathcal{E}\} \\ &= T(\varphi \circ f). \end{aligned}$$

To see that it is monotonic, suppose that $f \geq g$. For all $x \in \mathbb{R}$, $\{\omega : g(\omega) \geq x\} \subset \{\omega : f(\omega) \geq x\}$. Hence $\{\omega : f(\omega) \geq x\} \in \mathcal{E}$ implies $\{\omega : g(\omega) \geq x\} \in \mathcal{E}$. This implies that $\inf\{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\} \geq \inf\{x : \{\omega : g(\omega) \geq x\} \in \mathcal{E}\}$, or that $Tf \geq Tg$.

Conversely, suppose that T is ordinally covariant and monotonic. Define the family $\mathcal{E} \equiv \{E \in \Sigma : T1_E = 0\}$. By the Lemma, for all $E \in \Sigma$, $T1_E \in \{0, 1\}$. As $T1_\varnothing = T0 = 0$, and as $T1_\Omega = T1 = 1$, $\varnothing \in \mathcal{E}$ and $\Omega \notin \mathcal{E}$. Further, if $A \in \mathcal{E}$ and $B \subset A$, then $1_B \leq 1_A$, so that $T1_B \leq T1_A = 0$, from whence we conclude that $B \in \mathcal{E}$. Hence $\mathcal{E}$ is an ideal.

We verify that for all $E \in \Sigma$, $T1_E = \inf\{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}$. Let $E \in \mathcal{E}$. For all $x > 0$, $\{\omega : 1_E(\omega) \geq x\} \subset E$, so that $\{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}$. Therefore, $\inf\{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\} \leq 0$. However, for all $x < 0$, $\{\omega : 1_E(\omega) \geq x\} = \Omega \notin \mathcal{E}$. Hence, we may conclude that $\inf\{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\} = 0$, so that $T1_E = \inf\{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\}$. Suppose instead that $E \notin \mathcal{E}$. Then for all $x > 1$, $\{\omega : 1_E(\omega) \geq x\} = \varnothing$, so that $\inf\{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\} \leq 1$. But for all $x < 1$, $E \subset \{\omega : 1_E(\omega) \geq x\}$, so that $\{\omega : 1_E(\omega) \geq x\} \notin \mathcal{E}$. Hence $\inf\{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\} = 1$. Therefore, $T1_E = \inf\{x : \{\omega : 1_E(\omega) \geq x\} \in \mathcal{E}\}$. Next, we extend the result from indicator functions to all functions.

Let $f \in \mathcal{F}$ be arbitrary, and set $x^*(f) = \inf\{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}$. Let $\varepsilon > 0$ be arbitrary. Then $\{\omega : f(\omega) \geq x^*(f) + \varepsilon\} \in \mathcal{E}$ by definition of $x^*(f)$. Let $g^\varepsilon \in \mathcal{F}$ be defined as

$$g^\varepsilon \equiv \begin{cases} \sup f \text{ for } \omega : f(\omega) \geq x^*(f) + \varepsilon \\ x^*(f) + \varepsilon \text{ otherwise} \end{cases} .$$

Then $f \leq g^\varepsilon$, so that by monotonicity, $Tf \leq Tg^\varepsilon$. Note that $\{\omega : f(\omega) \geq x^*(f) + \varepsilon\} \in \mathcal{E}$, so that $\{\omega : g^\varepsilon(\omega) \geq \sup f\} \in \mathcal{E}$. As g^ε is an ordinal transformation of the indicator function of $\{\omega : f(\omega) \geq x^*(f) + \varepsilon\}$, we may conclude $Tg^\varepsilon = x^*(f) + \varepsilon$. As ε is arbitrary, $Tf \leq x^*(f)$.

Let $\varepsilon > 0$ be arbitrary. Let $h^\varepsilon \in \mathcal{F}$ be defined as

$$h^\varepsilon \equiv \begin{cases} \inf f \text{ for } \omega : f(\omega) < x^*(f) - \varepsilon \\ x^*(f) - \varepsilon \text{ otherwise} \end{cases} .$$

Then $f \geq h^\varepsilon$. Moreover, $\{\omega : f(\omega) \geq x^*(f) - \varepsilon\} \notin \mathcal{E}$. But $f(\omega) \geq x^*(f) - \varepsilon$ if and only if $h^\varepsilon(\omega) \geq x^*(f) - \varepsilon$. Therefore, $\{\omega : h^\varepsilon(\omega) \geq x^*(f) - \varepsilon\} \notin \mathcal{E}$. As h^ε is an ordinal transformation of the indicator function of $\{\omega : f(\omega) \geq x^*(f) - \varepsilon\}$, we may conclude $Th^\varepsilon = x^*(f) - \varepsilon$. By monotonicity, $Tf \geq x^*(f) - \varepsilon$. As ε is arbitrary, $Tf \geq x^*(f)$.

Therefore $Tf = \inf\{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\}$. ■

Note that for an ideal $\mathcal{E}$,

$$\inf\{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\} = \inf\{x : \{\omega : f(\omega) > x\} \in \mathcal{E}\}.$$

Thus, there is no distinction between working with strict upper contour sets of f and weak upper contour sets of f. Moreover, a similar representation in terms of lower contour sets of functions is also possible.

For any functional T satisfying our axioms, the collection of sets $\mathcal{E}$ discussed in the preceding representation is clearly unique. In the rest of this section, we discuss several alternative representations and some examples.

A **capacity** is a set function $v : \Sigma \to \mathbb{R}$ such that $v(\varnothing) = 0$, $\nu(\Omega) > 0$, and $A \subset B$ implies $v(A) \leq v(B)$. Capacities are often used to represent subjective notions of belief. Capacities need not be additive. We will require that a capacity be normalized so that $v(\Omega) = 1$. Our preceding theorem can be equivalently re-expressed as the following:

Corollary 1: A functional T satisfies ordinal covariance and monotonicity if and only if there exists a normalized capacity v and a real number $\alpha \in [0, 1)$ such that $Tf = \inf\{x : v(\{\omega : f(\omega) \geq x\}) \leq \alpha\}$.

Proof. It is clear that if T has the desired representation, then T is both ordinally covariant and monotonic.

