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## DYNAMIC URBAN MODELS: AGGLOMERATION AND GROWTH

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## Abstract

Theoretical models of urban growth are surveyed in a common framework. Exogenous growth models, where growth in some capital stock as a function of investment is assumed, are examined first. Then endogenous growth models, where use of some factor by a firm increases the productivity of other firms, are studied. These are all modes with perfect competition among agents. Next, models with imperfect competition are discussed. There are two varieties: those employing a monopolistic competition approach to product differentiation, and those employing explicit externalities but lacking some markets. Finally, avenues for future research are explored. Correlations between agglomeration and growth in the various models and data are compared.

Keywords: Agglomerative Activity; Marshallian Externalities; Matching; Urban Growth

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## 1. Introduction

Over the past two centuries, long-term trends of urbanization of population and sustained economic growth across both developed and developing countries are clear. This urbanization trend features an on-going increase in the number as well as the size of cities.<sup>1</sup> The observation is robust despite some sharp differences in the patterns of agglomeration, particularly between U.S. and Asian/European cities in the past three or four decades.<sup>2</sup> What are the determinants of the rates city growth and the speed of spatial agglomeration? Why do some cities rise and some fall in the process of economic development?<sup>3</sup> In spite of the lack of a complete microeconomic structure mimicking the real world economy, various dynamic urban models have attempted to address these important issues. The primary purpose of this critical survey is to synthesize the existing literature on agglomeration and growth so as to promote better understanding of the underlying driving forces of spatial agglomeration and the channels through which agglomerative activity fosters urban growth.

There have been several recent developments uncovering key determinants of spatial agglomeration as well as establishing empirical regularities concerning cities and growth. As stressed by Marshall (1895), Kuznets (1962), Pred (1966), and Jacobs (1969), knowledge spillovers are one primary force for agglomerative activities. Jaffe, Trajtenberg and Henderson (1993) show that knowledge

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<sup>1</sup>For example, based on a consistent, old census definition, the number of American cities with population of 2,500 or more increased from 30 in 1800 to more than 4,000 in 1950. While urban population as a percentage of total population rose from about 6% to 60% over the same period, the population of the largest city, New York, increased from approximately 0.1 million to 10 million.

<sup>2</sup>Among many others, a crucial difference has been the suburbanization trend (within metropolitan areas) in the U.S. since 1960, accompanied by the decay of central cities. This phenomenon is not prevalent in European countries, for instance.

<sup>3</sup>For instance, there have been dramatic changes in the national rankings of U.S. cities over the past century. Notable cities rising in the rankings include Lexington, Los Angeles, Nashville, New Orleans and San Antonio, whereas cities falling in the rankings include Albany, Baltimore, Louisville, Pittsburgh, and St. Louis. In contrast, Eaton and Eckstein (1997) find few new agglomerations added to the stock of 35 cities in France (1876-1990) and 40 in Japan (1925-1985), with *no* cities vanishing (ranked below 50) over the respective time periods.

spillovers are localized in the sense that patents are more likely to cite previous patents from the same area and spillovers can cross national borders only with delay. Glaeser, Kallal, Scheinkman and Shleifer (1992) and Henderson, Kuncoro and Turner (1995) find that spillovers occur both within and between industries and that characteristics of urban areas play a role in the location decisions of industries. While Rauch (1993), Saxenian (1994) and Glaeser (1999) stress the role of cities in transmitting knowledge, Helsley and Strange (2002) describe how agglomeration facilitates innovation by lowering the costs of innovation through a higher density of factor suppliers.

Just what is the relationship between agglomeration and growth in cities? To address this issue, one must go beyond a simple measure of urban population to examine more thoroughly employment and income measures of a given city. Consider the following table.<sup>4</sup>

Table 1.1. Population, Employment and Median Income in Selected Cities

City/Year		1990	2000	Percent Change (%)
Los Angeles	Population	3,485,398	3,694,834	6.0
	Employment	1,670,488	1,532,074	-8.3
	Median Income	40,346	36,687	-9.0
Chicago	Population	2,783,726	2,895,964	4.0
	Employment	1,207,108	1,220,040	1.1
	Median Income	34,314	38,625	12.6
Akron	Population	223,019	217,088	-2.7
	Employment	94,103	99,310	5.5
	Median Income	29,066	31,835	9.5
San Antonio	Population	935,927	1,144,554	22.3
	Employment	389,727	488,747	25.4
	Median Income	30,769	36,214	17.7

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<sup>4</sup>Source: Authors' computations from U.S. Census Bureau data <http://www.census.gov/Press-Release/www/2002/dp.comptables.html> where median income is in 1999 dollars per household. As this is a survey of theory, we will not perform formal econometric tests.

There is a clear positive correlation between city employment and median income, though no clear relation with city population.<sup>5</sup>

To study further the relationship between agglomeration and growth, consider the next table.<sup>6</sup>

Table 1.2. Agglomeration Versus Growth: A First Look

City/Year		1990	2000	Percent Change (%)
Los Angeles	City Share of County Employment	.397	.388	-2.3
	Median County Income	45617	42189	-7.5
Chicago	City Share of County Employment	.500	.500	0
	Median County Income	42627	45922	7.7
Akron	City Share of County Employment	.4	.38	-.05
	Median County Income	37830	42304	11.8
San Antonio	City Share of County Employment	.78	.82	5.1
	Median County Income	33824	38328	13.3

If the change in city share of employment is a proxy for the change in agglomeration, and the change in real median income is a proxy for growth, then the link between agglomeration and growth is clearly more complicated than what we see from the first table. In the second table, there is no apparent link between agglomeration of jobs and growth.

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<sup>5</sup>Obviously, one could use metropolitan area statistics, but this tends to mask migration of population and employment from the city to the suburbs. For example, 2000 census data provided in <http://www.census.gov/population/cen2000/phc-t3/tab02.pdf> shows us that the population of St. Louis increased by 4.5% in the last decade, while we know that population has exited St. Louis city at a rapid rate. The next table addresses some of these issues in more detail.

