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A MODEL OF ELECTIONS WITH SPATIAL AND DISTRIBUTIVE PREFERENCES

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Abstract

This paper introduces a model where elections are games where voters have preferences over a public good (policy platforms) and a private good (transfers). The model produces the standard social choice results such as core convergence and policy separation. Furthermore, by introducing transfers, I am able to make more precise predictions about candidate locations and their dynamics than is possible under the standard spatial model. Another purpose of this paper lies in the creation of favored groups in elections.

Ultimately, it is important to characterize political behavior while considering the different preferences that might exist in the constituents. By incorporating utility for private goods into standard utility assumptions, this model introduces these considerations into the standard spatial model, allowing us to have a richer and more nuanced look into elections.

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1 Introduction

This paper examines candidate policy platforms under spatial competition in a simple voting game. The motivation of this is to understand the spatial model when we introduce new assumptions about the preference of the voters. A standard result in the spatial model shows that an equilibrium typically does not exist. Even when they do, e.g., in $\mathbb{R}^1$ with single peaked preferences, or when the Plott conditions are satisfied, the prediction of a majority winner typically is not very interesting. In a fundamental result of voting theory, Ledyard proved that in equilibrium with voting costs, candidates' policy platforms will converge and no one would vote [6]. Another fundamental result by McKelvey showed that, when the core does not exist, and that if the policy space is continuous, one can get reach any policy platform via some sequence [8]. These results are certainly not a realistic model of actual elections, when we almost never see identical policy platforms and the election cycles are much less common than would be predicted by the Downsian model.

Ideological position, of course, need not the only preferences voters have over candidates. Another prominent modeling choice of elections involves vote buying, or equivalently as a redistribution game. These models view elections as a game where voters' preferences lie over private goods, i.e., transfers, whereas the Downsian model views it as a game over public goods, i.e., ideology. The vote buying approach began with the seminal paper of Baron and Ferejohn [1]. Their model in turn was based on the Rubinstein bargaining model, in which agents bargain over the distribution of a fixed amount of private goods. They model the bargaining process as one where legislators are recognized, make a proposal, and vote. Compared to the Downsian world, the bargaining approach provides precise predictions about equilibrium behavior when the Downsian model would predict chaos. A prominent vote buying models is that of Myerson [10]. In the Myerson model candidates compete for voters by promising to voters a draw from a random distribution that is identical for all voters. Voters votes for the candidate under whom they received the best realization of the random distribution. Lizzeri extends this model to a two period world, where candidates compete in both periods.

Here I combine the two ideas and model elections as a game where voters have preferences over both policy platforms and distributional considerations. Simply put, although there may always be some policy position that will defeat an incumbent, there are a number of advantages that the incumbent that can help in an election. By combining a spatial model and a distributive model, I am able to derive ideological policy separation.

The intuition is straightforward: given the incumbent's position, the challenger wants to maximize the amount the incumbent will have to pay to "buy off" the voters.

The model here is similar to that of Jackson and Moselle [5]. They considered a legislative voting game where legislators have preferences on a distributive dimension as well as an ideological dimension. The legislators play a Baron-Ferejohn bargaining game with a random recognition rule, where the game ends when a proposal receives a majority of the vote. Their paper is motivated by a similar observation, that legislators have preferences over both ideology as well as distribution, and that the distribution aspect adds to the standard Downsian model both realism and predictive power.

The present model differs from Jackson and Moselle in that I assume a much more structured game, namely, a Stackelberg game in which there exists a leader and an entrant, instead of the potentially infinitely repeated bargaining game considered in most bargaining games. Whereas an infinitely repeated bargaining game has a natural interpretation for a legislative game, a sequential model makes more sense in an election context. Namely, the incumbent's policy platform is likely to be quite similar to his location from the previous election. A challenger enters and chooses a policy platform. In the traditional Downsian model, this generally means certain loss to the incumbent, unless the core exists and the incumbent is located there. Here we introduce a distributive dimension. The simple structure of this game also allows this model to be generalized to take account of multi-dimensional ideological policy space, which was not considered in Jackson and Moselle.

Another purpose of this paper lies in the creation of favored groups in elections. This is similarly the motivation of Myerson [10]. Whereas Myerson's motivation is to explain differences in offers to otherwise identical voters, I am interested in the trade-offs that candidates make on the distributive dimension versus the ideological dimension. In the section where I extend the model to one where the challenger has distributive income, I will give conditions under which the incumbent will distribute to various voters.

This model also is similar to Bernhardt and Ingberman [2] in spirit. As Bernhardt and Ingberman noted, the results in the classical Downsian model holds because of the extreme assumptions about the structure of the game in which the agents are playing. If we relax those assumptions, the predictions of the model change accordingly. This model therefore can be seen as a complementary effort from that of Bernhardt and Ingberman. Whereas they adds in uncertainty to the Downsian world, this model adds another dimension on which voters have preferences over.

Wuffle et al. [12] presented a model based on the idea that incumbents have certain advantages that the challengers may not. Specifically, in their model, they assume that the incumbent can move within some area of his proposed ideological position, or equivalently, that voters have fat indifference curves. They proposed "Finagle's Law," which states that "No matter what happens, you can always come out ahead if you just know how to finagle." In the Finagle model, incumbents pick a position, but can subsequently alter their policy position, i.e., "Finagling." The incumbents have limitations on how far they are able to "finagle." With this they were able to find the "Finagling point," defined as a point that minimizes the distance that the candidate must move, or "finagle" in order to beat the challenger. Somewhat surprisingly, under the assumption of Euclidean preferences, the "Finagling Point" for three voters is located at the same point as the optimal incumbent location in the current model.

The Finagling model can be seen as a complementary effort to explain the incumbency advantage observed empirically. Unlike the Finagle model, however, the model presented here makes precise predictions on where the challenger would locate. This allows me to draw some implications about the dynamics of the model. In a three voter case. I am able to show that the optimal incumbent position is an unstable equilibrium. That is, should the incumbent ever lose the election (which happens with positive probability by construction) at θ, candidates' ideological position will come back to the θ.

There still exist, however, several difficulties with this interpretation of elections. First, pork, once distributed to the voters, is gone. We therefore run into the commitment problem—voters cannot credibly commit to voting for either candidate. The same problem exists if we interpret it as a case of the incumbent making *promises* of pork. We can suppose that there exists some "warm-glow" effect of delivery pork in the here and now. Under this interpretation, incumbents have the advantage of delivery pork immediately rather than in the future. The interpretation that I prefer is one in which candidates have varying "qualities" of delivering pork, i.e., one is Robert Byrd and the other is John Doe. By giving transfers, a candidate is displaying his ability of delivering future transfers. Ultimately, it is important to characterize political behavior while considering the different preferences that might exist in the constituents. By incorporating utility for private goods, e.g., pork, into standard utility assumptions, it will perhaps give us a richer and more nuanced into elections.

2 The Election Model

Candidates There are 2 candidates, an incumbent I and a challenger C. The candidates are office motivated and expected utility maximizers. Denote χ_{win} be the indicator function of whether the candidate wins; let $u_j(\chi_{win}) = 1$ if $\chi_{win} = 1$, otherwise $u_j(\chi_{win}) = 0$ if $\chi_{win} = 0$.

There is an ideological policy space $\Theta \subset \mathbb{R}^m$. Assume Θ to be compact. Candidates choose an ideological position $\theta \in \Theta$.

In addition to the ideological position, candidates choose a transfer vector t to the voters. The budget of the transfers is denoted B, so that $\sum_i t_i \leq B$. Furthermore, the distributions of the budgets F_j, for $j \in \{I, C\}$ are common knowledge and are assumed to have bounded support $[0, T_j]$, for $j \in \{I, C\}$.

Denote $(\theta_j, t_j) \in \Theta \times \mathbb{R}^n$ to be the policy position of the candidates.

Voters Each voter has preferences over the ideological location, which is public, and the transfer, which is private. So preferences are represented by a utility function $u_i : \Theta \times \mathbb{R} \mapsto \mathbb{R}$. The utility function $u_i(\theta, t)$ is nonnegative, continuous, and strictly increasing in t_i, for every $\theta \in \Theta$.

The voters votes for candidate j if $u_i(\theta_j, t_j) > u_i(\theta_k, \theta_k)$, for $j, k \in \{I, C\}$, $j \neq k$.

Voters' preferences over the ideological location and transfers are separable. Thus, $\forall (\theta, t), (\theta', t')$,

$$u_i(\theta, t_i) > u_i(\theta', t_i) \Leftrightarrow u_i(\theta, t'_i) > u_i(\theta', t'_i).$$

In addition, voters have an ideal point θ_i^*, such that $u_i(\theta_i^*, t_i) > u_i(\theta, t_i), \forall t_i, \theta \neq \theta_i^*$.

For tractability, I will assume throughout that voters' utility function are quasi-linear in transfer. The utility function is hence $u_i(\theta, t_i) = v(\theta_i^*, \theta) + t_i$, where $v(\theta_i^*, \theta)$ is assume to be pseudo-concave. That is, if $\forall \theta, \theta' \in \Theta$ with $\theta \neq \theta'$,$\bigtriangledown u_i(\theta)(\theta' - \theta) \leq 0 \Rightarrow u_i(\theta') < u_i(\theta)$. In some cases I am able to derive results only for the Euclidean utility function, where $u_i(\theta, t_i) = \|\theta_i^* - \theta\| + t_i$.

The Election There are two dates. At date 1 there are three stages.