Conversely, given the ideal $\mathcal{E}$ described in the preceding theorem, define $v(E) = 0$ if $E \in \mathcal{E}$ and $v(E) = 1$ if $E \notin \mathcal{E}$. Let $\alpha = 1/2$. Then $\mathcal{E} = \{E \in \Sigma : v(E) \leq \alpha\}$. ■

Therefore, any ordinally covariant and monotonic function can be written as a quantile with respect to a possibly non-additive set function. In fact, they can be written as quantiles with respect to set functions which take only the values zero and one. Moreover, they can also be written as the Choquet integral with respect to the capacity given in the preceding corollary; thus if ν is the capacity defined in the preceding corollary,

$$Tf = \int_\Omega f(\omega)\, d\nu(\omega),$$

where the integration is in the sense of Choquet.[1]

Corollary 2: A functional T is ordinally covariant and monotonic if and only if there exists a normalized capacity ν taking values in $\{0, 1\}$ such that

$$Tf = \int_\Omega f(\omega)\, d\nu(\omega).$$

[1] The Choquet integral is defined as:

$$\int_\Omega f d\nu = \int_0^\infty \nu(\{\omega : f(\omega) \geq x\})\, dx + \int_{-\infty}^0 \nu[(\{\omega : f(\omega) \geq x\}) - 1]\, dx.$$

Proof. Let T be ordinally covariant and monotonic, and let $\mathcal{E}$ be its associated ideal. As in the preceding corollary, define $\nu(E) = 1_{E \notin \mathcal{E}}$. Clearly ν is a normalized capacity taking values in $\{0, 1\}$. We claim that $Tf = \int_\Omega f(\omega)\, d\nu(\omega)$. First, suppose that $Tf \geq 0$. Then, for all $x < 0$, $\{\omega : f(\omega) \geq x\} \notin \mathcal{E}$. In particular, this implies that for all $x < 0$, $\nu(\{\omega : f(\omega) \geq x\}) = 1$. Hence $\int_{-\infty}^{0} [\nu(\{\omega : f(\omega) \geq x\}) - 1]\, dx = \int_{-\infty}^{0} 0 dx = 0$. Moreover, for all $x > Tf$, $\{\omega : f(\omega) \geq x\} \in \mathcal{E}$, again by definition. Finally, for all $x < Tf$, $\{\omega : f(\omega) \geq x\} \notin \mathcal{E}$. Therefore, $\int_0^\infty v(\{\omega : f(\omega) \geq x\})\, dx = \int_0^{Tf} 1 dx = Tf$. The case for which $Tf < 0$ is proved similarly.

For the converse direction, observe that if ν is a normalized capacity taking values in $\{0, 1\}$, then $\int_\Omega f(\omega)\, d\nu(\omega) = \inf\{x : \nu(\{\omega : f(\omega) \geq x\}) = 0\}$. Therefore, we define $\mathcal{E} \equiv \{E \in \Sigma : \nu(E) = 0\}$ and we are done by Theorem 1. ■

The preceding corollary implies, in particular, that any ordinally covariant and monotonic functional is comonotonically additive. This means that for any $f, g \in \mathcal{F}$ which are **comonotonic** (i.e. for all $\omega, \omega' \in \Omega$, $(f(\omega) - f(\omega'))(g(\omega) - g(\omega')) \geq 0$), $T(f + g) = Tf + Tg$. See, for example, Schmeidler [16].

For more on quantile functions, see Denneberg [5].

The following are a few examples of ordinally covariant and monotonic functionals. They are used to illustrate the theorem, and also to give examples of "generalized quantiles."

Example 1: Fix any state $\omega^* \in \Omega$ and define $Tf \equiv f(\omega^*)$. Then it is clear that T is both ordinally covariant and monotonic; so there must be a representation in the sense of Theorem 1. The appropriate monotonic family of sets is given by $\mathcal{E} \equiv \{E \in \Sigma : \omega^* \notin E\}$.

Example 2: Another example is that of the supremum, whereby $Tf \equiv \sup_\omega f(\omega)$. This functional results when $\mathcal{E} \equiv \{\varnothing\}$. The infimum, $Tf \equiv \inf_\omega f(\omega)$, results when $\mathcal{E} \equiv \Sigma \backslash \varnothing$. The essential supremum on a probability space (X, Σ, p) results when $\mathcal{E} \equiv \{E \in \Sigma : p(E) = 0\}$. The essential infimum results when $\mathcal{E} \equiv \{E \in \Sigma : p(E) < 1\}$.

Example 3: More generally, for any fixed $F \in \Sigma$, $Tf \equiv \sup_{\omega \in F} f(\omega)$ results from the collection of subsets of the complement of F; so that $\mathcal{E} \equiv \{E \in \Sigma : E \subset \Omega \backslash F\}$. Further, for any fixed $F \in \Sigma$, $Tf \equiv \inf_{\omega \in F} f(\omega)$ results from the collection of subsets that do not contain F, so that $\mathcal{E} \equiv \{E \in \Sigma : (\Omega \backslash E) \cap F \neq \varnothing\}$.

Example 4: Suppose that $\Omega \equiv \mathbb{N}$ and that $\Sigma \equiv 2^{\mathbb{N}}$. Define $Tf \equiv \liminf_{n \to \infty} f(n)$. Then it is clear again that T satisfies each of our axioms. Therefore, it must have an associated collection of sets $\mathcal{E}$. This collection of sets consists of all sets whose complements have infinite cardinality. Thus, $\mathcal{E} \equiv \{E \in \Sigma : |\Omega \backslash E| = +\infty\}$. Also, the functional $Tf \equiv \limsup_{n \to \infty} f(n)$ satisfies each of our axioms. The collection of sets corresponding to this functional consists of all finite sets. Thus, $\mathcal{E} \equiv \{E \in \Sigma : |E| < +\infty\}$. In particular, both the lim inf and lim sup can be written as expectations according to some capacity. The former observation was first made by Marinacci [10], p. 1012.

Example 5: If $\{f_\lambda\}_{\lambda\in\Lambda}$ is a family of ordinally covariant, monotonic functionals with corresponding collections of sets $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda}$, then $\bigwedge_{\lambda\in\Lambda} f_\lambda$ (the pointwise infimum) is as well, with corresponding collection $\bigcup_{\lambda\in\Lambda}\mathcal{E}_\lambda$. The functional $\bigvee_{\lambda\in\Lambda} f_\lambda$, the pointwise supremum, results from the collection $\bigcap_{\lambda\in\Lambda}\mathcal{E}_\lambda$.

The preceding examples give us another method of characterizing the ordinally covariant and monotonic functionals. Say a collection of sets $\mathcal{E}$ is **generated by $\mathbf{F}\in\boldsymbol{\Sigma}$** if $\mathcal{E}\equiv\{E\in\Sigma : E\subset F\}$. We will write the collection of sets generated by F as $\mathcal{E}_F$. By example 3, if T satisfies our axioms with corresponding collection of sets $\mathcal{E}_F$, then $Tf=\sup_{\omega\in\Omega\setminus F} f(\omega)$. Moreover, it is obvious that any ideal $\mathcal{E}$ is the union of collections of sets generated by the elements of $\mathcal{E}$. Thus, $\mathcal{E}\equiv\bigcup_{F\in\mathcal{E}}\mathcal{E}_F$. Example 5 tells us that any T must therefore be the pointwise infimum over a collection of supremum functionals.