<sup>6</sup>Source: Authors' computations from U.S. Census Bureau data <http://www.census.gov/Press-Release/www/2002/dp.comptables.html> where median income is in 1999 dollars per household.

Consider next the general trend of urbanization since 1950 in the following table.<sup>7</sup>

Table 1.3. Urbanization Trend in the U.S., 1950-1990

Year	Percent of Population that is Urban
1950	64.0
1960	69.9
1970	73.6
1980	73.7
1990	75.2

So we can see that population has been moving from rural to urban areas, resulting in agglomeration at least in a macro sense. Between the 1980 census and the 1990 census, 137 cities with population greater than 100,000 experienced population growth, while 56 experienced population loss.<sup>8</sup>

Conceptually, we regard a city as a “settlement that consistently generates its economic growth from its own local economy” (cf. Jacobs 1969, p. 262). As both the economic growth of cities and the agglomeration of economic agents are apparently endogenous, a formal model containing these endogenous variables, as well as exogenous variables, is required to understand how they are related. Of course, differences in exogenous variables across cities can help explain different patterns of agglomeration and growth. Our purpose here is to provide an integrative and simple framework for studying the agglomeration of economic agents and the growth of city economies.

We limit the scope of our survey to models of real dynamics and growth. By this, we mean that there is a variable, usually the change in a capital stock, controlled by an agent in the current period that affects feasible allocations in the future. Models not falling into this description include those that examine disequilibrium migration dynamics, for instance stability of a static migration equilibrium.

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<sup>7</sup>Source: <http://www.census.gov/population/censusdata/table4.pdf>

<sup>8</sup>Source: <http://www.census.gov/population/censusdata/c1008090pc.txt>

Another class not falling into this description is the class of models that are essentially static in nature, but some exogenous parameter such as population is presumed to grow at a rate related to time. The static equilibrium is examined in each period. Such models are essentially an extended comparative statics exercise, as the change in the parameter is not the result of the choice of an agent.

In Section 2, we develop an integrated canonical Walrasian framework, synthesizing a variety of conventional models of exogenous urban growth studied by Fujita (1976, 1982), Anas (1978, 1992), Kanemoto (1980), Henderson and Ioannides (1981) and Miyao (1982). We characterize the steady-state equilibrium using three approaches: (i) the Solow-Swan aggregate production approach, (ii) the Allais-Phelps golden rule solution, and (iii) the Ramsey-Cass-Koopmans optimal exogenous growth framework. By utilizing an integrated setup, we allow easy cross-model comparisons.

In Section 3, we generalize the integrated framework constructed in Section 2 to allow for endogenous growth so that we can examine thoroughly two-way dynamic interactions between spatial agglomeration and urban development. In particular, we focus on models with perfect competition, including Palivos and Wang (1996), Eaton and Eckstein (1997), Black and Henderson (1999), Lin, Mai and Wang (2002), Rossi-Hansberg and Wright (2003). We fully explore a simple one-sector framework that highlights the trade-off between increasing returns from production externalities (a centripetal force of city formation) and transportation costs (a centrifugal force), followed by an illustration of a more general two-sector setup.

In Section 4, we consider dynamic urban growth models without a perfectly competitive labor, intermediate good, or consumption good market. The first part of this section introduces imperfect market structures, concentrating in particular on the case of monopolistic competition. Different from the perfect competition setup, these models consider increasing returns from differentiated products as the centripetal force of city formation. The second part of this section shifts attention to non-Walrasian setups that incorporate market frictions into the urban growth framework. This is motivated by its useful implications for urban employment, as illustrated in Helsley and Strange (1990), Abdel-Rahman and

Wang (1995), Coulson, Laing and Wang (2000), and Brueckner and Zenou (2003). Specifically, we focus on a matching model of agglomeration and growth based on a recent study by Berliant, Reed and Wang (2003), where the role of horizontal knowledge exchange in spatial interactions is explored. We illustrate why improvements in the effectiveness of knowledge exchange may lead to higher growth and examine the channels through which an increase in the rate of economic growth may lead to spatial agglomeration. The conclusions obtained herein will lend theoretical support to the empirical facts presented in Section 1.

In Section 5, we provide critiques of the literature and avenues for future research. In particular, we elaborate on the size and number of black boxes in the models mentioned in Sections 3-6. Also, we would like to point out a problem with the definition of social optimum compared to that of Pareto optimum, and how this might be fixed. Finally, we discuss ways to formulate testable hypotheses that can distinguish between the different models. Such empirical tests may help explain the workings of urban growth models in explaining the dynamic process of city formation and development.

## **2. From Solow-Swan to Ramsey Urban Growth Models**

Modern growth theory was spurred by two seminal contributions by Harrod (1939) and von Neumann (1937) that provide mathematical treatment to long-run economic growth in a Ricardian economy. Harrod's steady-state equilibrium is inherently unstable, as it requires a knife-edge condition specifying a fixed relationship between three exogenous constants (the fixed capital-output ratio, the constant population growth rate and the exogenous saving rate) that can hold true only for a set of parameters of measure zero. This undesirable property gave rise to the birth of neoclassical growth theory, led by Solow (1956) and Swan (1956), allowing a variable capital-output ratio using a neoclassical aggregate production function to generate a well-defined nondegenerate steady-state equilibrium. Over the next three decades, this aggregate production function was the heart of exogenous growth theory.



In this section, we develop an integrated framework, synthesizing a variety of urban growth models. We characterize the steady-state equilibrium using the Solow-Swan aggregate production approach and the golden rule solution as well as the optimal exogenous growth framework. The primary purpose of utilizing an integrated setup is to enable cross-model comparison in a parsimonious manner.