Stage 0 Nature chooses B_I, B_C, and reveals to the candidates $F_I(B_I), F_C(B_C)$, assumed to be common knowledge.

Stage 1 The incumbent (or nature), chooses $\theta_I \in \Theta$.

Stage 2 The challenger chooses $\theta_C \in \Theta$ conditional on θ_I.

At the date 1, nature reveals to the incumbent B_I, and the challenger B_C. There are two stages in date 2.

Stage 1 The incumbent chooses a transfer vector $t_I \in \mathbb{R}^n$.

Stage 2 The challenger chooses a transfer vector $t_C \in \mathbb{R}^n$, conditional on t_I.

Stage 3 The voters vote for the candidate that maximizes their utility.[1]

3 Challenger with No Transfers

In this section I will present the results where the challenger has no transfers; thus $B_C = 0$. I will show that even when the challenger has no money, we are able to obtain separation of policy platforms. Because of the tractability of the problem, I am also able to derive results based on multi-dimensional policy space, as well as dynamics of the model.

3.1 Equilibrium characterization

We will backward induct and solve for the subgame perfect equilibrium. Because the challenger has no transfers to distribute to the voters, we can start with the incumbent's optimal vote buying strategy.

Incumbent's vote buying strategy Let the minimum amount of money that the incumbent needs to win the election be

$$t^*(\theta_C, \theta_I) \equiv \min_t \sum_i t_i : |\{u_i(\theta_I, t) > u_i(\theta_C, 0)\}| \geq n/2 \tag{1}$$

[1]To avoid open-set nonexistence problems, I will assume that ties are always resolved in favor of the incumbent.

The incumbent will win the election if the realized value $t_I > t^*$, and lose otherwise.

Challenger's ideological positioning strategy The challenger's utility given θ_I and the reaction function of the incumbent, is

$$
\begin{aligned}
u_C(\theta_C, \theta_I, t^*(\cdot)) &= \Pr\left(T_I < t^*(\theta_{C,}\theta_I)\right) u(\chi_{win} = 1) \\
&+ \Pr(T_I \geq t^*(\theta_C, \theta_I)) u(\chi_{win} = 0) \\
&= F(t^*(\theta_{C,}\theta_I))
\end{aligned} \tag{2}
$$

This is an increasing function of t. Therefore the optimal positioning strategy is

$$
\theta_C^*(\theta_I, t) = \max_{\theta_C} t^*(\theta_{C,}\theta_I) \tag{3}
$$

Incumbent's ideological positioning strategy If the incumbent could position himself, he would be placed such that he maximize his utility given the reaction function of the challenger and his own reaction function in the vote buying period. The incumbent's utility is given by

$$
u_I(\theta_I, \theta_C^*(\cdot), t(\cdot)) = \Pr(T_I \geq t^*(\theta_C^*(\theta_I), \theta_I)) u_I(\chi_{win} = 1) \tag{4}
$$
$$
+ \Pr(T_I \leq t^*(\theta_C^*(\theta_I), \theta_I)) u_I(\chi_{win} = 0) \tag{5}
$$
$$
= 1 - F(t^*(\theta_C, \theta_I))
$$

which is a decreasing function of t^*, so the optimal positioning strategy is given by

$$
\theta_I^*(\theta_C^*(\cdot), t^*(\cdot)) = \min_{\theta_I} t(\theta_C, \theta_I) \tag{6}
$$

Existence of equilibrium in this model depends critically on the sequential nature of the game. Had we assumed that this to be a simultaneous move game, we would run into the "Colonel Blotto" problem [10], where the equilibrium is almost impossible to solve for.

3.2 Ideological Position in One Dimension

Black's theorem proves that, in one dimensional ideological policy space with single peaked preferences, the core is non-empty [8]. In particular, the core is located at the median voter's ideal point, denoted θ_m.

3.2.1 Non-convergence to core

I prove here that in the current model, if the incumbent is not located at θ_m, the challenger will not either. Assume throughout wolog that $\theta_I < \theta_m$, where θ_m is the median of the distribution.

Proposition 1 (Euclidean Preferences) *Suppose that voters had Euclidean utility in the ideological dimension, and that ideal points are distributed according to F, assumed to be continuous and strictly increasing. Let F be restricted wolog to $[0,1]$. Then if the incumbent locates at $\theta_I \neq \theta_m$, the challenger will also locate such that $\theta_C \neq \theta_m$.*

Proof. Suppose not, and that $\theta_C = \theta_m$. Also assume wolog that $\theta_I < \theta_m$. The probability that the challenger wins is $F(\tau(\theta_C))$, which is strictly increasing in τ. Let $\theta' = \theta_I + \frac{\theta_C - \theta_I}{2}$, then

$$u_C(\theta_C) = \max_{\theta_C} \int_{\theta'}^{\theta_C} (\theta - \theta')\, dF(\theta) + \int_{\theta_C}^{\theta_m} (\theta_C - \theta_I)\, dF(\theta), \forall \theta_I < \theta_C \leq \theta_m$$

Apply Leibnitz's rule, and differentiate $u'_C(\theta_C)$ we have FOC

$$\frac{1}{2}\left((-\theta_C + \theta_I) f(\theta_C) + 2\int_{\theta_C}^{\theta_m} f(\theta) d\theta - \int_{\theta'}^{\theta_C} f(\theta) d\theta\right) \equiv 0 \tag{7}$$

Since $\theta_C = \theta_m$, the middle term is 0, so

$$(-\theta_C^* + \theta_I) f(\theta_C^*) = \int_{\theta'}^{\theta_C^*} f(\theta) d\theta$$

But $\theta_C \geq \theta_I$, so $(-\theta_C + \theta_I) \leq 0$, which implies that the $LHS \leq 0$. The $RHS \geq 0$ however, with zero at $\theta' = \theta_I + \frac{\theta_C - \theta_I}{2}$, or that $\theta_I = \theta_C = \theta_m$, a contradiction.

Next we check the SOC. Unfortunately there may be several local maxima. However, we can still show that $\theta_C^* \not> \theta_m$. Let $\theta_C = \theta_m$, then

$$u_C(\theta_C) = \int_{\theta'}^{\theta_m} (\theta - \theta') f(\theta) d\theta, \forall \theta_I > \theta_m$$

Differentiating $u_C(\theta_C)$, we have FOC

$$-\frac{1}{2} \int_{\theta'}^{\theta_m} f(\theta) d\theta$$

which is negative everywhere. Thus $\theta_C \in (\theta_I, \theta_m)$. ■

Example 1 (Uniform distribution) *Suppose that voters have Euclidean preferences, and that ideal points are distributed uniformly over* $[0, 1]$*, we then have*

$$\begin{aligned} \tau(\theta_C) &= \int_{\theta'}^{\theta_C} (\theta - \theta')\, d\theta + \int_{\theta_C}^{\theta_m} (\theta_C - \theta_I)\, d\theta \\ &= -\frac{1}{8}(\theta_C - \theta_I)(7\theta_C + \theta_I - 8\theta_m) \end{aligned}$$

FOC gives

$$\begin{aligned} -\frac{7}{4}\theta_C^* + \frac{3}{4}\theta_I + \theta_m &\equiv 0 \\ \theta_C^* &= \frac{3}{7}\theta_I + \frac{4}{7}\theta_m. \end{aligned} \tag{8}$$

The uniform distribution of voters also gives a global maximum, since $\tau''(\theta_C) = -\frac{7}{4}$*. The challenger's ideological position is therefore a convex combination of the incumbent's ideological position and the median voter's ideal point. Suppose that* $\theta_I = 0, \theta_m = 1$*, then* $\theta_C^* \approx 0.57$*.*

Next I generalize this result to pseudo-concave preferences.

Lemma 1 *Suppose that voters' preferences are pseudo-concave and that their ideal points are distributed according to* F*, assumed to be continuous. If* $\theta_I < \theta_m$*, then* $\theta_C^*(\theta_I) < \theta_m$*.*

Proof. Suppose not, and that $\theta_C' = \theta_C^*(\theta_I) > \theta_m$. Then let $\theta_C = \theta_m$. It must be that $m(\theta_C') > m(\theta_C)$. By pseudo-concavity of the utility function, the incumbent will the closest voters not in his coalition. So

$$t^*(\theta_C) = \int_{m(\theta_C)}^{\theta_m} \left[u(\theta, \theta_C) - u(\theta, \theta_I)\right] f(\theta) d\theta \tag{9}$$

$$t^*(\theta'_C) = \int_{m(\theta'_C)}^{\theta_m} \left[u(\theta, \theta'_C) - u(\theta, \theta_I)\right] f(\theta) d\theta. \tag{10}$$

where $\int_{m(\theta_C)}^{\theta_m} f(\theta)d\theta > \int_{m(\theta'_C)}^{\theta_m} f(\theta)d\theta$ as $m(\theta'_C) > m(\theta_C)$.

Furthermore I will prove that 9 is point-wise greater than 10. By single peakedness property of pseudo-concave preferences,

$$[u(\theta, \theta_C) - u(\theta, \theta_I)] - [u(\theta, \theta'_C) - u(\theta, \theta_I)] > 0,$$

as $\theta < \theta_m = \theta_C < \theta'_C$. ■

The above proposition states that, given the incumbent's ideological location, the challenger will not "jump" to the other side of the median.