Corollary 3: A functional T satisfies ordinal covariance and monotonicity if and only if there exists a nonempty family $\{E_\lambda\}_{\lambda\in\Lambda}\subset\Sigma$ such that for all $\lambda\in\Lambda$, $E_\lambda\neq\varnothing$, for which

$$Tf\equiv\inf_{\lambda\in\Lambda}\left\{\sup_{\omega\in E_\lambda} f(\omega)\right\}.$$

Corollary 3 gives us a "lim sup" style representation for any ordinally covariant and monotonic functional (including the lim inf on the set of sequences). The lim sup results when $\{E_\lambda\}$ is the family of all cofinite sets; the lim inf results when $\{E_\lambda\}$ is the family of all infinite sets. Moreover, it tells us in a sense that the supremum and infimum functionals are the most "basic" of all functionals satisfying the two properties. Every other functional satisfying them is a composition of these two.

Corollary 4 can be similarly obtained:

Corollary 4: A functional T satisfies ordinal covariance and monotonicity if and only if there exists a nonempty family $\{E_\gamma\}_{\gamma\in\Gamma}\subset\Sigma$ such that for all $\gamma\in\Gamma$, $E_\gamma\neq\varnothing$, for which

$$Tf\equiv\sup_{\gamma\in\Gamma}\left\{\inf_{\omega\in E_\gamma} f(\omega)\right\}.$$

One more obvious remark: Example 5 establishes that the set of functionals satisfying the two axioms forms a complete lattice under the pointwise ordering of functionals. Thus, $T\leq T'$ if and only if for all $f\in\mathcal{F}$, $Tf\leq T'f$. It is obvious that $T\leq T'$ if and only if the collection of sets $\mathcal{E}$ corresponding to T contains the collection of sets $\mathcal{E}'$ corresponding to T'.

3 Quantiles with an exogenous probability

In this section, we suppose that a probability measure p is exogenously specified on (Ω, Σ). The probability measure p together with a function f induces a probability distribution over the reals. When such data is given, it is natural to ask when our functional is monotonic with respect to the stochastic dominance relation on the induced distributions.

Let $\Delta(\Omega, \Sigma)$ denote the set of countably additive probability distributions on (Ω, Σ). For all $f \in \mathcal{F}$ and $p \in \Delta(\Omega, \Sigma)$, define $\pi_{p,f} \in \Delta(\mathbb{R}, \mathcal{B})$ by $\pi_{p,f}(E) \equiv p(\{\omega : \omega \in f^{-1}(E)\})$. The relation $\geq_{FOSD}$ refers to first order stochastic dominance on $\Delta(\mathbb{R}, \mathcal{B})$.

***p*-Monotonicity:** For all $f, g \in \mathcal{F}$, if $\pi_{p,f} \geq_{FOSD} \pi_{p,g}$, then $Tf \geq Tg$.

Say a functional T is a **lower quantile with respect to p** if there exists $\alpha \in [0, 1)$ such that

$$Tf = \inf\{x : p(\{\omega : f(\omega) \geq x\}) \leq \alpha\}.$$

Say it is an **upper quantile with respect to p** if there exists $\alpha \in (0, 1]$ such that

$$Tf = \sup\{x : p(\{\omega : f(\omega) \geq x\}) \geq \alpha\}.$$

We will say that T is a **quantile with respect to p** if it is either a lower or upper quantile with respect to p.

Theorem 2: T satisfies ordinal covariance and p-monotonicity if and only if it is a quantile with respect to p.

Proof. That quantiles with respect to p are p-monotonic is well-known. Therefore, we will prove the converse statement. Let T be any functional satisfying the axioms. Clearly, p-monotonicity implies monotonicity. There exists a representation in the sense of Theorem 1. Recall the definition of $\mathcal{E}$ from Theorem 1. In this environment, if $E \in \mathcal{E}$, then if $p(F) \leq p(E)$, $\pi_{p,1_E} \geq_{FOSD} \pi_{p,1_F}$, so that $F \in \mathcal{E}$. Thus, $\mathcal{E}$ must be of one of the following two forms:

$$\mathcal{E} \equiv \{E \in \Sigma : p(E) \leq \alpha\} \text{ for some } \alpha$$

or

$$\mathcal{E} \equiv \{E \in \Sigma : p(E) < \alpha\} \text{ for some } \alpha.$$

In the first case, T is a lower quantile. In the second case,

$$\begin{aligned} Tf &= \inf\{x : p(\{\omega : f(\omega) \geq x\}) < \alpha\} \\ &= \sup\{x : p(\{\omega : f(\omega) \geq x\}) \geq \alpha\}, \end{aligned}$$

so that T is an upper quantile. ∎

4 Quantiles with an endogenous probability

Theorem 1 shows that every ordinally covariant and monotonic functional can be represented as a quantile with respect to some capacity. The question arises as to whether this is equivalent to being represented by a quantile with respect to some probability. Unfortunately, the next example demonstrates that this is not the case.

Example 6: Let $\Omega = \{1, 2, 3, 4\}$, and $\Sigma = 2^{\Omega}$. Suppose that T is ordinally covariant and monotonic. Let its associated ideal be $\mathcal{E} = \{\{1, 2\}, \{3, 4\}, \{1\}, \{2\}, \{3\}, \{4\}, \varnothing\}$. We claim that there does not exist a probability measure p and an $\alpha \in [0, 1]$ such that $\mathcal{E} = \{E \in \Sigma : p(E) \leq \alpha\}$. To see why, suppose that there do exist such a p and α. Then as $p(\{1, 2\}) + p(\{3, 4\}) = 1$, it must be the case that either $p(\{1, 2\}) \geq 1/2$ or $p(\{3, 4\}) \geq 1/2$. Either way, this implies that $\alpha \geq 1/2$. But similarly, it must be the case that either $p(\{1, 3\}) \leq 1/2$ or $p(\{2, 4\}) \leq 1/2$. But neither of these sets are included in $\mathcal{E}$, which is a contradiction.

We will say that (in a finite measurable space $(\Omega, 2^{\Omega})$) a functional T is a **probabilistic quantile** if there exists a probability measure p on $(\Omega, 2^{\Omega})$ and a real number $\alpha \in [0, 1)$ such that

$$Tf = \inf\{x : p(\{\omega : f(\omega) \geq x\}) \leq \alpha\}.$$

The next axiom states that it is impossible to use the functional to induce someone to choose a sequence of zero-one bets that do at least as poorly as another sequence in the aggregate.

Betting consistency: Let $\{A_1, ..., A_n\} \subset 2^{\Omega}$ and $\{B_1, ..., B_n\} \subset 2^{\Omega}$ such that $\sum 1_{A_i} \geq \sum 1_{B_i}$. Then there exists $i \in \{1, ..., n\}$ such that $T1_{A_i} \geq T1_{B_i}$.

One might think that the motivation for this axiom can also be used to motivate the following:

Risk consistency: Let $\{f_1, ..., f_n\} \subset \mathcal{F}$ and $\{g_1, ..., g_n\} \subset \mathcal{F}$ such that $\sum f_i \geq \sum g_i$. Then there exists $i \in \{1, ..., n\}$ such that $Tf_i \geq Tg_i$.