### 2.1. The Aggregate Production Approach to Urban Growth

Consider a stylized neoclassical production function with final goods output ( $Y$ ) produced using reproducible physical capital ( $K$ ) and raw labor ( $N$ ):  $Y = F(K, L)$ , where effective labor ( $L = AN$ ) is raw labor augmented by a Harrod-neutral (labor-augmenting) technology and  $A > 0$  is a scaling factor reflecting the current state of technology. The production function  $F$  is strictly increasing and strictly concave in each argument, is continuously differentiable with derivatives denoted by subscripts, and satisfies constant-returns-to-scale ( $F(aK, aL) = aF(K, L)$  for all  $a > 0$ ), a boundary condition ( $F(0, 0) = 0$ ), and Inada conditions ( $\lim_{K \rightarrow 0} F_K(K, L) = \infty$ ,  $\lim_{L \rightarrow 0} F_L(K, L) = \infty$ ,  $\lim_{K \rightarrow \infty} F_K(K, L) = 0$ , and  $\lim_{L \rightarrow \infty} F_L(K, L) = 0$ ). Using constant returns, we can write the output per worker in efficiency units ( $y = Y/L$ ) as a well-defined function of the capital-effective labor ratio ( $k = K/L$ ):  $y = f(k)$ , where  $f(k) = F(k, 1)$ .

Let  $S$  denote aggregate savings,  $I$  gross investment,  $\gamma$  the (constant) rate of technical progress, and  $n$  the (constant) rate of population growth. The Solow-Swan model can be summarized by the following three fundamental relationships (all in per worker forms):

- (i) (fixed savings rate,  $s$ )  $S/L = sf(k)$ ;
- (ii) (full employment)  $\dot{L}/L = n + \gamma$ ;
- (iii) (capital accumulation)  $\dot{k}/k = I/L - n - \gamma$ .

A loanable funds market equilibrium requires that savings per worker equals investment per worker, that is,  $S/L = I/L$ , or,

$$\dot{k} = sf(k) - (n + \gamma)k \quad (2.1)$$

In the steady-state, capital per effective unit of labor reaches a constant (i.e.,  $\dot{k} = 0$ ). It follows immediately that

$$sf(k) = (n + \gamma)k \quad (2.2)$$

determining the steady-state equilibrium level of capital accumulation.

In the 1960s and 1970s, urban economists who emphasize the supply-side often adopted the Solow-Swan framework (see a long list of papers cited by Miyao 1987). It is not the purpose of this paper to present individual contributions of this vintaged literature. Rather, we would like to highlight the spirit of this conventional urban growth framework by simply modifying two fundamental relationships. Consider an area of interest - a city, a community or a region. Let  $w$  and  $r$  denote the (real) wage rate (in effective units) and the (real) interest rate, respectively, and  $w^A$  and  $r^A$ , the (exogenous) corresponding rates outside the area of interest (sometimes referred to as the national average or the agricultural/rural rates). In an open city setup, we can postulate that labor migration depends on the wage gap  $((w - w^A)/w^A)$  whereas capital flows are determined by the interest rate differential  $(r - r^A)$ . Then we have:

$$\dot{N} = nN + b^N(w/w^A - 1)N \quad (2.3)$$

$$\dot{K} = sF(K, AN) + b^K(r - r^A) \quad (2.4)$$

Thus, (2.1) becomes:

$$\dot{k} = sf(k) - [(n + \gamma) + b^N(w/w^A - 1) - b^K(r - r^A)]k = sf(k) - (v + \gamma)k \quad (2.5)$$

where  $v = I/K = n + b^N(w/w^A - 1) - b^K(r - r^A)$  measures the investment ratio. Equation (2.5) can be used together with the envelope conditions ( $w = f(k) - kf_k(k)$  and  $r = f_k(k)$ ) to yield the following:

$$\left(s - \frac{b^N}{w^A}k\right)f(k) + \left(b^K + \frac{b^N}{w^A}k\right)f_k(k)k = [(n + \gamma) + b^Kr^A - b^N]k \quad (2.6)$$

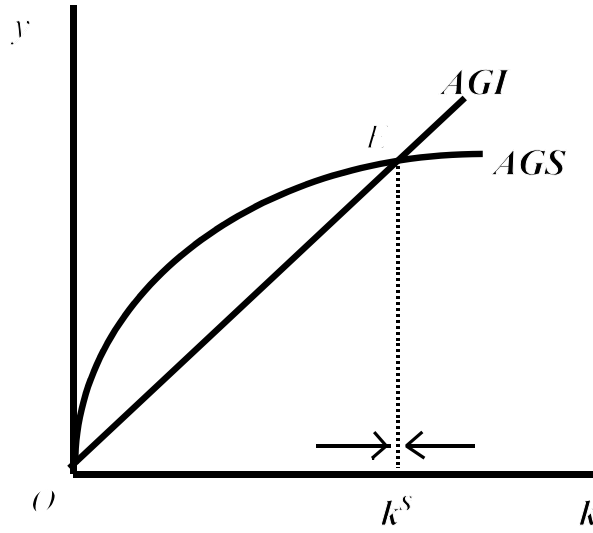


Figure 2.1. Steady-State Equilibrium in the Solow-Swan Model of Urban Growth

It is convenient to call the LHS of (2.6) “adjusted gross savings” (AGS) and the RHS “adjusted gross investment” (AGI), which can be plotted in Figure 2.1 to pin down the steady-state equilibrium value of capital per effective labor unit ( $k^S$ ). A useful example is the Cobb-Douglas production function with  $f(k) = k^\alpha$ , where the capital income share,  $\alpha \in (0, 1)$ , is constant over time. In this case, the steady-state equilibrium solution can be conveniently derived as:  $k^S = [s/(v+\gamma)]^{1/(1-\alpha)}$ . Next we discuss the comparative statics of this model. We can see that the AGS locus shifts out when the saving rate ( $s$ ) increases, the outside alternative wage ( $w^A$ ) is higher, or the state of the city is more sensitive to capital flows but less to labor migration (higher  $b^K$ , lower  $b^N$ ). On the other hand, the AGI locus rotates clockwise in response to 1) a decrease in the population growth rate ( $n$ ), or 2) a decrease in the outside alternative interest rate ( $r^A$ ), or 3) a decrease in the sensitivity of the city to capital flows (lower  $b^K$ ), or 4) to an increase in the sensitivity to labor migration (higher  $b^N$ ). In all of these cases, equilibrium achieves greater accumulation of capital and per capita output ( $\Delta y$ ). When the rate of technical progress ( $\gamma$ ) increases, however, the steady-state value of output per effective unit of labor ( $y$ ) drops, but per capita output rises.