Proposition 2 (Pseudo-concave preferences) *Suppose that voters have pseudo-concave utility functions in the ideological dimension, and that ideal points are distributed according to F, assumed to be continuous and strictly increasing. Let F be restricted wolog to* $[0, 1]$. *Then if the incumbent locates at $\theta_I \neq \theta_m$, the challenger will also locate such that $\theta_C \neq \theta_m$. Furthermore, $\theta^*_C \in (\theta_I, \theta_m)$.*

Proof. By lemma 1, $\theta_C \leq \theta_m$, thus we only have to consider the case that $\theta^*_C \in (\theta_I, \theta_m]$. Let $m(\theta_C) \equiv \{\theta : u(\theta, \theta_C) = u(\theta, \theta_I)\}$.

$$u_C(\theta_C) = \int_{m(\theta_C)}^{\theta_m} \left[u(\theta, \theta_C) - u(\theta, \theta_I)\right] f(\theta) d\theta$$

Taking FOC gives

$$\int_{m(\theta_C)}^{\theta_m} u_\theta(\theta, \theta_C) f(t) d\theta + f(m(\theta_C))(-u(m(\theta_C), \theta_C)) \tag{11}$$
$$+ u(m(\theta_C), \theta_I))m'(\theta_C)$$

Setting $\theta_C = \theta_m$, expression 11 becomes

$$\begin{aligned} &\int_{m(\theta_C)}^{\theta_m} u_\theta(\theta, \theta_m) f(t) d\theta + f(m(\theta_m))(-u(m(\theta_m), \theta_m)) \\ &+ u(m(\theta_m), \theta_I)) m'(\theta_m) \\ &= \int_{m(\theta_C)}^{\theta_m} u_\theta(\theta, \theta_m) f(t) d\theta \end{aligned} \tag{12}$$

By the definition of $m(\theta_C)$, the second term in 12 is equal to 0. Expression 12 is negative, since by the definition if single peakedness, if $\theta' < \theta < \theta_m \Leftrightarrow u(\theta) > u(\theta')$.

To prove that $\theta_C^*(\theta_I) \in (\theta_I, \theta_m)$, set $\theta_C = \theta_I$. Expression 11 is equal to $\int_{m(\theta_I)}^{\theta_m} u_\theta(\theta, \theta_I) d\theta$, and is positive. Since $u_C(\theta_C)$ is continuous on (θ_I, θ_m), it has an interior maximum in (θ_I, θ_m). ■

3.2.2 Asymptotic convergence to core

Although the proposition 1 challenger does not locate on the core, it does show that the challenger moves closer to the core relative to the incumbent, since that is the only way the challenger could win.

Example 2 (Uniform distribution) *Suppose that voters have Euclidean utility function and that ideal points are distributed uniformly over* $[0, 1]$,*then by equation 8 the rate of convergence is* $3/7$.

We wish to show that $\lim_{k \to \infty} \theta_C^\infty(\theta_I) = \theta_m$.

Proposition 3 (pseudo-concave preferences) *Suppose that voters' preferences are pseudo-concave, and that their ideal points are distributed according to* F*, assumed to be continuous. Suppose also that* $\theta_C'(\theta) \geq 0$*. Let* F *be restricted wolog to* $[0, 1]$*. Then* $\forall \varepsilon, \exists \delta$ *such that* $\theta_C^\infty(\theta_I) - \theta_m < \varepsilon, \forall \theta \in \Theta$.[2]

Proof. Define θ^* as the point of convergence, by the fact that $\left\{\theta_C^k\right\}_{k=1}^\infty$ is a monotone sequence and is bounded at θ_m. We thus have $0 < \theta^* - \theta_C(\theta^* - \varepsilon) < \varepsilon$. By taking a first

[2] The proof of this proposition is incomplete. There should be some theorem that I can use but don't know.

order Taylor expansion,

$$\begin{aligned}
&0 < \theta^* - \theta_C(\theta^* - \varepsilon) < \varepsilon \\
&\approx \theta^* - [\theta_C(\theta^*) + \varepsilon\theta'_C(\theta^*)] \\
&= \theta^* - \theta_C(\theta^*) - \varepsilon\theta'_C(\theta^*) < \varepsilon
\end{aligned}$$

By assumption $\theta'_C(\theta^*) \geq 0$, so

$$\begin{aligned}
\theta^* - \theta_C(\theta^*) - \varepsilon C &< \varepsilon \\
\theta^* - \theta_C(\theta^*) &< \varepsilon(1 + C)
\end{aligned}$$

■

The technical condition that $\theta'_C(\theta) \geq 0$ is necessary for the result that candidate positions converge toward the median. However, I was not able to prove this for the general pseudo-concave case.

3.2.3 Extension to multiple stage game

Proposition 4 (non-commitment) *Denote the vector of distributive resources the incumbent spends by a. At stage 1, the incumbent will not commit any distributive resources, such that $\sum_i a_i = 0$ and that $a_i \geq 0, \forall i$.*

Proof. I will show that committing resources at stage 1 is a weakly dominated strategy. Let $\theta_C = \theta$ be the position of the challenger if the incumbent follows the equilibrium strategy, then by optimality the incumbent will spend $t(\theta) = t$. Now suppose that the incumbent deviates and spends a in the first and second period, respectively, such that $\sum_i a_i > 0$. Also denote $t_a = t - a$. Let $A = P_\theta(\theta_I)$. Let θ' be the position of the challenger when the incumbent deviates. Denote $B = P_{\theta'}(\theta_I)$. If $\theta' \neq \theta$, then it must be that

$$\begin{aligned}
\sum_{i \in B \cap A} a_i + t_i(\theta') + \sum_{j \in B \setminus A} t_j(\theta') &\geq \sum_{i \in A} a_i + t_{ai}(\theta) \\
&= \sum_{i \in A} t_i(\theta)
\end{aligned}$$

Therefore spending a positive amount at stage 1 is a weakly dominated strategy. ■

It is easy to see why the above proposition is true. Given the zero sum nature of the game, if the incumbent's choice affects the challenger's choice, then it must be for the worse to the incumbent. If the incumbent is buying voters before the challenger moves, he is essentially moving the voters ideal points so that they satisfy the Plott conditions, i.e., such that the incumbent's position is in the core. On the other hand, if the incumbent waits until the challenger has moved, he will only have to buy a subset of the voters, because the winset is the union of disjoint sets.

Proposition 4 shows that we can extend a game where the challenger has no money to one in which there are an arbitrary number of periods. It suffices to note that, as long as the incumbent has the last move, he can always wait until after the challenger moves to decide on the distributional dimension.

3.3 Multiple Ideological Dimensions

Throughout the previous section we have assumed that the incumbents cannot move to the median voter's ideal point. It is conceivable that the distribution of the voters have changed since the last election, due to for example, immigration, new industry, etc. If we were to relax this assumption and allow the incumbents to move before the challenger, then the incumbent will simply move to the median voter's ideal point. Since by assumption indifferent voters vote for the incumbent, the incumbent wins with probability 1.[3]

Here I extend the ideological space to multiple dimensions. As mentioned in the introduction, extension of the ideological space to $\mathbb{R}^n$ introduces voting cycles and the emptiness of the core. It is here that transfers can serve to a great degree to provide stability as well as pin down the policy position of the voters.

First I introduce a very simple result concerning incumbent ideological position when the core exists.

Proposition 5 *Suppose that $\mathcal{C}(\mathbf{R}) \neq \varnothing$, then $\theta_I^* \in \mathcal{C}(\mathbf{R})$.*

[3] Even if we assume that the challenger wins ties, the only case where the incumbent will lose is if $B_I = 0$, which occurs with probability 0.

Proof. Let $x \in \mathcal{C}(\mathbf{R})$, by the definition of the core, $|\{i : u_i(y) > u_i(x)\}| \leq n/2, \forall y \in X$. And so the incumbent wins with probability 1 by spending nothing. Assume not, and that $\theta_I^* = x' \notin \mathcal{C}(\mathbf{R})$, then $\exists y \in X$ such that $|\{i : u_i(y) > u_i(x)\}| > n/2$. Thus $\forall y \in X, t^*(x) > 0 > t^*(x')$. ■

3.3.1 Empty Core

Next I will show that if the core does not exist, the challenger will not hold the same ideological position as the incumbent. This result is not a simple consequence of the assumption that the incumbent will always win if they hold the same positions. Indeed, the result will hold even if we assume that the challenger always wins. Formally, I claim $\forall \theta_I, \exists \varepsilon > 0$ such that $\theta_C^* \notin N_e(\theta_I)$, where N_ε is a epsilon neighborhood around θ_I.

Definition 1 *Denote*

$$P_\theta(\theta') = \{i : u_i(\theta) > u_i(\theta')\}$$
$$winset(\theta) \equiv \left\{\theta' : |P_{\theta'}(\theta)| > \frac{n}{2}\right\}.$$

In words, $P_\theta(\theta')$ is set of voters who prefer θ to θ'. The winset of the ideological position θ is the set of ideological positions that would defeat θ under majority rule.

Proposition 6 (Non-convergence in platform) *Suppose that voters utilities are pseudo-concave, that the core is empty, then $\forall \theta_I, \exists \varepsilon > 0, \theta \notin N_\varepsilon(\theta_I)$ such that $u_i(\theta) > u_i(\theta'), \forall \theta' \in N_\varepsilon(\theta_I), \forall i \in P_\theta(\theta_I)$.*

Proof. Case 1: $\theta_I \notin closure(\text{winset}(\theta_I))$. Trivial.