Risk consistency states that there does not exist a sequence of comparisons such that for all i, g_i is recommended over f_i, but in aggregate does at least as poorly as the f sequence.

The probabilistic quantiles violate risk consistency.

Example 7: Let $\Omega = \{1, 2, 3\}$ and $\Sigma = 2^{\Omega}$. Let T be the median functional. For $i = 1, 2, 3$, let $f_i(\omega) = 10$ if $\omega = i$ and $f_i(\omega) = -1$ otherwise. For $i = 1, 2, 3$, let g_i be the constant function which is everywhere equal to 0. Then $\sum f_i$ is the constant function which is everywhere equal to 8, so that $\sum f_i > \sum g_i$. However, for each f_i, $Tf_i = -1 < 0 = Tg_i$. Thus risk consistency is violated.

Theorem 3: A functional T satisfies ordinal covariance, monotonicity, and betting consistency if and only if it is a probabilistic quantile.

Proof. To see that the probabilistic quantiles satisfy betting consistency, let T be a probabilistic quantile. There exist corresponding p and α. Let $\{A_1, ..., A_n\} \subset 2^\Omega$ and $\{B_1, ..., B_n\} \subset 2^\Omega$ for which $\sum 1_{A_i} \geq \sum 1_{B_i}$. Suppose, by means of contradiction, that for all $i \in \{1, ..., n\}$, $T1_{B_i} > T1_{A_i}$. This is only possible if for all $i \in \{1, ..., n\}$, $T1_{B_i} = 1$ and $T1_{A_i} = 0$. Hence, for all $i \in \{1, ..., n\}$, $p(B_i) > \alpha$ and $p(A_i) \leq \alpha$. As $\sum 1_{A_i} \geq \sum 1_{B_i}$, $\int_\Omega \sum 1_{A_i}(\omega)\, d\mu(\omega) \geq \int_\Omega \sum 1_{B_i}(\omega)\, d\mu(\omega)$. But $\int_\Omega \sum 1_{B_i}(\omega)\, d\mu(\omega) > n\alpha$ and $n\alpha \geq \int_\Omega \sum 1_{A_i}(\omega)\, d\mu(\omega)$, a contradiction.

Conversely, let T be any functional satisfying the axioms. Let $\mathcal{E}$ be the ideal from the representation in Theorem 1. We want to show that there exists a probability measure p and a number $\alpha \in [0, 1]$ such that $\mathcal{E} = \{E \in \Sigma : p(E) \leq \alpha\}$. The existence of such a probability measure is equivalent to the existence of a $[\alpha, p]$ solution of the following linear program: for all $E \in \mathcal{E}$, $(1, -1_E) \cdot [\alpha, p] \geq 0$; for all $E \notin \mathcal{E}$, $(-1, 1_E) \cdot [\alpha, p] > 0$; for all ω, $\left(0, 1_{\{\omega\}}\right) \cdot [\alpha, p] \geq 0$, and $(0, 1_\Omega) \cdot [\alpha, p] > 0$. If this system of linear inequalities does not have a solution, then there must exist (by the Theorem of the Alternative) nonnegative integers for each of the preceding constraints, so that $\sum_{E \in \mathcal{E}} n_E (1, -1_E) + \sum_{E \notin \mathcal{E}} n_E (-1, 1_E) + \sum_\omega n_\omega \left(0, 1_{\{\omega\}}\right) + m(0, 1_S) = 0$. Furthermore, one of the integers associated with one of the strict inequalities must be positive.

Therefore, we may also conclude that at least one of the n_E corresponding to an $E \in \mathcal{E}$ must be positive. Moreover, in order to equal zero, $\sum_{E \in \mathcal{E}} n_E = \sum_{E \notin \mathcal{E}} n_E$. Define $n \equiv \sum_{E \in \mathcal{E}} n_E$, and list out all of the sets $E \in \mathcal{E}$ a total of n_E times each to form a sequence $\{A_1, ..., A_n\} \subset \mathcal{E}$. List out all of the sets $E \notin \mathcal{E}$ a total of n_E times each to form a sequence $\{B_1, ..., B_n\} \subset 2^\Omega \backslash \mathcal{E}$. By definition of $\mathcal{E}$, for all $i = 1, ..., n$, $T1_{A_i} < T1_{B_i}$. As the constraints sum to zero, and the weights n_ω and m are nonnegative, $\sum_i 1_{A_i} \geq \sum_i 1_{B_i}$. This contradicts betting consistency.

Therefore, there exists a probability measure p and a number α such that $\mathcal{E} = \{E \in \Sigma : p(E) \leq \alpha\}$. ■

5 Medians

Think of a finite and odd set of states. In such an environment, the median functional enjoys the following property:

Negation covariance: For all $f \in \mathcal{F}$, $T(-f) = -Tf$.

This axiom states that the behavior of a functional is symmetric, in that it treats losses the same as it treats gains.

Say a collection of sets $\mathcal{E} \subset \Sigma$ is **strong** if $E \in \mathcal{E}$ if and only if $\Omega \backslash E \notin \mathcal{E}$. If a collection of sets is monotonic and strong, it might be interpreted as a collection of sets that have "weight" weakly less than 1/2. In this sense, the functions described in the next theorem form a class of "generalized medians."

Theorem 4: A functional T satisfies ordinal covariance, monotonicity, and negation covariance if and only if there exists a strong ideal of sets $\mathcal{E} \subset \Sigma$ such that $\varnothing \in \mathcal{E}$ and $\Omega \notin \mathcal{E}$ for which

$$Tf = \inf \{x : \{\omega : f(x) \geq \omega\} \in \mathcal{E}\}.$$

Proof. We have already defined the ideal $\mathcal{E}$ in previous proofs. We here show that $\mathcal{E}$ is strong. So, suppose that $E \in \mathcal{E}$. Then by definition, $T1_E = 0$. By negation covariance, $T(-1_E) = 0$. Let $\varphi(x) = x + 1$. Then $\varphi \circ (-1_E) = 1_{\Omega \backslash E}$. Moreover, $\varphi(T(-1_E)) = 1$. Therefore, $T1_{\Omega \backslash E} = 1$, so that $\Omega \backslash E \notin \mathcal{E}$. Conversely, suppose that $\Omega \backslash E \notin \mathcal{E}$. Then $T1_{\Omega \backslash E} = 1$. By methods similar to the preceding, we may obtain that $T(-1_E) = 0$, and by negation covariance, $T1_E = 0$. Hence $E \in \mathcal{E}$.