Since per capita output depends solely on the rate of technical progress, any growth effects discussed above are more precisely short-run (transitional) effects. Thus, it is important to examine the stability properties of the model. Conventionally, it is widely claimed that the steady-state equilibrium of urban growth is a saddle (cf. Miyao 1987). We would like to point out such a claim is unfortunately incorrect. To see this, differentiate (2.5) with respect to  $k$  and manipulate it by utilizing the steady-state equilibrium relationship (2.6) to obtain:

$$\frac{d\dot{k}}{dk} = \left( b^k + \frac{b^N}{w^A} k \right) f_{kk} k - b^N - \frac{s}{k} w < 0 \quad (2.7)$$

This implies the steady-state equilibrium is a sink (globally stable). That is, capital per effective labor will always converge monotonically to its steady-state value.

In summary, we conclude this subsection with the following comparative statics:

- (i) a higher savings rate or a lower population growth rate promotes a city's capital accumulation, thereby fostering short-run urban growth in per capita output;
- (ii) while an increase in outside wages reduces net population growth and raises city growth, an increase in outside interest rates lowers the supply of local funds and suppresses short-run growth;
- (iii) when the city's wages fall below the national average or its interest rates exceed the national level, less barriers to interregional labor migration or capital flows tend to increase per capita output growth in transition;
- (iv) in contrast with the above-mentioned factors, local technical progress advances the city's output growth not only in transition, but in the long run;
- (v) all regions adopting the same technology converge to the same steady-state in per capita output.

It is worthwhile to discuss the consequences of incorporating agglomeration economies or urban congestion next. First, should there be scale economies in the production function, all but the result concerning interregional migration remain qualitatively unchanged. While the presence of local scale

economies favors a more centralized interregional migration policy, the presence of global scale economies tends to discourage such a policy. Second, what urban congestion (in the form of traffic or pollution) add to the basic framework is the social cost associated with capital accumulation and population growth. Thus, the consideration of congestion provides a justification for a more centralized interregional migration policy.

## 2.2. The Golden Rule Solution

This supply-oriented neoclassical growth model relies entirely on the mechanics of the aggregate production technology without an explicit account of a representative agent's optimizing behavior. To remedy this problem, one may apply the golden rule solution (or a "maximum maximorum") proposed by von Neumann (1937), Allais (1947) and Phelps (1966). Specifically, the golden rule is reached when the steady-state level of consumption per worker ( $c$ ) is maximized. Utilizing the steady-state equilibrium relationship derived from the supply-oriented problem (equation (2.5) with  $\dot{k} = 0$ ), we have:

$$\max_k c = A[f(k) - (v + \gamma)] = A\{f(k) - [(n + \gamma) + b^N(w/w^A - 1) - b^K(r - r^A)]k\} \quad (2.8)$$

The first-order condition is:

$$f_k(k) = v + \gamma = (n + \gamma) + b^N(w/w^A - 1) - b^K(r - r^A) \quad (2.9)$$

It is easily seen from (2.9) that the golden rule accumulation of capital per effective worker ( $k^G$ ) may be higher or lower than the Solow-Swan steady state ( $k^S$ ), depending on whether the city's net interest income exceeds local savings or not. Under the Cobb-Douglas production function, the golden rule solution for capital accumulation is:  $k^G = [\alpha/(v + \gamma)]^{1/(1-\alpha)}$ . Thus, the Solow-Swan steady state over-accumulates capital relative to the golden rule if the exogenous savings rate is higher than the capital income share. How would this open city compare to a closed city in the absence of labor migration or capital flows? The latter case is equivalent to assuming that  $w = w^A$  and  $r = r^A$ . From (2.9), we can see that an open city reaches a higher level of output per capita if the local interest rate is higher than the

national average and the local wage rate is lower than the national average. Finally, we substitute the envelope conditions ( $w = f(k) - kf_k(k)$  and  $r = f_k(k)$ ) into (2.9) to obtain:

$$-\frac{b^N}{w^A} f(k) + \left(1 + b^K + \frac{b^N}{w^A}\right) f_k(k) k = (n + \gamma) + b^K r^A - b^N \quad (2.10)$$

By comparing (2.10) with (2.6), one can see that except for the result concerning changes in the exogenous savings rate, all the comparative statics obtained in Section 2.1 remain qualitatively valid for the golden rule solution.

### 2.3. The Optimal Exogenous Growth Framework

An obvious problem associated with the golden rule solution is that individuals ignore the time path of consumption, focusing on nothing but its steady-state level. When the rate of subjective time preference is positive, the solution is misleading, as it ignores an intertemporal cost of physical capital investment. This shortcoming has been noted by Cass (1965) and Koopmans (1965), who suggest a return to a pivotal but largely overlooked work by Ramsey (1928). Since then, this optimal exogenous growth model has become the predominant framework in urban growth theory (e.g., see Fujita 1976, 1982, Anas 1978, 1992, Kanemoto 1980, Henderson and Ioannides 1981 and Miyao 1987).

Specifically, denote by  $\rho > 0$  the subjective time preference rate and  $u(c)$  as the felicity function. Taking factor prices as parametrically given, the representative agent in the city of interest faces the following optimization problem:

$$\begin{aligned} \max_c \quad & U = \int_0^\infty u(c) e^{-\rho t} dt \\ \text{s.t.} \quad & \dot{k} = f(k) - (v + \gamma)k - \frac{c}{A} \end{aligned} \quad (2.11)$$

where  $A(0) = 1$ ,  $K(0)/N(0) = k_0 > 0$  is given, and recall that  $v = n + b^N(w/w^A - 1) - b^K(r - r^A)$ . Thus,

what concerns the individual is the lifetime utility ( $U$ ) that is associated with the properly discounted valuation of the path of consumption over their entire life span.