Case 2: $\theta_I \in closure(\text{winset}(\theta_I))$. Assume that $\exists \theta_I$ such that $\forall \varepsilon, \theta \notin N_\varepsilon(\theta_I), \exists \theta' \in N_\varepsilon(\theta_I), i \in P_\theta(\theta_I)$ such that. $u_i(\theta) < u_i(\theta')$. In particular, there exists a $\theta^* \in \text{winset}(\theta_I) \cap N_\varepsilon(\theta_I)$ such that. $u_i(\theta') > u_i(\theta^*)$. Since ε is arbitrary, this must hold for all $\hat{\theta} \in (\theta_I, \theta')$. But $u(\cdot)$ is assumed continuous, so $\lim_{\theta \to \theta_I} u_i(\theta) = u_i(\theta_I)$. This implies that $\exists \theta \in \text{winset}(\theta_I)$ such that $\triangledown u_i(\theta_I) \cdot (\theta - \theta_I) > 0$. ■

Example 3 *Figure 1 shows an example where $m = 2, n = 3$. Normalize incumbent's policy position is to $(0, 0)$. The voters' ideal points are $(7, 26), (11, -26), (-24, -20)$.*

The challenger's probability of election is maximized by choosing a position in the winset of θ_I that would require the incumbent to pay the greatest transfer to the voters.

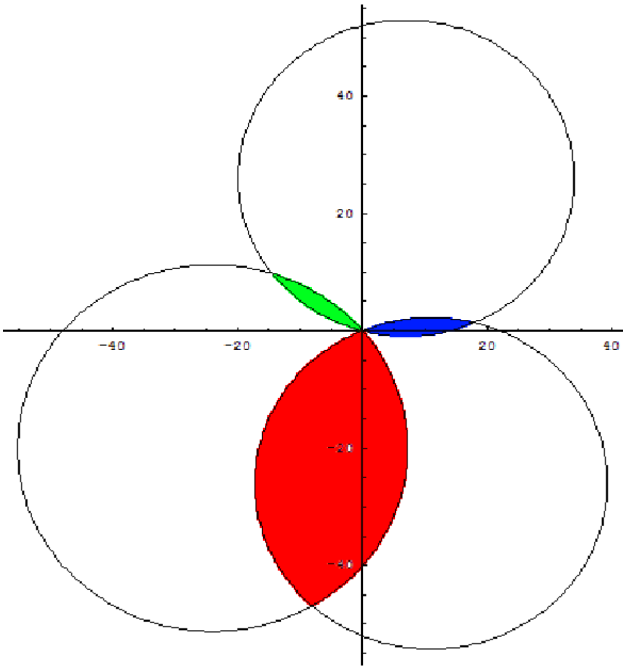


Figure 1: Indifference curves through the incumbent's ideological position (normalized to (0,0)). The colored regions denote the winsets.

We maximize the transfers by choosing a point that maximizes the difference within the colored regions, which is between the ideal points $(11, -26), (-24, -20)$. *Because the incumbent only needs to buy off one voter, the challenger will choose the midpoint of the line segment*

$$y + 26 = -\frac{6}{35}x - 11 \cap \frac{(x-11)^2}{||(11,-26)||}\frac{(y+26)^2}{||(11,-26)||} \cap \frac{(x-24)^2}{||(-24,-20||}\frac{(y-20)^2}{||(-24,-20||},$$

located at $(17.5, -23)$

The incumbent needs to buy off at least one voter. The minimum amount of transfer that is needed is easily calculated to be

$$\frac{||(11,-26)|| + ||(-24,-20|| - ||(11,-26) - (-24,-20)||}{2} \approx 11.98$$

And the probability that the challenger will win is $F(11.98)$.

3.4 Case with 3 Voters

3.4.1 Optimal Incumbent Positioning

First I will show the predictions of the model with 3 voters. Similar to [4], the principle of the strategy is one similar to the "no soft-spot" strategy. Surprisingly, despite the vastly different motivations of the models, the optimal incumbent position has the same solution as the Finagle point. This, however, is an artifact of the assumption that the utility function of the votes is linear with respect to the transfer and are identical for all voters. I follow Wuffle et al. [12] and sketch the algorithm to find the optimal ideological location for the incumbent.

Let $\overline{AB}$ denote the Euclidean distance of the line segment connecting A and B. The sizes of the winsets given X are

$$\begin{aligned} g_1(X) &= \overline{XA} + \overline{XB} - c \\ g_2(X) &= \overline{XA} + \overline{XC} - b \\ g_3(X) &= \overline{XB} + \overline{XC} - a \end{aligned}$$

where $\overline{XA}$ is the Euclidean distance from X to A, and XA as the line segment connecting X and A. Since Euclidean distance, the minimum function, the maximum function, and the composition of convex functions are all convex, we have

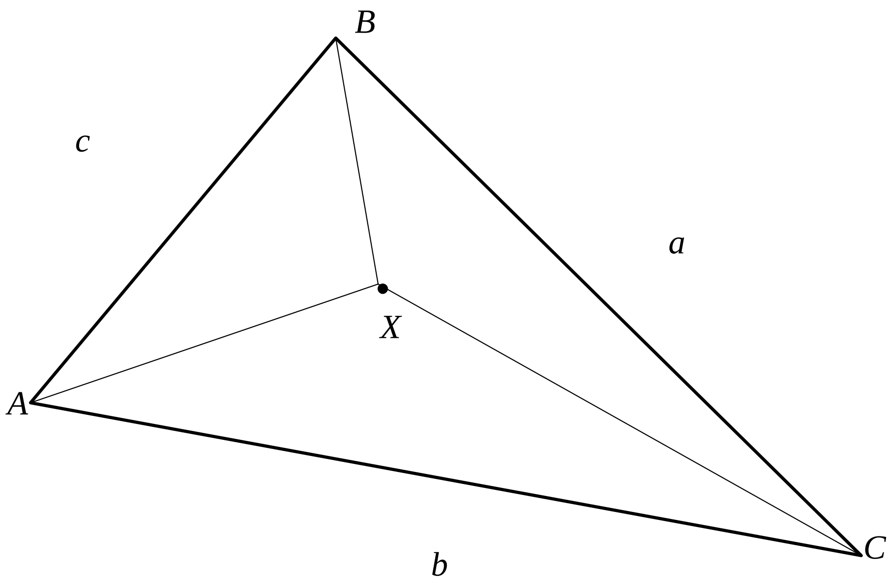
$$h(X) \equiv \{X : g_1(X) = g_2(X) = g_3(X) = \delta\}^4$$

I take a shortcut in finding the geometric solution, and instead find the algebraic solution. Reorient and normalize the triangle such that the vertices are $(a_1, a_2), (-\frac{1}{2}, 0), (\frac{1}{2}, 0)$. There is a unique solution to $H(X^*) = (x_1^*, x_2^*) = \left(a_1, \frac{1-4a_2^2}{4a_2}\right)$.

I will not dwell on the optimal incumbent position given three voters. Wuffle et al. explored a number of properties of the Finagling circle. In particular, they showed that the finagle radius is proportional to the half-winset, which is proportional to the winset.[5] Furthermore, the finagle radius is maximized relative to the yolk when the triangle is equilateral, and that, as the triangle becomes more acute or obtuse, we can make the

[4] For a proof of the optimality of $h(X)$, see [12].

[5] The half-winset is defined as the set of points which are obtained by uniformly reducing each ray in the winset of that point by half. [12]



finagle radius, or equivalently, the size of the winset, arbitrarily small. The intuition is straightforward, as a angles become smaller, the triangle moves toward becoming colinear, satisfying the conditions for core existence.

3.4.2 Cycles

Here I show that although the optimal incumbent position is in the interior of the Pareto set, it is not stable. That is, once the incumbent is defeated, the policy platform will never get back to that position. Furthermore, if the voters have Euclidean preferences, the ideological policy platform will follow a closed cycle along the 3 contract curves.

First note that, as long as the core symmetry conditions are not met, any coalition within the Pareto set has blocking coalition of size 2. I will prove that as long as the incumbent is positioned in the Pareto set, the challenger's optimal ideological position will be on one of the three contract curves between the voters. The proposition below shows that, if the incumbent is located in the interior of the Pareto set, then as soon as that incumbent loses, the ideological platform will never move to the interior of the Pareto set again.

Lemma 2 *Define the contract set of the voters as* $C(i,j) = \{\theta : \bigtriangledown u_i(\theta) = -\alpha \bigtriangledown u_j(\theta)$*, for some* $l, k \geq 0$ *(but not both zero)*$\}$*. Suppose that* $\mathcal{C}(R) = \varnothing$*,* $n = 3$*, and that utility functions of the voters are pseudo-concave, then the* $\cap_{i \neq j} C(i,j) = \varnothing, \forall i, j \in \{1, 2, 3\}$.