To see that any such function satisfies negation covariance, let $f \in \mathcal{F}$. Then the following string of equalities is true:

$$\begin{aligned}
T(-f) &= \inf \{x : \{\omega : -f(\omega) \geq x\} \in \mathcal{E}\} \\
&= \inf \{x : \{\omega : f(\omega) \leq -x\} \in \mathcal{E}\} \\
&= -\sup \{-x : \{\omega : f(\omega) \leq -x\} \in \mathcal{E}\} \\
&= -\sup \{x : \{\omega : f(\omega) \leq x\} \in \mathcal{E}\} \\
&= -\sup \{x : \{\omega : f(\omega) > x\} \notin \mathcal{E}\} \\
&= -\inf \{x : \{\omega : f(\omega) \geq x\} \in \mathcal{E}\} = -Tf.
\end{aligned}$$

Here, the fifth equality follows from strongness. ■

Example 8: It is easy to see from our previous example that the function $Tf \equiv f(\omega^*)$ is negation covariant.

Example 9: In an environment in which $\Omega \equiv \{\omega_1, \omega_2, \omega_3\}$, it is easy to use our theorem to check that there are exactly four functionals satisfying ordinal covariance, monotonicity, and negation covariance. These functionals are the median, and for each i, the function $Tf \equiv f(\omega_i)$.

Example 10: Recall that an ultrafilter is a collection of sets $\mathcal{U}$ that satisfies $i)$ $\varnothing \notin \mathcal{U}$, $ii)$ $E \in \mathcal{U}$ and $E \subset F \implies F \in \mathcal{U}$, $iii)$ $E, F \in \mathcal{U} \Rightarrow E \cap F \in \mathcal{U}$, and $iv)$ $E \in \mathcal{U} \Leftrightarrow \Omega \backslash E \notin \mathcal{U}$. An ultrafilter $\mathcal{U}$ is free if $\bigcap_{E \in \mathcal{U}} E = \varnothing$. An especially interesting example occurs when $\Omega \equiv \mathbb{N}$, $\Sigma \equiv 2^{\mathbb{N}}$, and $\mathcal{E}$ is the set of complements of elements of some free ultrafilter $\mathcal{U}$; i.e., $\mathcal{E} \equiv \{E \in \Sigma : \Omega \backslash E \in \mathcal{U}\}$.[2] Then it is

[2] Existence of free ultrafilters on infinite sets is implied by the Axiom of Choice (Aliprantis and Border, 2.16 [1]).

obvious that $\mathcal{E}$ is both monotonic and strong. This functional has the interesting feature that for all f,g which coincide except on a finite number of points, $Tf = Tg$.

It is interesting to ask whether on finite state spaces, every "generalized" median is a probabilistic quantile. We do not know the answer to this question.

6 Restricted classes of transformations

Ordinal covariance is very strong. It is therefore of interest to understand what happens when we restrict our class of permitted transformations. Suppose we require covariance only under concave transformations.

Concave covariance: For all $f \in \mathcal{F}$ and all strictly increasing and concave $\varphi : \mathbb{R} \to \mathbb{R}$, $T(\varphi \circ f) = \varphi(Tf)$.

Then our first lemma fails under the weakened condition of concave covariance, and all that can be said in general of concave covariant functionals is that $Tf \in [\inf f, \sup f]$. However, together with monotonicity, all of our results continue to hold. This is a consequence of the following simple proposition.

Proposition: Suppose that T satisfies concave covariance. Then for all $E \in \Sigma$, $T1_E \in \{0,1\}$.

Proof. Let $E \in \Sigma$. Let φ be defined as

$$\varphi(x) \equiv \left\{ \begin{array}{c} \frac{3x}{2} \text{ for } x < 1/2 \\ \frac{x+1}{2} \text{ for } x \geq 1/2 \end{array} \right\}.$$

It is clear that φ is both strictly increasing and concave. Moreover, $\varphi(0) = 0$ and $\varphi(1) = 1$. Hence, $\varphi \circ 1_E = 1_E$. By concave covariance, conclude $\varphi(T1_E) = T(\varphi \circ 1_E) = T1_E$. But the only fixed points of φ are 0 and 1. Hence, $T1_E \in \{0,1\}$. ■

The proof of this Proposition requires nothing stronger than invariance under the function φ described there. Perhaps the weakest family containing this function is the set of functions that are piecewise linear in at most two pieces, and concave. The full force of ordinal covariance plays no role in the remainder of any of our proofs.

7 Ranking utility streams

One application of the results in this paper lie, as already mentioned, in the social choice/decision theory literature. Suppose that $\Omega = N$, where N is some set of agents. An element $f \in \mathcal{F}$ specifies a utility value for each agent in society; the interpretation being that this is the utility of choosing some alternative. A functional now returns as output a social utility of a given alternative. Monotonicity in this environment is a weak Pareto condition. Ordinal covariance is a strong version of the requirement that units of utility are meaningless except for their ordinal content.

Suppose that N is finite and consider the following axiom:

Anonymity: For all $f \in \mathcal{F}$ and all bijections from $\sigma : N \to N$, $T(f \circ \sigma) = Tf$.

A simple corollary of our main result is that a functional is ordinally covariant, monotonic, and anonymous if and only if it is a rank-order dictatorship. This means there exists some k such that Tf is the k^{th} order statistic of f. This type of result is already well-known, see [3, 7, 8, 13, 14]. Corollaries 3 and 4 above can be used to establish a non-anonymous version of these results, whereby such an aggregation rule is monotonic with respect to the Pareto relation and ordinally covariant if and only if it is a "maximinimax" rule. Thus, there must exist a family of coalitions $\mathcal{N} \subset 2^N$ such that

$$Tf = \min_{E \in \mathcal{N}} \left\{ \max_{i \in E} f(i) \right\}.$$

If the set $\mathcal{N}$ contains all singleton coalitions, then this is a Rawlsian-style maximin aggregation rule (see Rawls [12]). By setting $\mathcal{N} = \{N\}$, we obtain a maximax rule, and by setting it equal to coalitions of equal size we obtain the rank order dictatorships.

Our Theorem 1 allows us to generalize these types of results to the theory of ranking infinite utility streams. Traditionally, the theory of ranking infinite utility streams has taken as primitive a countably infinite set of agents, and has required conditions on functionals mapping from utility streams to the reals. Classical results in this theory tell us that such a functional cannot be continuous (in the sup-norm topology), anonymous (in almost any sense), and compatible with the strong Pareto relation (for example, see Diamond [6]). For a recent impossibility result along these lines without any continuity assumption, see Basu and Mitra [4], and the references therein. The impossibility obtains because we require the existence of a functional, as opposed to an ordinal ranking of utility streams. In a sense, the real numbers are not a large enough set to embed all sequences in a strictly monotonic way. If *numerical* measurement is not desired, then possibility results obtain (see Svensson [18]).

We maintain the requirement that social preference be representable by a numerical functional, but weaken the strong Pareto principle. Our version of the Pareto principle will be our monotonicity condition.

We introduce a consistency condition that we believe has not yet been stated in the literature. Consider some finite collection of utility streams. Suppose they are all ranked as equivalent according to the functional T. Suppose we form a new utility stream which is composed of the original, finite collection of utility streams, in the sense that every element in one of the original utility streams is mapped to some element in the new utility stream, and conversely. We require that the new utility stream so constructed is ranked as equivalent to all of the original utility streams. This condition encompasses the strongest notions of anonymity existing in the literature. However, it is a very natural condition if it is believed that the sequencing of when utility values are faced is completely irrelevant. Conditions related to this are pervasive in the social choice literature on variable populations and in fair allocation theory, where the population is typically finite (for example, see Thomson [19] or Young [21]).