Denoting  $\lambda$  as the co-state variable associated with the capital evolution equation of the city (2.11), we can set up the current-value Hamiltonian as:

$$\mathcal{H} = u(c) + \lambda [f(k) - (v + \gamma)k - c/A] \quad (2.12)$$

Straightforward application of the Pontryagin Maximum Principle yields:

$$Au_c(c) = \lambda \quad (2.13)$$

$$\frac{\dot{\lambda}}{\lambda} = \rho + v + \gamma - f_k(k) \quad (2.14)$$

While the first-order condition (2.13) ensures intertemporal consumption efficiency, the Euler equation (2.14) governs the evolution of the shadow price of capital. The transversality condition is:

$$\lim_{t \rightarrow \infty} \lambda(t)k(t)e^{-\rho t} = 0 \quad (2.15)$$

Totally differentiating (2.13) and utilizing (2.14) to eliminate the co-state variable, we obtain the “Keynes-Ramsey equation”:

$$\dot{c} = \sigma(c)c[f_k(k) - (\rho + v)] \quad (2.16)$$

where  $\sigma(c) = -\frac{u_c}{u_{cc}c} > 0$  measures the Fisherian intertemporal elasticity of substitution. That is, consumption per worker grows over time if the marginal product of capital exceeds its user cost ( $\rho + v$ ).

Equations (2.11) and (2.16) constitute the dynamical system of this optimal urban growth model in  $(c, k)$  space. In the steady state ( $\dot{k} = 0$  and  $\dot{c}/c = \gamma$ ), we get:

$$f_k(k) = [\rho - (1 - \sigma^{-1})\gamma] + z = (\rho + n + \sigma^{-1}) + b^N(w/w^A - 1) - b^K(r - r^A) \quad (2.17)$$

$$c = A[f(k) - zk] = Af(k) - A[(n + \gamma) + b^N(w/w^A - 1) - b^K(r - r^A)]k \quad (2.18)$$

Equation (2.17) is the modified golden rule that equates the marginal product of capital with its user cost, whereas (2.18) equates net output per worker with consumption per worker. The dynamics of the system can be depicted in Figure 2.2 below.

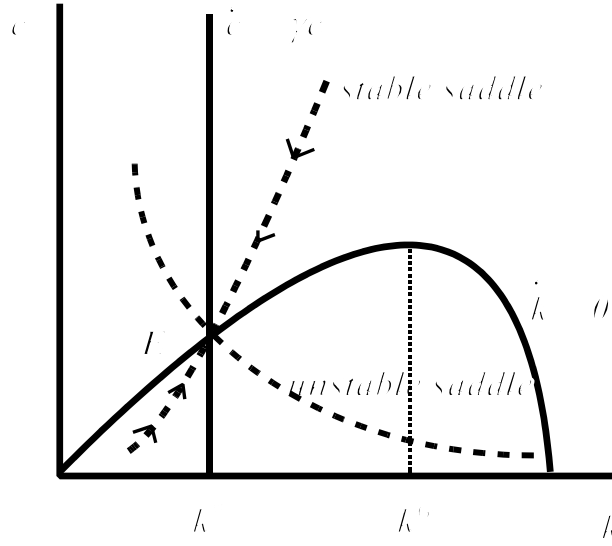


Figure 2.2. Equilibrium Dynamics in the Optimal Urban Growth Model

Equation (2.16) clearly indicates that the dynamics of consumption per worker evaluated at the steady state is independent of  $c$ , so the  $\dot{c} = \gamma c$  locus is vertical. Totally differentiating (2.11) and evaluating it at the steady state yields a hump-shaped  $\dot{k} = 0$  locus, where the peak corresponds to the conventional golden rule solution of capital per effective worker ( $k^G$ ). The intersection of these two loci (point E) determines the steady-state optimal growth equilibrium value of capital per effective worker ( $k^*$ ). Notice that along the unstable saddle, either the nonnegativity constraint on consumption per worker ( $c > 0$ ) or the transversality condition (2.15) will be violated. The dynamic equilibrium is thus uniquely determined by the stable saddle along which  $c$  (the control variable) must jump to the corresponding value for each given value of  $k$  (the state variable).

Comparing (2.17) with (2.9), one can see that the former has an addition of the term  $\rho - (1-\sigma^{-1})\gamma$ , which is positive under the transversality condition. By diminishing marginal product, it follows immediately that relative to the optimal growth solution, the golden rule solution must over-accumulate physical capital in the steady state and is hence dynamically inefficient. In addition to previous



comparative-static results (which continue to hold qualitatively), we also find that either an increase in the time preference rate or a decrease in the intertemporal elasticity discourage steady-state capital accumulation and lowers per capita output. As before, the rate of output growth is solely driven by the exogenous technical progress rate. Furthermore, we can characterize the optimal saving rate in the city, which falls (rises) with capital per worker if the intertemporal elasticity of substitution is sufficiently high (low) relative to the capital income share,  $kf_k(k)/f(k)$ .

In spite of its substantial influence on the theory of urban dynamics, exogenous growth models have several shortcomings. For brevity, we only discuss three. First, since the rate of exogenous technical progress is the lone driving force of the advancement of the city economy, little has been added to better understanding of the determinants of city growth. Second, along the stable saddle, cities with the same technology converge monotonically to the steady state, thus failing to explain why some cities rise but others fall. Finally, with regard to urban policy issues, all the instruments can only affect city growth in transition unless they can affect the rate of technological change.

### **3. From Exogenous to Endogenous Urban Growth Models**

Since the seminal work by Romer (1986), growth theorists have revived interest in long-run development by devoting efforts towards understanding the underlying forces for economic advancement under the so-called endogenous growth framework. A central feature of the endogenous growth model is to consider the marginal product of all reproducible factors (including human, knowledge, physical and research capitals) to be bounded below by a positive constant. This is ensured when the aggregate production function exhibits:

- (i) constant returns (cf. Rebelo 1991, Bond, Wang and Yip 1996, and Benhabib, Meng and Nishimura 2000),
- (ii) asymptotically constant returns (cf. Pitchford 1960 and Jones and Manuelli 1990),
- (iii) or increasing returns (cf. Romer 1986, Lucas 1988, and Boldrin and Rustichini 1994),

with respect to all reproducible factors simultaneously.