Proof. Suppose not, and that $\exists \theta^*, i \neq j \neq k$ such that $\theta^* \in C(i,j) \cap C(j,k)$. Then it must be that $\bigtriangledown u_i(\theta) = -\alpha \bigtriangledown u_j(\theta), \forall i,j$. But by the fact that $\mathcal{C}(R) = \varnothing$, there must exist i,j such that $\bigtriangledown u_i(\theta) \neq -\alpha \bigtriangledown u_j(\theta)$. ■

Proposition 7 *Suppose that $n = 3$, $\mathcal{C}(R) = \varnothing$, and that θ_I is in the Pareto set, then $\theta^*_C(\theta_I) \in C(i.j)$, for some i,j.*

Proof. Because θ_I is in the Pareto set and $n = 3$, $\max|\text{winset}| = \min|\text{winset}| = 2$. Therefore the incumbent only has to buy 1 voter to win with the minimum buying power of $\min_{i \in P_{\theta'}(\theta)} u_i(\theta') - u_i(\theta)$. The challenger will choose from $\{\theta' : \min_{i \in P_{\theta'}(\theta)} u_i(\theta') - u_i(\theta)\}$. Assume that $\theta \equiv \theta^*_C(\theta_I) \notin C(i,j), \forall i,j$. Then $\exists \theta'$ such that $\bigtriangledown u_i(\theta)(\theta' - \theta) > 0$ for at least 2 voters. Thus the challenger can improve upon his positioning by choosing θ'' in the direction of $\bigtriangledown u_i(\theta)(\theta' - \theta)$, contradicting the optimality of θ. ■

Euclidean Preferences The above proposition showed that the contract curves of the voters are attractive for pseudo-concave preferences. The next proposition shows that if voters have Euclidean preferences, the policy path follows a closed cycle.

Proposition 8 (Closed Cycle) *If there are 3 voters, who have Euclidean preferences, the policy positions of the candidates will follow a closed cycle.*

Proof. I will prove the proposition using Banach's Fixed Point Theorem, which states that if X is a non-empty complete metric space and $f : X \mapsto X$ is a strict contraction, then f has a unique fixed point x^*. Furthermore, $x^* = \lim_{k \to \infty} f^k(x), \forall x \in X$.

First note that because the core is empty, the winset is non-empty, for all θ. Therefore by the assumption that the incumbent's budget is distributed over the interval $[0,b]$, the probability that the challenger will win is strictly positive. By lemma 2, the challenger's ideological position will always be at the contract curve of exactly two of the three voters. Lemma 3 shows that the cycle will occur in only one direction. That is, if $f_1(\alpha) = \beta$, then $f_2(\beta) = \gamma$, and $f_3(\gamma) = \alpha$.

Let $\beta = f_1(\alpha)$, $\gamma = f_2(\beta)$, $\alpha = f_3(\gamma)$. Define the composition function $g(\alpha) = f_3 \circ f_2 \circ f_1(\alpha) = \alpha$. To prove that this function is a contraction, we need

$$\left| \frac{dg}{d\alpha} \right| < 1, \forall \alpha \in \Theta. \tag{13}$$

Take $f_1(\alpha)$ first, by the law of cosine (see figure 2),

$$\beta = \frac{1}{b}\left((1-\alpha)a - \frac{1}{2}\left[(1-\alpha)a + \sqrt{(\alpha a)^2 + c^2 - 2\alpha c\cos B} - b\right]\right).$$

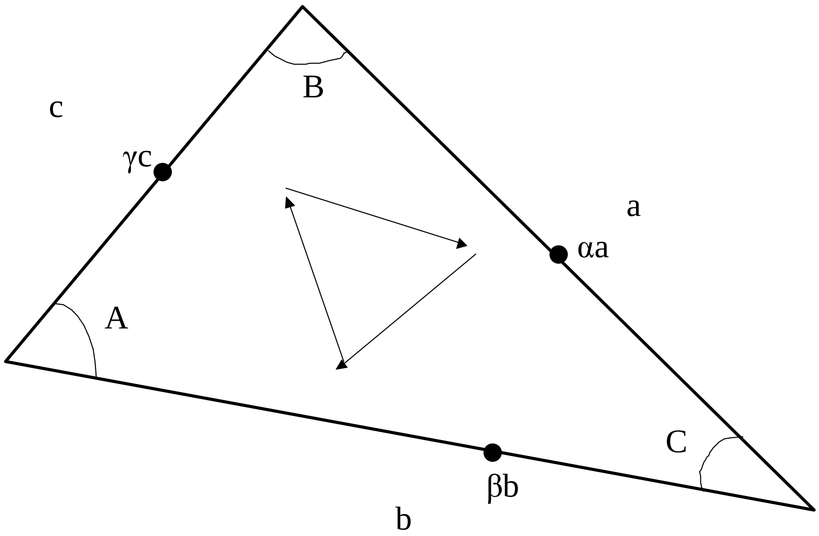


Figure 2: Cycle

It follows that

$$\begin{aligned} b\frac{df_1(\alpha)}{d_\alpha} &= -a - \frac{1}{2}\left[-a + \frac{1}{2}\frac{2\alpha a^2 - 2ac\cos B}{\sqrt{(\alpha a)^2 + c^2 - 2\alpha c\cos B}}\right] \\ &= -\frac{a}{2}\left[1 + \frac{\alpha a - c\cos B}{\sqrt{(\alpha a)^2 + c^2 - 2\alpha c\cos B}}\right] \end{aligned} \tag{14}$$

The fraction in expression 14 is

$$\frac{\alpha a - c\cos B}{\sqrt{(\alpha a)^2 + c^2 - 2\alpha c\cos B}} = \sin\theta, \text{ for some } \theta,$$

where $\sin\theta = 1$ iff $\theta = \pi/2$ iff the ideal points are colinear.

$$-\frac{a}{b} < f_1'(\alpha) = -\frac{a}{2b}\left[1 + \sin\theta\right] < 0, \forall\alpha \tag{15}$$

This implies that the mapping from side BC to side AC reaches a maximum in β when $\alpha = 0$, which implies that $\beta b \leq \frac{1}{2}(a + b - c)$. Thus $\theta_C(\theta_C(\theta_I)) \in AB$. By symmetry of equation 15, this implies that θ_C will be mapped back to BC.

Now I will prove 13. By symmetry, we have $0 > f_2'(\beta) > -\frac{b}{c}, \dot{0} > f_3'(\gamma) > -\frac{c}{a}$. Therefore, $0 > g'(\alpha) > -1$, and $g'(\alpha)$ is a contraction whose range is within its domain $[0, \frac{1}{2b}(a+c-b)]$. Therefore by Banach's fixed point theorem, a fixed point exists and is globally attractive. ■

Lemma 3 (Direction of Cycle) *Let $BC \equiv \{\theta : \theta \in \alpha B + (1-\alpha)C, \forall \alpha \in [0,1]\}, \theta_C(\theta_I) \in BC$, then $\theta_C \in AC$ if $\alpha a \leq \frac{1}{2}(a+c-b)$. By symmetry, $\theta_C \in AB$ if $\beta b \leq \frac{1}{2}(a+b-c)$, and $\theta_C \in BC$ if $\theta_C \in BC$ if $\gamma c \leq \frac{1}{2}(b+c-a)$.*

Proof. For any $\theta \in BC$,

$$\int d\theta' = (1-\alpha)a + \sqrt{(\alpha a)^2 + c^2 - 2\alpha c \cos B} - b, \forall \theta' \in \text{winset}(\theta) \cap AC$$

$$\int d\theta'' = \alpha a + \sqrt{(\alpha a)^2 + c^2 - 2\alpha c \cos B} - a, \forall \theta'' \in \text{winset}(\theta) \cap AB$$

Thus, challenger will choose the ideological position θ_C in $\max\{\int d\theta', \int d\theta''\}$. ■

Example 4 (Equilateral Triangle) *Here I will show an example with three votes with euclidean preferences whose ideal points form an equilateral triangle. The orbit formed by the model is quite surprising, as it does not seem to bear any relationship to the yolk besides both being in the Pareto set.*

Without loss of generality we can transform the ideal points of the voters to $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, $(0, \frac{1}{\sqrt{3}})$, $(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$. The optimal location of the incumbent, if the incumbent were free to choose his ideological position, he would trivially choose the center of the triangle, located at $(0,0)$. But the cycle shows that it moves away from $(0,0)$.

Conjecture 1 *With Euclidean preferences, the optimal path of the candidates will follow a closed cycle.*

Remark 1 *I was unable to solve for the optimal path of the candidates with $n > 3$. I was able to numerically solve for the optimal path of candidates' ideological location. Below I show an example with 21 voters.*

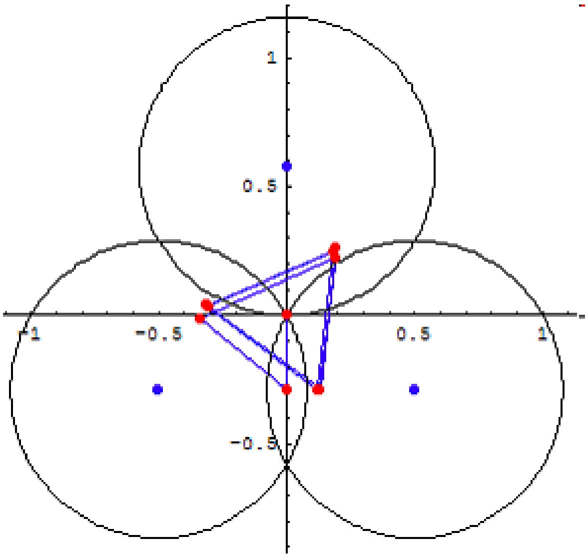


Figure 3: An example ideological policy cycle where voters' ideal points form an equilateral triangle centered on $(0, 0)$, with edge lengths 1. The blue dots represent voter ideal points; red dots the optimal challenger location. Incumbent initially located at $(0, 0)$.