Here, we suppose that $\Omega = \mathbb{N}$ and $\Sigma = 2^{\mathbb{N}}$.

Consistency: Let $\{f_i\}_{i=1}^{K} \subset \mathcal{F}$, where $K < +\infty$. Suppose that for all $i, j \in \{1, ..., K\}$, $Tf_i = Tf_j$. Let $\sigma : \mathbb{N} \to \mathbb{N} \times K$ be a bijection. Define $f_\sigma(\omega) \equiv f_{\sigma_2(\omega)}(\sigma_1(\omega))$. Then $Tf_\sigma = Tf_1$.

The strength of the axiom allows us to prove the following characterization theorem:

Theorem 5: A functional T satisfies ordinal covariance, monotonicity, and consistency if and only if it is either the supremum, the infimum, the limit superior, or the limit inferior.

Proof. It is easy to check that each of the four functionals discussed above satisfies the three axioms. Conversely, suppose that T satisfies the three axioms. Let $\mathcal{E}$ be its corresponding collection of sets as delivered by Theorem 1.

We will establish three facts. First, for all $E, F \in \Sigma$ for which $|E| < +\infty$ and $|F| < +\infty$, $E \in \mathcal{E} \Leftrightarrow F \in \mathcal{E}$. Secondly, for all $E, F \in \Sigma$ for which $|E| = +\infty$ and $|\Omega \backslash E| = +\infty$, and $|F| = +\infty$ and $|\Omega \backslash F| = +\infty$, $E \in \mathcal{E} \Leftrightarrow F \in \mathcal{E}$. Lastly, for all $E, F \in \Sigma$ for which $|\Omega \backslash E| < +\infty$ and $|\Omega \backslash F| < +\infty$, $E \in \mathcal{E} \Leftrightarrow F \in \mathcal{E}$.

To establish the first fact, we show that $T1_{\{1\}} = T1_{\{1,...,K\}}$ for all $K < +\infty$. But this is trivial; fix $f_i = 1_{\{1\}}$ for $i = 1, ..., K$. For all $k \leq K$, let $\sigma(mK + k) = (m + 1, k)$. Then clearly $f_\sigma = 1_{\{1,...,K\}}$. By consistency, $T1_{\{1,...,K\}} = T1_{\{1\}}$. For any finite set $E \in \Sigma$, let σ be a bijection such that $\sigma(\{1, ..., |E|\}) = E$. Then $T1_{\{1,...,|E|\}} = T1_E$. Therefore, the first fact is true.

To establish the second fact, let $E, F \in \Sigma$ which satisfy $|E| = +\infty$ and $|\Omega \backslash E| = +\infty$, and $|F| = +\infty$ and $|\Omega \backslash F| = +\infty$. List the elements of E as $E = \{e_1, e_2, ...\}$ and the elements of F as $F = \{f_1, f_2, ...\}$. List the elements of $\Omega \backslash E$ as $\{e'_1, e'_2, ...\}$ and the elements of $\Omega \backslash F$ as $\{f'_1, f'_2, ...\}$. For all i, let $\sigma(e_i) = f_i$ and let $\sigma(e'_i) = f'_i$. Then $1_E = (1_F)_\sigma$. Hence, by consistency, $T1_E = T1_F$ and the second fact is true.

The last fact follows similarly to the first fact. Therefore, all three facts are true.

If $\mathcal{E}$ contains only the empty set, then it is the supremum functional. If it contains any finite set, but no infinite sets, then by fact 1 it contains all finite sets, and is the limit superior. If it contains an infinite set, but no cofinite set, then it contains all infinite sets whose complements are infinite and all finite sets by fact 2, and therefore it is the limit inferior. Otherwise, if it contains any cofinite set, then it contains all cofinite sets by fact 3 and hence contains all sets but Ω, and is the infimum. ■

Lauwers [9] axiomatizes the infimum rule using axioms very closely related to ours. He works in a completely ordinal framework; however. He imposes the requirements of monotonicity and ordinal covariance just as we do. In fact, his version of monotonicity is slightly stronger. Moreover, he also requires invariance with respect to arbitrary permutations. In addition to this, however, he requires continuity in the sup-norm topology and a condition that he calls the "repetition-approximation" principle. Very roughly, the repetition approximation principle requires that the value of a functional applied to a sequence can be approximated arbitrarily closely by the "value" of finite subsequences of the sequence. He rules out the supremum functional by the use of a mild equity axiom.

8 Risk measures and the Value at Risk

In this section, we discuss the theory of risk measures. In this environment, Ω is a set of states of the world. The true state is unknown. Elements of $\mathcal{F}$ are risks, or state-contingent monetary outcomes. A risk measure is a functional that ranks risks. Usually, a risk measure attributes a lower value to a risk that is judged as better, and we will follow this convention. Thus, we will take a risk measure to be a functional $R : \mathcal{F} \to \mathbb{R}$; whereby $Rf < Rg$ is the statement that risk f is recommended over risk g for an arbitrary decision maker.

Anti-monotonicity: For all $f, g \in \mathcal{F}$, $f \leq g$ implies $Tf \geq Tg$.

One particular risk measure that has received much attention is the Value at Risk (VaR). Given an underlying probability space, the VaR functions as a quantile. For some fixed α, it recommends those risks that have a higher α-quantile. To our knowledge, there has been no justification for the use the VaR, other than its inherent simplicity. Indeed; much of the literature has taken to criticizing the VaR. See especially Artzner et al [2] for a well-known alternative to the VaR.

The following definition also comes from Artzner et al. [2]. One definition for an α-VaR with a given probability measure p on(Ω, Σ) is

$$VaR_{\alpha}(f) = -\inf \{x : p(\{\omega : f(\omega) \leq x\}) > \alpha\}.$$

Note that the VaR_{α} is the negation of an α-quantile with an exogenously specified probability. Thus, our axioms can therefore be used to deliver an axiomatic justification for the use of the VaR. Of course, this justification is only as useful as the axioms are meaningful.

Artzner et al. refer to a model with an exogenously specified probability as "model dependent." All known versions of the VaR are model dependent. In the following analysis, we will discuss natural analogues of the VaR which are (in the Artzner et al. terminology) "model independent." Thus, they are risk measures which have no exogenously specified probability measure.

Artzner et al. define a risk measure to be **coherent** if it satisfies the following three axioms (in addition to antimonotonicity):

Translation covariance: For all $f \in \mathcal{F}$ and all $b \in \mathbb{R}$, $R(f+b) = Rf - b$.

Positive homogeneity: For all $f \in \mathcal{F}$ and all $\alpha > 0$, $R(\alpha f) = \alpha Rf$.