The development of endogenous growth theory based on the one-sector Ramsey-Cass-Koopmans setup consists of important contributions by Romer (1986), Rebelo (1991), and Jones and Manuelli (1990). In these papers, general capital is interpreted as either knowledge capital or a combination of both physical and human capital. When the model exhibits constant returns (cf. Rebelo 1991), the economy jumps instantaneously onto the balanced growth path (BGP) along which consumption, capital and output all grow at a common rate. Yet, even under constant returns, one may obtain very rich growth dynamics in two- or multi-sectoral setting, following the classic work by Uzawa (1965). Typical examples can be found in the two-sector, constant-returns endogenous growth model of Bond, Wang and Yip (1996) or in the two-sector, social constant-returns (private diminishing-returns) endogenous growth model of Benhabib, Meng and Nishimura (2000). In these models, there is a capital good that is a perfect substitute for the final consumption good (normally referred to as physical capital) and a non-consumable pure capital good (usually referred to as human capital, knowledge capital or research capital). As far as research capital (or inventive activity) is concerned, it is natural to permit some degree of monopoly power, as suggested by Shell (1966). Accordingly, one may consider R&D and growth in a monopolistically competitive framework (cf. Romer 1990 and Grossman and Helpman 1991) or in a monopoly setup (cf. Aghion and Howitt 1992). We summarize this literature in the table below.

Table 3.1. Summary of Endogenous Growth Models

Returns to Scale	One-Sector	Multi-Sector
Constant or Asymptotically Constant	Rebelo (1991) Jones and Manuelli (1990)	Bond, Wang and Yip (1996) Benhabib, Meng and Nishimura (2000)
Increasing	Romer (1986)	Lucas (1988) Romer (1990) Grossman and Helpman (1991) Stokey (1991) Aghion and Howitt (1992) Boldrin and Rustichini (1994)

In this Section, we will discuss an array of endogenous growth models of cities, organized by the number of sectors in the model and the way the population size of the city is determined.

### 3.1. A Basic One-Sector Endogenous Urban Growth Model

There is no doubt that the simplest form of the production function satisfying the required property for endogenous growth is the so-called AK-model constructed by Rebelo (1991). Specifically, output is assumed to be linear in a general capital input:  $Y = AK$  with  $A > 0$ . In terms of urban economics where endogenous population is a critical issue, however, this production function fails to capture the implications of an endogenous labor force. Thus, a more appropriate form is that follows the spirit of Romer (1986) is:  $Y = AK^\alpha N^{1-\alpha} \bar{K}^{1-\alpha}$ , where  $A$  is assumed to be a positive constant (i.e., the exogenous technical progress rate  $\gamma$  is set to zero),  $\bar{K} = K$  is the aggregate level of capital in the city and  $\alpha \in (0, 1)$ . Notice that this production function exhibits constant returns in all reproducible factors ( $K$  and  $\bar{K}$ ) and constant returns in all private factors ( $K$  and  $N$ ). While the former ensures a positive marginal product of reproducible capital, the latter guarantees 100% distribution of gross revenues to private factors and zero profit in equilibrium. The term  $\bar{K}$  is designed to capture the Marshallian externality in a given city, created by uncompensated positive spillovers of general capital as suggested by Jacobs (1969). Such spillovers may be consequences of knowledge overlaps, peer-group learning effects, neighborhood externalities and/or invention applications. In the context of urban economics, the general capital stock can be seen as consisting of physical capital (structures) or human capital (knowledge), depending on the application.

To accept balanced growth, consider the felicity function to take the form of constant elasticity of intertemporal substitution ( $\sigma$ ):  $u(C/N) = [(C/N)^{1-\sigma} - 1]/(1 - \sigma)$ , where  $\sigma > 1$ . Since per capita output is equal to  $Y/N = A(K/N)^\alpha (\bar{K}/N)^{1-\alpha} N^{1-\alpha}$ , we can specify the optimization problem of a

representative agent in the urban area as:

$$\begin{aligned} \max_{C/N} \quad & U = \int_0^\infty \frac{(C/N)^{1-\sigma^{-1}} - 1}{1-\sigma^{-1}} e^{-\rho t} dt \\ \text{s.t.} \quad & \left(\frac{\dot{K}}{N}\right) = A\left(\frac{K}{N}\right)^\alpha \left(\frac{\bar{K}}{N}\right)^{1-\alpha} N^{1-\alpha} - \mathbf{v} \frac{K}{N} - \frac{C}{N} \end{aligned} \quad (3.1)$$

where  $K(0)/N(0) = k_0 > 0$  is given, and recall that  $\mathbf{v} = n + b^N(w/w^A - 1) - b^K(r - r^A)$  measures the investment ratio. Moreover, since wages are growing and unbounded with general capital in this setup, it is crucial to assume that the alternative wage will also grow at the same rate as the city's aggregate capital stock so that a BGP is attainable. More explicitly, we let:  $w^A = w_0 \bar{K}$ , which is taken as parametrically given by each individual agent, with the coefficient  $w_0$  to be determined in the balanced growth equilibrium.

Following similar techniques, it is not difficult to show that all per capita quantities (consumption, capital and output) exhibit common growth at rate  $\theta$ , and the modified golden rule can be written as:

$$\left(\frac{\dot{C}}{N}\right) = \sigma\left(\frac{C}{N}\right) [\alpha A N^{1-\alpha} - (\rho + \mathbf{v})] \quad (3.2)$$

That is, the consideration of city-wide positive spillovers creates a “scale effect” in the sense that a larger population size spurs city growth.