4 Challenger with money

In this section, I extend the case where $T_C > 0$, such that $B_C \in [0, T_C]$. Because of the complexity of the problem, I will present results for one dimensional ideological policy space. This becomes then an extension of the Groseclose and Snyder [4] model. I will therefore retain much of the terminology in the Groseclose and Snyder model. The intuition behind the Groseclose and Snyder is quite simple. Given any allocation of transfers by the incumbent, the challenger will buy off the cheapest voters. The incumbent's optimal strategy is to make all voters equally expensive to buy.[6]

The optimal buying strategies in the Groseclose and Snyder model are characterized by a series of cutoffs given the incumbent budget. They considered the case when the first move knows the budget of the second mover, whereas in the current model the first mover (incumbent) knows only the distribution of the second mover's (challenger) budget. Below I will consider all cases and calculate the challenger's optimal ideological location maximizing the candidate's expected utility by integrating over the choice of the incumbent's optimal buying strategy. Define the minimum amount the challenger needs to win by $t^*_C(\theta_I, \theta_C, t_I)$[7]. Since the incumbent knows only the distribution of the

[6] For proof of optimality, see Groseclose and Snyder [4].

[7] Following Groseclose and Snyder, I will not characterize B's equilibrium strategies, because they're less interesting. And since all the information is revealed in the game, it becomes a trivial problem for

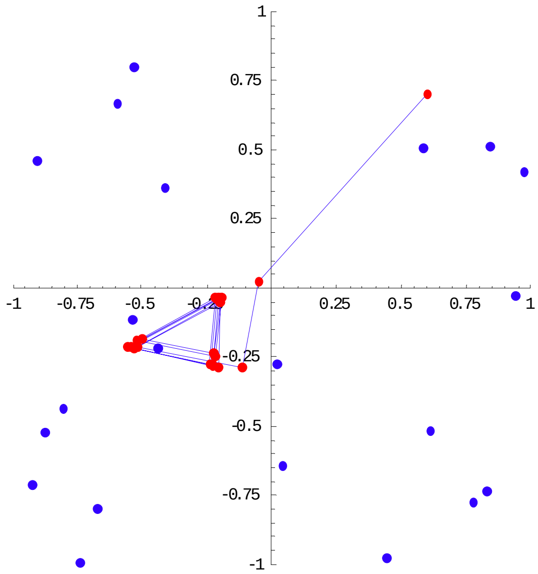


Figure 4: Ideological location path for 21 voters with randomly distributed ideal points on $[0,1]^2$: The blue dots represent voter ideal points. the red dots represent optimal challenger location. Incumbent initially located at $(0.65, 0.74)$.

challenger's budget, his expected utility is: $u_I(t_I, \theta_C, t_C^*(\cdot)) = \int_0^{T_C} F_C(t < t_C^*(\theta_C, \theta_I, t_I)$. Given the optimal buying strategy, which I will characterize below, the expected utility of the challenger is $u_C(\theta_C, \theta_I, t_I^*(\cdot), t_C^*(\cdot)) = \int_0^{T_I} F_C\left[t > t_C^*(\theta_C, \theta_I, t_I)\right] f(t)dt$.The challenger therefore chooses $\theta_C^*(\theta_I) = \max_{\theta_C} \int_0^{T_I} F\left[t < t^*(\theta_C, \theta_I, t_I)\right] f(t)dt$.

In the next few sections, I will find the optimal challenger location over the range incumbent budgets characterized by these cut-points. I will then expect over the incumbent budget, and find the expected optimal challenger. There are 4 cases of optimal incumbent voting buying function to consider, characterized by cutoff points. I assume that θ_I is given exogenously, where $\theta_I < \theta_m$. The universal case I will prove for arbitrary distributions. The others I will restrict attention to the uniform. Groseclose and Snyder deals with uniform distribution throughout.

4.1 Universal

Here I will assume that voters have Euclidean preferences and that voters are distributed on a uniform distribution. In a universal coalition, the candidates buy all voters, and bring the voters' utility to the same level. Assume voters are distributed according to

B: either he can win or he can't with the realization of t_C. For details, see [4].

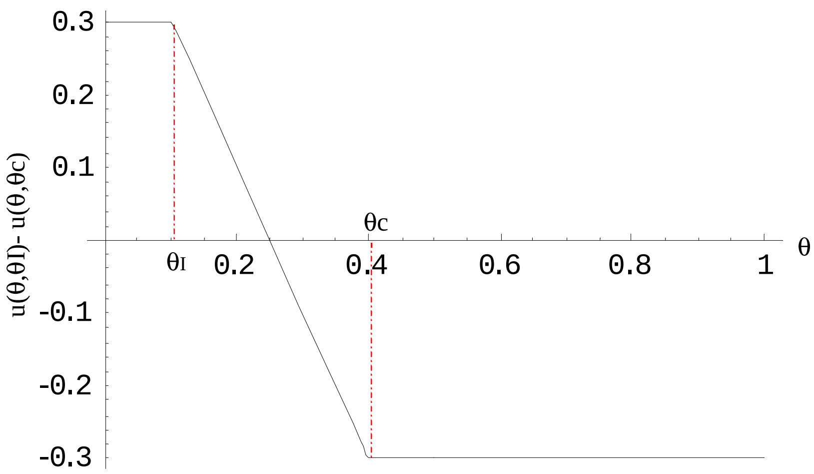


Figure 5: Voter utility for Euclidean preferences under uniform distribution where $\theta_C = 0.1, \theta_I = 0.4$.

the cdf $F(\theta)$, which is differentiable and has an associated density $f(\theta)$. The incumbent can therefore bring all voters to $v(\theta_C)$, where $v(\theta_C)$ is defined as

$$\begin{aligned} v(\theta_C) = B_I - \int_{\theta_I}^{\theta_C} (\theta_C - \theta_I - d(\theta, \theta_I, \theta_C)) f(\theta)\, d\theta \\ - \int_{\theta_C}^{1} 2(\theta_C - \theta_I) f(\theta) d\theta + \theta_C - \theta_I \end{aligned}$$

where $d(\theta, \theta_I, \theta_C) = -|\theta - \theta_I| + |\theta - \theta_C|$. We wish to find $\min_{\theta_C} v(\theta_C)$. Taking FOC, we have $1 - 2\int_{\theta_C^*}^{1} f(\theta) d\theta = 0$. Rearranging terms, and $F(\theta_C^*) = 1/2$, implying that θ_C^* is located at the median. Taking SOC gives us $v''(\theta_C) = 2f(\theta_C) \geq 0$.

Graphically, figure 6 shows that the incumbent's spending can be separated into 3 areas. In regions II and III, the incumbent raises all voters to the level of the maximum of his supporters; in region I, the incumbent raises everyone to the same level up to the maximum allowed by his budget.

Example 5 *If voters are distributed uniformly, then*

$$\begin{aligned} v(\theta_C) &= B - (\theta_C - \theta_I)^2 - (1 - \theta_C) 2 (\theta_C - \theta_I) + (\theta_C - \theta_I) \\ \theta_C^* &= 1/2. \end{aligned}$$

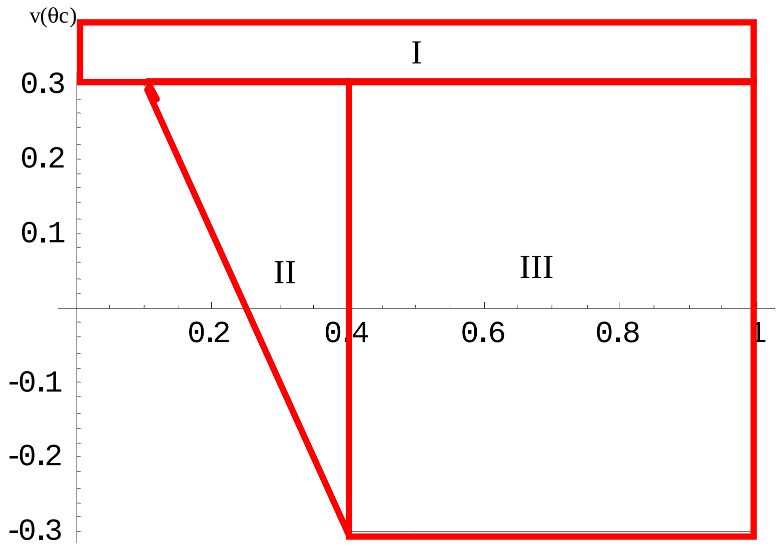


Figure 6: Universal Coalition: The incumbent moves everyone to the same level of utility, up to the maximum allowed by his budget.

4.1.1 Quadratic Preferences

If voters have quadratic ideological preferences, there is an equally intuitive solution

$$v(\theta_C) = B - \int_0^1 \left[d(0, \theta_I, \theta_C) - d(\theta, \theta_I, \theta_C) \right] f(\theta) d\theta + d(0, \theta_I, \theta_C)$$

where $d(\theta, \theta_I, \theta_C) \equiv -\left(\theta - \theta_I\right)^2 + \left(\theta - \theta_C\right)^2$. Taking FOC condition with respect to θ_C, we have $2\theta_C^* - 2\int_0^1 \theta f(\theta) d\theta \equiv 0$. So $\theta_C^* = \int_0^1 \theta f(\theta) d\theta$. Thus unlike the case with Euclidean preferences, if voters have quadratic preferences, and the incumbent can implement a flooded strategy, the mean of the distribution. SOC condition equals 2, ensuring us that we have a minimum.

4.2 Flooded

Groseclose and Snyder defines a *flooded, non-universalistic* coalition if the incumbent buys a majority of voters, but not all voters. Figure 7 show this for a uniform distribution of voters.