Subadditivity: For all $f, g \in \mathcal{F}$, $Rf + Rg \geq R(f+g)$.

The first two of these are easily seen to be related to our ordinal covariance requirement. Subadditivity, on the other hand, is a condition that is usually interpreted as meaning that "hedging" two risks results in a less risky scenario.

A risk measure is defined by Artzner et al. as being a function which specifies for every risk, the minimal amount of money needed (in every state) to make the risk "acceptable." One way of reading this is that a risk measure gives a number that should be viewed as equivalent certain loss incurred from accepting the risk.

We will reframe our ordinal covariance axiom to state that risk measures are covariant with respect to arbitrary increasing transformations of monetary payoffs. There are several justifications for such an axiom. One obvious (and practical) justification is the presence of income taxes. One may treat the payoffs a risk gives as *after tax* income, but in so doing, we ignore the very real possibility that the same risk may give very different after tax incomes to different agents. Again, we desire our risk measure to be applicable to *any* decision maker. If a risk has the potential to put an agent in another tax bracket, the real payoffs to this agent are altered in a nonlinear fashion. In such a situation, if a risk measure is not covariant with respect to arbitrary monotonic transformations, the measure may very well make an incorrect recommendation for this agent in terms of after tax payoffs. To rule this possibility out, we require that the risk measure is invariant under increasing transformations. Thus, whether risks are measured in before or after-tax incomes are irrelevant.

A more conceptual justification is as follows. Classical decision theories, including those of von Neumann and Morgenstern [20] and Savage [15], deal with a decision maker who may or may not be facing an exogenous probability measure over a state space. However, both of these theories deliver a (typically monotonic) index over monetary outcomes that can be used to evaluate the value of a risk in an expected utility fashion. This index can be used to construct agent-specific notions of loss. For an agent with corresponding index u and some probability measure p over the state space (Ω, Σ), the agent evaluates risks according to the functional $U(f) \equiv E_p[u \circ f]$. If we want a risk measure to be a type of aggregate, or a recommendation for an arbitrary agent, it follows that the risk measure of f should be the minimal amount of utility given to an agent with utility function u that makes this agent able to accept the risk. Formally, the risk f is treated as losing Rf for sure. In other words, the monetary 'loss' associated with this risk is $-Rf$. The utility of the loss is thus $u(-Rf)$. In other words, the amount of utility that an agent with utility function u should be compensated by in order to accept the risk is $-u(-Rf)$. But this is precisely how we defined $R(u \circ f)$. Thus, we should require that $R(u \circ f) = -u(-Rf)$.

Ordinal risk covariance: For all $f \in \mathcal{F}$ and all $\varphi : \mathbb{R} \to \mathbb{R}$ strictly increasing and continuous, $R(\varphi \circ f) = -\varphi(-Rf)$.

Of course, one may require the preceding axiom to hold only with respect to all increasing, concave functions. This does not change any of the results.

A common criticism of the VaR is that it violates the subadditivity property (which is usually interpreted as a type of "hedging" property). However; the VaR always presupposes an exogenously given probability measure. We will show how to derive subadditive analogues of the VaR if an exogenous probability measure is not given.

In order to provide our proposition, we first need the following definition. A **filter** of Σ is a collection $\mathcal{G}$ of subsets such that $i)$ $\Omega \in \mathcal{G}$, $\varnothing \notin \mathcal{G}$, $ii)$ $A \in \mathcal{G}$ and $A \subset B \Longrightarrow B \in \mathcal{G}$, and $iii)$ $A, B \in \mathcal{G} \Longrightarrow A \cap B \in \mathcal{G}$. Note that for any filter $\mathcal{G}$, the set function $1_{\mathcal{G}} : \Sigma \to \mathbb{R}$ defined by $1_{\mathcal{G}}(E) = 1 \Leftrightarrow E \in \mathcal{G}$ is a normalized $\{0,1\}$ capacity.

Proposition: A risk measure R satisfies antimonotonicity, ordinal risk covariance, and subadditivity if and only if there exists some filter $\mathcal{G}$ of Σ such that $Rf = -\int_{\Omega} f(\omega)\, d1_{\mathcal{G}}(\omega)$.

Proof. Suppose that R satisfies the properties. Define $T = -R$. Then T is easily seen to be ordinally covariant and monotonic. Moreover, T is now superadditive as opposed to subadditive. By Corollary 2, we know that T may be represented as the integral with respect to some normalized $\{0,1\}$-capacity, say ν. We will show that $\nu = 1_{\mathcal{G}}$ for some filter of subsets.

By Proposition 3 of Schmeidler [16], T is superadditive if and only if ν is convex, so that for all $E, F \in \Sigma$, $\nu(E \cup F) + \nu(E \cap F) \geq \nu(E) + \nu(F)$. We will show that any convex, $\{0,1\}$-capacity is the indicator function of some filter $\mathcal{G}$. Thus, define $\mathcal{G} \equiv \{E \in \Sigma : \nu(E) = 1\}$. Clearly, $\mathcal{G}$ satisfies conditions $i)$ and $ii)$ listed in the definition of filter. To see that it satisfies condition $iii)$, suppose that $A, B \in \mathcal{G}$, but by means of contradiction that $A \cap B \notin \mathcal{G}$. Then $2 = \nu(A) + \nu(B) \leq \nu(A \cap B) + \nu(A \cup B) = 1$, a contradiction. Therefore, $\mathcal{G}$ is a filter.

Now suppose that $\mathcal{G}$ is a filter. We will show that $1_{\mathcal{G}}$ is convex. Let $A, B \in \Sigma$. We check three cases; the fourth is symmetric.

Suppose that $A, B \in \mathcal{G}$. Then $1_{\mathcal{G}}(A) + 1_{\mathcal{G}}(B) = 2$. But by condition $ii)$, $A \cup B \in \mathcal{G}$ and by condition $iii)$, $A \cap B \in \mathcal{G}$ so that $1_{\mathcal{G}}(A) + 1_{\mathcal{G}}(B) = 2 = 1_{\mathcal{G}}(A \cup B) + 1_{\mathcal{G}}(A \cap B)$.

Suppose that $A \in \mathcal{G}$ and $B \notin \mathcal{G}$. Then by condition $ii)$, $A \cup B \in \mathcal{G}$. Therefore, $1_{\mathcal{G}}(A) + 1_{\mathcal{G}}(B) = 1 \leq 1_{\mathcal{G}}(A \cup B) + 1_{\mathcal{G}}(A \cap B)$.