In equilibrium, factors must be paid at their marginal products:  $w = (1-\alpha)A N^{-\alpha} K$  and  $r = \alpha A N^{1-\alpha}$ . Thus, we can rewrite the investment ratio as a function of  $N$  alone:

$$\mathbf{v}(N) = (n + b^K r^A - b^N) + [b^N (1-\alpha) A / w_0] N^{-\alpha} - b^K \alpha A N^{1-\alpha} \quad (3.3)$$

which is strictly decreasing and strictly convex in  $N$ . It is useful to note that in the exogenous growth framework, the investment ratio is simply a constant in the steady state. Thus, the dependency of the investment ratio on the size of city population is again a consequence of the scale effect created by the

Marshallian externality.

In the conventional exogenous growth model, the system becomes stationary once all growing variables are divided by the technology scaling factor. In contrast, the technology scaling factor is fixed in the present setup where the source of growth arises endogenously from the accumulation of general capital in the absence of diminishing returns. As a consequence, it is convenient to transform the system by using the “great ratios” - in particular, the consumption-capital ration,  $\chi \equiv C/K$ . Combining (3.1) and (3.2), one gets:

$$\dot{\chi} = \chi [\chi - (1 - \alpha\sigma)\alpha A N^{1-\alpha} + (1 - \sigma)v(N) - \sigma\rho] \quad (3.4)$$

which depends on  $z$  and  $N$ . Next, substituting the equilibrium values of wages into (2.3) yield:

$$\dot{N} = \{(n - b^N) + [b^N(1 - \alpha)A/w_0]N^{-\alpha}\}N \quad (3.5)$$

which is driven by  $N$  alone.

Along a BGP, consumption and capital grow at a common rate and there must be no net migration nor net capital flows interregionally. That is, we must have:  $\dot{z} = 0$ ,  $\dot{N} = nN$ , and  $w_0 = [\alpha^\alpha(1 - \alpha)^{1-\alpha}A(r^A)^{-\alpha}]^{1/(1-\alpha)}$ . We can substitute  $w_0$  into (3.5) to obtain:

$$\dot{N} = \{(n - b^N) + b^N[r^A/(\alpha A)]^{\alpha/(1-\alpha)}N^{-\alpha}\}N \quad (3.6)$$

which implies the rate of city population growth is decreasing in the size of the city, eventually converging to a maximum bound. By straightforward manipulation, the BGP equilibrium values of  $(\chi^*, N^*)$  can be solved:

$$N = [r^A/(\alpha A)]^{1/(1-\alpha)} \quad (3.7)$$

$$\chi = (1 - \alpha\sigma)r^A + (1 - \sigma)n - \sigma\rho \quad (3.8)$$

Equations (3.4) and (3.6) constitute the dynamic system of  $(\chi, N)$ . The local dynamic properties of the BGP equilibrium can be characterized by evaluating the trace and the determinant of the corresponding Jacobian matrix at the BGP values  $(\chi^*, N^*)$ , which are, respectively,

$$\text{trace} = (n - \alpha b^N) + \chi \text{ and } \text{determinant} = (n - \alpha b^N)\chi \quad (3.9)$$

For a given natural rate of population growth  $n$ , the BGP is a saddle if and only if  $n < \alpha b^N$ . That is, saddle path stability requires that (i) the city is responsive to migration (so  $b^N$  is high enough) and (ii) the agglomerative externality from uncompensated spillovers is not too large (so  $\alpha$  is high enough).

Otherwise, when these conditions do not hold, namely when  $n > \alpha b^N$ , the city must either shrink continually or grow unboundedly over time, implying that the BGP equilibrium is a source (locally unstable). The BGP equilibrium and the transition path can be depicted in Figure 3.1 below for the case of  $\sigma < 1$ .

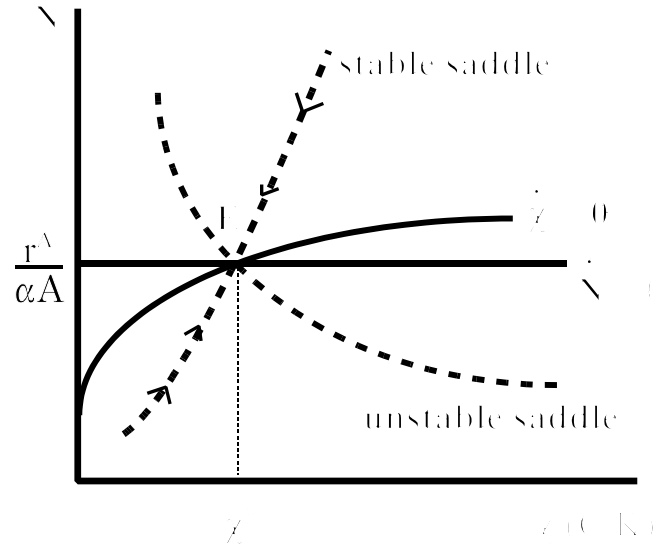


Figure 3.1. Equilibrium Dynamics in the Endogenous Urban Growth Model

### 3.2. A Modified One-Sector Model of Endogenous Urban Growth

The model delineated above is essentially a straightforward extension of the conventional exogenous urban growth model to allow general capital that is not subject to diminishing returns in the long run. While the evolution of city population depends on an ad hoc adjustment process based on interregional wage differentials, the model itself also lacks an explicit spatial structure. These shortcomings will be remedied in this Section, which follows closely the work by Palivos and Wang (1996).