I will shows, however, that in the present model, the incumbent will never want to implement a flooded coalition. Figure **??** shows why. Graphically, the incumbent will only wish to implement a flooded coalition if the regions R and Q are of the same size. In the

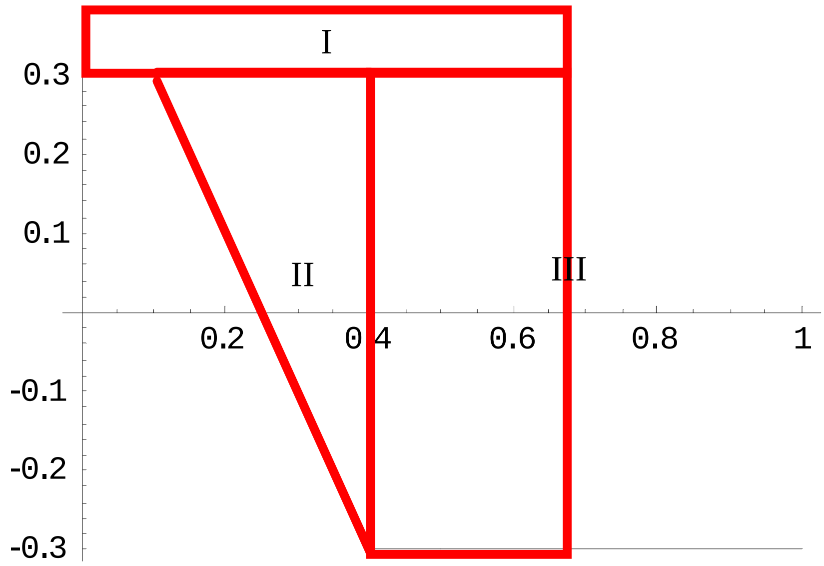


Figure 7: Flooded Coalition

present model, however, $\int_R d\theta \geq \int_0^{\theta_m}(\theta_C - \theta_I)d\theta = \frac{1}{2}(\theta_C - \theta_I) = \int_m^1(\theta_C - \theta_I)d\theta > \int_Q d\theta$. Therefore, the criterion for flooded optimal coalition is never satisfied. This is of course a result of choosing the Euclidean preferences and uniform distribution, which results in the inverse of the buying being not one-to-one in certain distributions.

4.3 Non-flooded

As in Groseclose and Snyder, there is a non-flooded equilibrium when $\theta_2 > 0$, and $B_C \leq \int_0^{\theta'} d(\theta, \theta_I, \theta_C)d\theta$, where $\theta' = \frac{\theta_I+\theta_C}{2}$, and θ_2 will be defined below in expression 16, then the incumbent will choose a non-flooded buying strategy. Geometrically, this occurs when that the size of R is equal to the size of Q. This guarantees that the equation has a maximum within (θ_I, θ_C).

The incumbent chooses the optimal number of voters to buy by choosing θ_2^* such that

$$\max_{\theta_I<\theta_2<\theta_C}(-2\theta_2 + \theta_I + \theta_C)\left(\frac{B_I - (\theta_C - \theta_2)^2}{2(\theta_C - \theta_2)} + \theta_C - \frac{1}{2}\right) \tag{16}$$

The challenger will find the location that minimizes the above expression over θ_C

$$\min_{\theta_C}(-2\theta_2^*(\theta_C) + \theta_I + \theta_C)\left(\frac{B_I - (\theta_C - \theta_2^*(\theta_C))^2}{2(\theta_C - \theta_2^*(\theta_C))} + \theta_C - \frac{1}{2}\right).$$

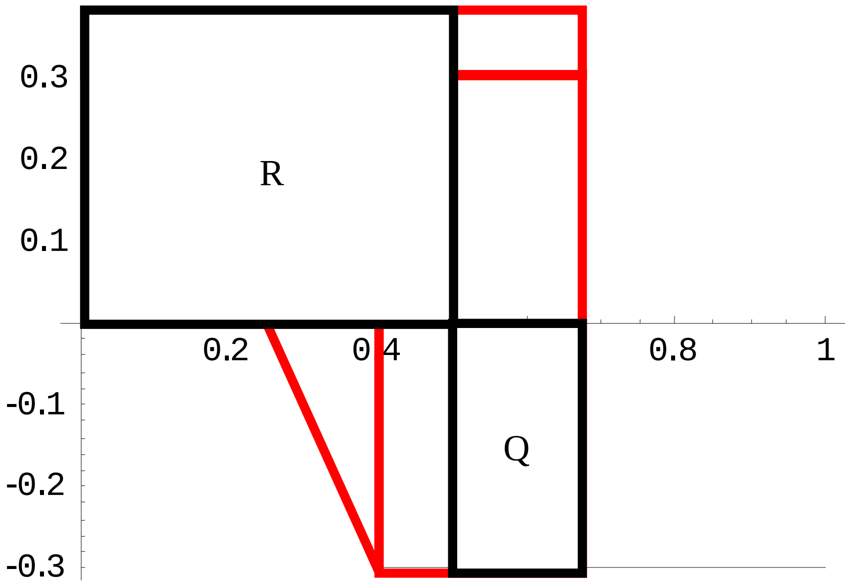


Figure 8: Why flooded coalition is not optimal.

The analytical solution of those equations however proved to be hopelessly complicated, and I was unable to find a way to simplify the expressions. Therefore I will resort to numerical solutions to find their optimal value.

4.4 Non-flooded(b)

Because of the inverse function in the case of the Euclidean preferences is not one-to-one, there exists a case that was not present in Groseclose and Snyder. Specifically, it occurs when express [?] does not have an interior maximum.

Geometrically, it occurs when the height of region R reaches $\theta_C - \theta_I$, at which point there is a discontinuity in the budget necessary to satisfy the Groseclose Snyder condition for non-flooded optimum. I am not able to obtain a closed-form solution for the $B_I(\theta_I)$ that does not satisfy the Groseclose and Snyder condition. I was able to obtain it numerically. For example, let $\theta_I = 0.3$, $B_I = 0.2$, then expression 16 has no interior maximum.

Proposition 9 *The buying strategy in non-flooded coalition is sub-optimal. Furthermore, define* $[0, \theta_3]$ *as the interval of voters that the incumbent will buy, the optimal strategy of the incumbent is to choose* $\max_{t(\cdot)} \theta_3(t(\cdot))$*, i.e., maximize the interval of voters in addition to the non-flooded coalition.*

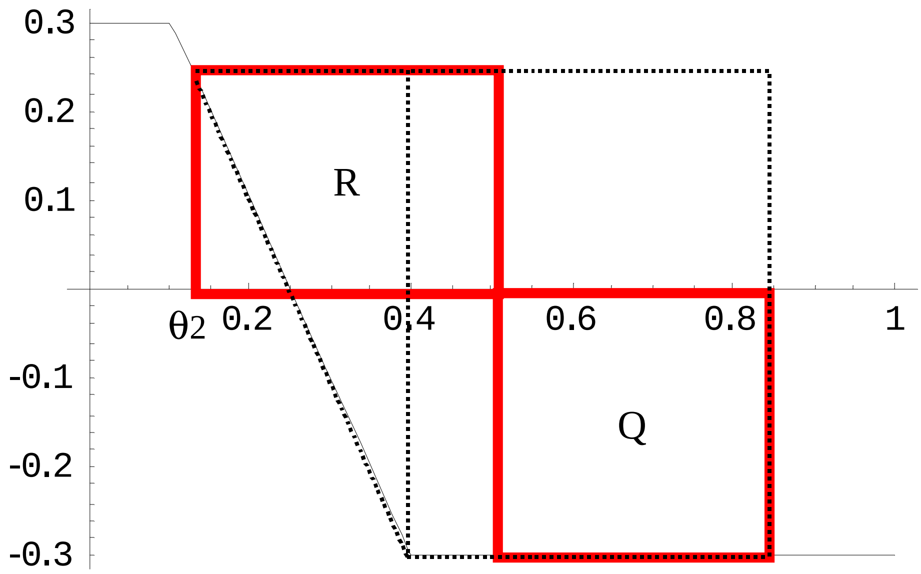


Figure 9: Non-flooded coalition: The incumbent pays voters in the dashed line region. For this strategy to be optimal, it must be that the red regions R and Q have equal size.

Proof. Denote the amount the incumbent adds to the utility of his coalition be y, the measure of voters he adds to the coalition be x, and let λ be the total budget. The amount the challenger needs to win is

$$L(x, y, \lambda) = (\theta_C - \theta_I + y)\left(\frac{1}{2} - \theta_I + x\right) - \lambda\left(y\left(1 - \theta_I\right) + x\left(2\theta_C - 2\theta_I + y\right) - \gamma\right)$$

where $\gamma = B_I - (1 - \theta_I - \theta_C)^2(\theta_C - \theta_I) + (\theta_C - \theta_I)^2$ is the budget leftover from implementing the non-flooded strategy. Solving for the FOC of $L_x = 0$, we have $y = -2(\theta_C - \theta_I)(\theta_I - x)$. Since by assumption $\max(x) = \theta_I$, this implies $y \leq 0$. Thus there is no interior solution for $x > 0$.