Suppose that $A \notin \mathcal{G}$ and $B \notin \mathcal{G}$. Then trivially, $1_{\mathcal{G}}(A) + 1_{\mathcal{G}}(B) = 0 \leq 1_{\mathcal{G}}(A \cup B) + 1_{\mathcal{G}}(A \cap B)$. ■

In finite sets Ω, all filters are **principal**, thus, there exists some $F \in \Sigma$ such that $\mathcal{G} = \{E \in \Sigma : F \subset E\}$. In this case, it is trivial to check that for such an environment, $Rf \equiv -\inf_{\omega \in F} f(\omega)$. So, for finite state spaces, the infimum functionals are the only ones that are anti-monotonic, ordinally risk covariant, and subadditive. However, for infinite sets, there are many more filters. Some filters are **free**, so that $\bigcap\limits_{E \in \mathcal{G}} E = \varnothing$. An example of such a filter is the filter of cofinite sets. If $\Omega = \mathbb{N}$, this filter gives us the negative of the $\liminf$ functional. Thus, in a sense, all anti-monotonic and ordinally risk covariant functionals that are also subadditive are generalizations of the concept of infimum over some set.

Monotonicity with respect to stochastic dominance is also an important criterion of a risk measure when an exogenous probability measure is given. Although there is little hope of obtaining a general characterization of the coherent risk measures that are monotonic with respect to stochastic dominance, at least one result exists in the literature on this subject.

A general theorem of Marinacci [11] implies that, under reasonable conditions, the only coherent risk measure that is monotonic with respect to stochastic dominance for an exogenously given measure p on (Ω, Σ) is the negative expected value functional.

Theorem (Marinacci): Let T be a coherent risk measure. Suppose that p is a nonatomic probability measure on (Ω, Σ) with respect to which T is strictly inversely monotonic with respect to stochastic dominance, and that there exists some $E \in \Sigma$ such that $T1_E \in (0, -1)$ and $T1_E + T1_{\Omega \setminus E} = -1$. Then $Tf = -\int_{\Omega} f(\omega)\, dp(\omega)$.

The condition required in the assumption of Marinacci's theorem is arguably weak. It merely requires that the state-space (Ω, Σ) is sufficiently "rich," so that the probability measure p is nonatomic, and that there is some event E for which risks that are measurable with respect to that event are evaluated in an expected value fashion.

9 Ordinal measurement

Thus far, we have always presupposed the existence of a functional $T : \mathcal{F} \to \mathbb{R}$. A natural question to ask is what happens when we only have an order structure $\preceq$ on $\mathcal{F}$.

In fact, it is easy to come up with axioms related to the preceding which only apply to the ordinal structure of T. We discuss these now.

Order: $\preceq$ is a weak order.

Ordinal invariance: For all $f, g \in \mathcal{F}$ and all strictly increasing and continuous $\varphi : \mathbb{R} \to \mathbb{R}$, $f \succeq g \Longleftrightarrow \varphi \circ f \succeq \varphi \circ g$.

This, in fact, is the natural interpretation of our ordinal covariance condition for functionals T. One can see that any ordinally covariant T generates an order structure on $\mathcal{F}$ which is ordinally invariant.

Monotonicity: For all $f, g \in \mathcal{F}$ for which $f \geq g$, $f \succeq g$.

These two axioms are not enough; however, to characterize order structures that may be represented by functionals satisfying ordinal covariance and monotonicity. An example should make this clear.

Example 11: Let $\preceq$ on $\mathcal{F}$ rank every pair of functions $f, g \in \mathcal{F}$ as equivalent. Then it is clear that $\preceq$ is ordinally invariant and monotonic.

The preceding example is simple to rule out through a classical type of axiom:

Non-degeneracy: There exist $f, g \in \mathcal{F}$ such that $f \succ g$.

Of course, these axioms are still not enough to characterize those order structures $\preceq$ that can be represented by functionals that are ordinally covariant and monotonic. One needs to rule out simple 'lexicographic' examples.[3] As we do not wish to impose any topological structure on $\mathcal{F}$, we do this through the following axiom. Here, we abuse notation and identify any constant function with the value that constant function takes.

Solvability: For all $f \in \mathcal{F}$, there exists $x \in \mathbb{R}$ such that $f \sim x$.

Any binary relation $\preceq$ satisfying these five axioms on $\mathcal{F}$ can be represented by a T satisfying ordinal covariance and monotonicity.

Theorem 6: A binary relation $\preceq$ satisfies order, ordinal invariance, monotonicity, non-degeneracy, and solvability if and only if there exists an ordinally covariant and monotonic $T : \mathcal{F} \to \mathbb{R}$ such that $f \succeq g \Leftrightarrow Tf \geq Tg$.

[3]One may not wish to rule these types of examples out in many applications. For example, lexicographic versions of minimax rules are an integral part of the theory of social choice. We do not know of any lexicographic notion of risk measure, but such objects are clearly worth investigating. In general, a theory of lexicographic quantiles may be interesting.

Proof. It is clear that any T satisfying ordinal covariance and monotonicity generates an order which is ordinally invariant, monotonic, non-degenerate, and solvable.

To prove the other direction, let $\preceq$ satisfy the five axioms. Since $\preceq$ is non-degenerate, there exist $f, g \in \mathcal{F}$ for which $f \succ g$. By solvability, there exist $x_f, x_g \in \mathbb{R}$ such that $x_f \sim f$ and $x_g \sim g$. As $\preceq$ is a weak order, $x_f \succ x_g$. By monotonicity, $x_f > x_g$. We claim that for all $x, y \in \mathbb{R}$ for which $x > y$, $x \succ y$. To see this, let $\varphi : \mathbb{R} \to \mathbb{R}$ be any strictly monotonic and continuous mapping for which $\varphi(x_f) = x$ and $\varphi(x_g) = y$. (Clearly such a mapping exists; indeed, we may even choose it to be concave). By ordinal invariance, $x \succ y$.

We now define $Tf = \{x \in \mathbb{R} : f \sim x\}$. Clearly, Tf is unique and well-defined. We claim that T is ordinally covariant and monotonic. Monotonicity is trivial; thus, suppose that $f \geq g$. Suppose that $x_f \sim f$ and $x_g \sim g$. By monotonicity of $\preceq$, $f \succeq g$. Therefore, if $x_f \prec x_g$, completeness and transitivity are violated. Hence, $x_f \succeq x_g$, from which we conclude that $x_f \geq x_g$, so that $Tf \geq Tg$.

To see that ordinal covariance is satisfied, let $f \in \mathcal{F}$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be strictly monotonic and continuous. By definition, $Tf \sim f$. By ordinal invariance of $\preceq$, $\varphi \circ Tf \sim \varphi \circ f$, so that $T(\varphi \circ Tf) = T(\varphi \circ f)$. But $T(\varphi \circ Tf) = \varphi(Tf)$. Hence $\varphi(Tf) = T(\varphi \circ f)$. ■

10 Conclusion

One important issue that we have not discussed is the extension of our functionals to unbounded functions. The axioms of monotonicity and ordinal covariance are no longer enough to characterize the generalized quantile type of functions that we axiomatize on the set of bounded functions. There are several issues here; one such issue is whether or not we wish to allow the functional to take infinite values. If we do not allow this, then the supremum and infimum functionals are no longer well-defined. Regardless, we leave this issue to future research (although we believe only mild additions to the axioms discussed here are necessary).

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