We now consider the two boundary solutions. Define $L1$ as the amount that the incumbent can raise the utility of his coalition, and $L2$ as the interval of voters not in the incumbent coalition that the incumbent can buy.

$$L1 = \left(\frac{\gamma}{1 - \theta_I} + \theta_C - \theta_I\right)\left(\frac{1}{2} - \theta_I\right)$$
$$L2 = \left(\frac{\gamma}{2(\theta_C - \theta_I)} + \frac{1}{2} - \theta_I\right)(\theta_C - \theta_I).$$

Subtracting $L2 - L1 = \frac{\theta_I \gamma}{2 - 2\theta_I} > 0$. Thus buying more voters is the optimal strategy. ■

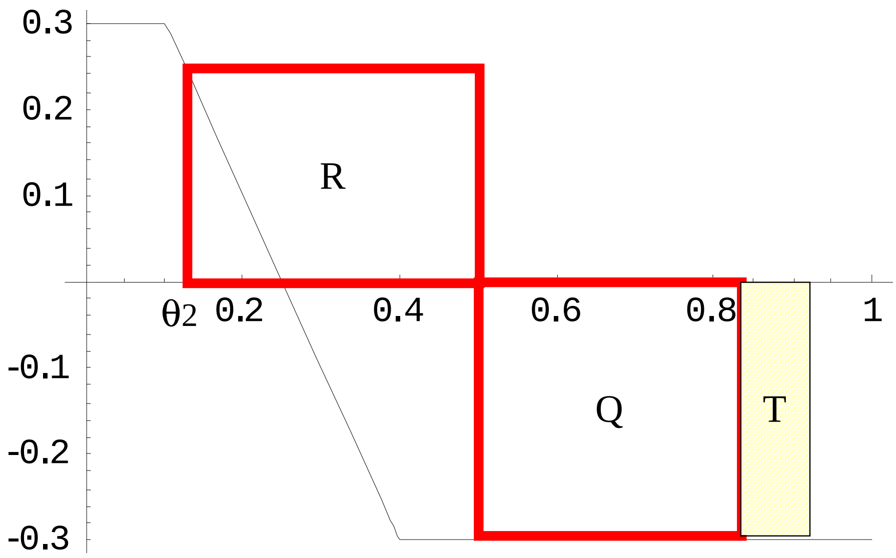


Figure 10: Non-flooded coalition (b)

In this case, the amount the challenger requires for a given θ_C is $t(\theta_{I,}\theta_C, B_I) = \frac{B_I-(\theta_C-\theta_I)^2}{2(\theta_C-\theta_I)} - \left(\frac{1}{2}-\theta_C\right), (\theta_C - \theta_I)$. The challenger therefore wishes to minimize this amount,

$$\begin{aligned} u(\theta_I, \theta_C, B) &= F(B_C > t(\theta_I, \theta_C, B_I)) \\ &= 1 - F\left(\left[\frac{B_I - (\theta_C - \theta_I)^2}{2(\theta_C - \theta_I)} - \left(\frac{1}{2} - \theta_C\right)\right](\theta_C - \theta_I)\right) \end{aligned}$$

So the challenger wishes to minimize $F(B_C > t(\theta_I, \theta_C, B_I))$. Taking FOC with respect to θ_C gives $-\frac{1}{2} + \theta_C \equiv 0$, thus $\theta_C = \frac{1}{2}$.

4.5 Lame Duck

There is the one more region in the model to complete the solution space. It occurs when the incumbent is too far away from the median and does not have enough money to buy over the median voter, and thus loses for sure if the challenger stands within the winset. To defeat a challenger who has no money, the incumbent needs to have a budget of size

$$B_I > \left(\frac{1}{2} - \theta_C\right)(\theta_C - \theta_I) + \frac{1}{4}(\theta_C - \theta_I)^2$$

The challenger chooses $\theta_C = \frac{1}{3}(\theta_I + 1)$ to maximize B_I, which implies that it must be that $B > \frac{1}{3}(\frac{1}{2} - \theta_I)^2$ to defeat any challenger.

I have thus characterized θ_C^* if the challenger had full information about the budget of the incumbent. Figure 11 presents the optimal ideological location for the incumbent given θ_I and B_I. Except for the non-flooded regions, I was able to find the analytical solution for the optimal challenger location. However, it is only in the non-flooded region where θ_C^* varies.

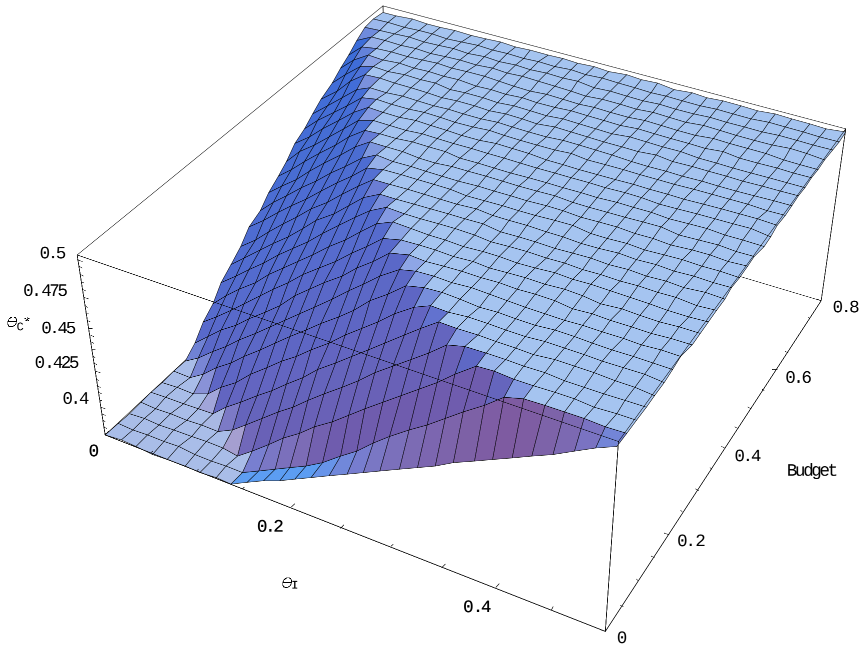


Figure 11: θ_C^* given budget of incumbent and θ_I.

4.6 Optimal Candidate Ideological Position

Because some of the cases were only solved numerically, I was not able to find the optimal expected analytically. Figure 12 shows the numerical solution to the optimal challenger location in the game.

The figure shows something very intuitive. Namely, that the challengers trades off the tension of building a bigger coalition by locating himself closer to the incumbent, versus maximizing the amount that the incumbent has to pay his winning coalition by locating

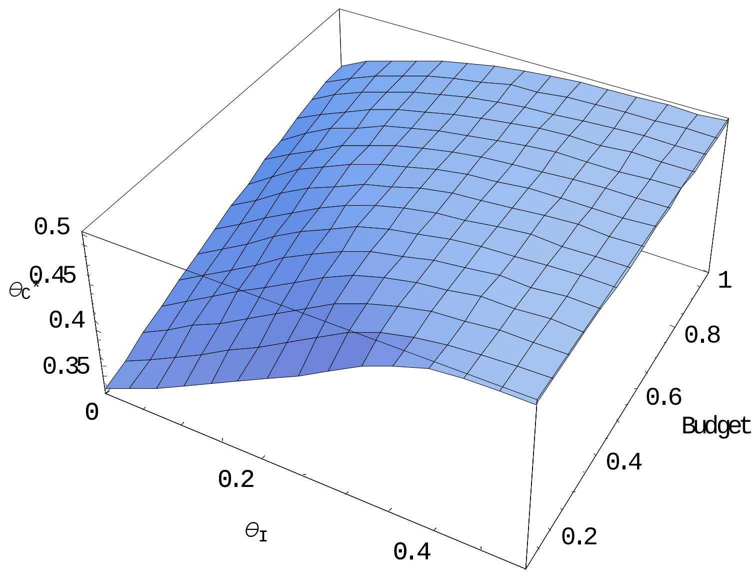


Figure 12: Optimal θ_C^* given θ_I and distribution of B_I.

closer to the median. The numerical results show that $\theta_C^*(\theta_I + \varepsilon, T_I) > \theta_C^*(\theta_I, T_I)$, and that $\theta_C^*(\theta_I, T_I + \varepsilon) > \theta_C^*(\theta_I, T_I)$.

It also shows that, in contrast to the case where the challenger budget equals 0, a challenger with transfers immediately moves to the median of the distribution under a wide range of parameters. It is also interesting that under quadratic preferences, when the incumbent is able to operate a universal buying rule, the challenger moves to the mean of the distribution instead of the median. This suggests that for any order statistic, we can also find some utility function that will induce the challenger to move toward that location.

5 Conclusion

In this paper I explored some implications to the spatial model if we introduce utility for private goods, e.g., transfers to the utility function of the voters. I find that this addition introduces a number of useful features into the standard model. The present model is able to pin down the optimal location of the incumbent in multi-dimensional ideological space where the core does not exist. It also creates ideological separation in both the single-dimensional and multi-dimensional case, although admittedly in the single dimensional case the incumbent is not acting optimally by location off the core. The model also shows that the incumbent will have an incentive to create super-majorities, a la Groseclose and Snyder, where there may exist voters who do receive no transfers.

One major area of this model which I did not pursue was to allow voter utility functions to differ. Throughout the model I have assumed that all voters have identical utility functions. Indeed the original motivation of this model was the conjecture that by allowing voter utility functions to differ, specifically a world where there are differences in the relative marginal utility for ideological location and transfers, candidates will exploit this difference by locating closer to the voters with high valuation for ideology and give more transfers to those with high valuation for transfers. That model however, proved to have several complications, the most important one being that in a sequential world like the present one, there is always a tendency to move toward to majority, regardless of differences of the utility functions. I.e., the candidates will both locate near *and* give transfers to voters with low(high) relative valuation for transfers, so long as they constitute a majority. Moving to a simultaneous world runs immediately to the "Colonel Blotto" problem where the Nash equilibrium is unsolvable for all practical considerations.

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