Consider a Lösch-Alonso circular monocentric city framework with completely inelastic demand for land and an exogenous, time-varying city border of  $b(t)$  at time  $t$ . For simplicity, it is assumed that there is no exogenous population growth ( $n = 0$ ), that is, any evolution of city population must be a result of migration. The fixed density of land used by a consumer at a location  $z \in [0, b(t)]$  away from the central business district (CBD)  $m(z) = m$  (for all  $z$ ) and hence the population density is  $2\pi z/m$ . The population identity is:  $N(t) = \int_0^{b(t)} (2\pi z/m) dz$ , or, equivalently,

$$b(t) = [mN(t)/\pi]^{1/2} \quad (3.10)$$

Let all production and transaction activities take place in the CBD. Let  $C(t)$  be the total amount of composite good produced by the economy at time  $t$ . Per capita land use is assumed to be constant across locations, and the only other good consumed by agents is composite good. Hence the gross per capita consumption of composite good, prior to the subtraction of transport cost from the CBD to the consumer's house, for all agents living in all locations is  $C(t)/N(t)$ . Assume that the transportation cost facing each agent takes a modified iceberg form:  $T(z, t) = \tau z C(t)/N(t)$ , where  $\tau > 0$  measures the transportation cost of one unit of consumption good per unit distance. Further assume that the city government sublets the publicly owned land to city residents at the competitively determined rent  $R(z, t)$ . Since all households are identical and the lot size is fixed, locational equilibrium implies  $T(z, t) + R(z, t)$  must be constant for all  $z \in [0, b(t)]$ . By normalizing the rural land rent to zero, we have:

$$R(z, t) = T(b, t) - T(z, t) = \tau [b(t) - z] [C(t)/N(t)] \quad (3.11)$$

which is, as one would expect, decreasing in the distance from the CBD.

Straightforward integration, together with the expression (3.10), yields the total transportation cost (TTC) and the total land rent (TLR):

$$TTC(t) = (2/3) \tau (m/\pi)^{1/2} [N(t)]^{3/2} [C(t)/N(t)] \quad (3.12)$$

$$TLR(t) = (1/3) \tau (m/\pi)^{1/2} [N(t)]^{3/2} [C(t)/N(t)] \quad (3.13)$$

Therefore, the resource constraint for the city economy in per capita form can be specified as:

$$\left(\frac{\dot{K}}{N}\right) = A \left(\frac{K}{N}\right)^\alpha \left(\frac{\bar{K}}{N}\right)^{1-\alpha} N^{1-\alpha} - \left[1 + \tau \left(\frac{m}{\pi}\right)^{1/2} N^{1/2}\right] \frac{C}{N} \quad (3.14)$$

where the last term is the sum of per capita spending, including consumption, transportation cost and land rent for each agent.

Thus, the representative agent's optimization is:

$$\max_{C/N} U = \int_0^\infty \frac{(C/N)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt \quad (3.15)$$

subject to (3.14). The modified golden rule now becomes:

$$\left(\frac{\dot{C}}{N}\right) = \left(\frac{C}{N}\right) \sigma (\alpha A N^{1-\alpha} - \rho) \quad (3.16)$$

Along a BGP, one can utilize (3.14) and (3.15) to compute the lifetime utility of the representative agent, for a given value of  $N$ , as:



$$\mathcal{U}(N) = - \frac{(1/\sigma)^{1/\sigma} [\sigma/(1-\sigma)]}{(k_0)^{(1-\sigma)/\sigma} \{\rho + [(1-\sigma)/\sigma] A \alpha N^{1-\alpha}\}} \left\{ \left[ \frac{1 + \tau(m/\pi)^{1/2} N^{1/2}}{\rho + (\sigma^{-1} - \alpha) A N^{1-\alpha}} \right]^{(1-\sigma)/\sigma} \right\} \equiv -U^D(N) \{U^C(N)\} \quad (3.17)$$

Thus, population size affects lifetime utility through two channels: (i) it raises individual welfare via a positive scale effect on city growth (embedded in  $U^D$  via the effective discount rate  $\rho + [(1-\sigma)/\sigma] A \alpha N^{1-\alpha}$ ) and (ii) it creates an ambiguous effect on welfare via its impact on the initial level of per capita consumption (captured by the term in curly brackets,  $U^C(N)$ ). If the initial consumption effect is locally negative around the BGP, there exists an individual welfare maximizing size of city population.

Let  $G$  denote government spending. One may construct a social optimum where the benevolent (individual welfare maximizing) city government chooses consumption, capital accumulation and population subject to the resource constraint and the (period by period) government budget constraint:

$$\left(\frac{\dot{K}}{N}\right) = A \left(\frac{K}{N}\right)^\alpha \left(\frac{\bar{K}}{N}\right)^{1-\alpha} N^{1-\alpha} - \left[ 1 + \frac{2}{3} \tau \left(\frac{m}{\pi}\right)^{1/2} N^{1/2} \right] \frac{C}{N} - \frac{G}{N} \quad (3.18)$$

$$G = \frac{1}{3} \tau \left(\frac{m}{\pi}\right)^{1/2} N^{3/2} \quad (3.19)$$

where the government spending is entirely financed by the total city land rent (the Henry George approach). The socially optimal consumption growth is governed by the following equation:

$$\left(\frac{\dot{C}}{N}\right) = \left(\frac{C}{N}\right) \sigma (A N^{1-\alpha} - \rho) \quad (3.20)$$

Comparing (3.20) with (3.16), one can easily see that the social rate of return on capital is higher than the private rate of return, thus yielding a higher city growth under the social optimum. Intuitively, this is due to the presence of the free-rider problem in a decentralized equilibrium where individual fails to account for the positive spillover effect of their investment in capital.

Moreover, the lifetime utility along a BGP is given by:

$$U^S(N) = U(N) \left[ \frac{\rho + (\sigma^{-1} - \alpha)AN^{1-\alpha}}{\rho + (\sigma^{-1} - 1)AN^{1-\alpha}} \right]^{(1-\sigma)/\sigma} \equiv U(N)\Upsilon(N) \quad (3.21)$$

Note that provided  $\sigma < 1$ , we have:  $U(N) < 0$ ,  $\Upsilon(N) < 1$  and  $d\Upsilon(N)/dN < 0$ . Manipulation of equations (3.17) and (3.21) thus implies a lower net marginal benefit of city size in the private sense (NMB) than that in the social sense (NMB<sup>S</sup>), as in Figure 3.2. From (3.16) and (3.20), we can also plot the corresponding city growth rates ( $\theta$  and  $\theta^S$ ) as (strictly increasing and strictly concave) functions of city population.

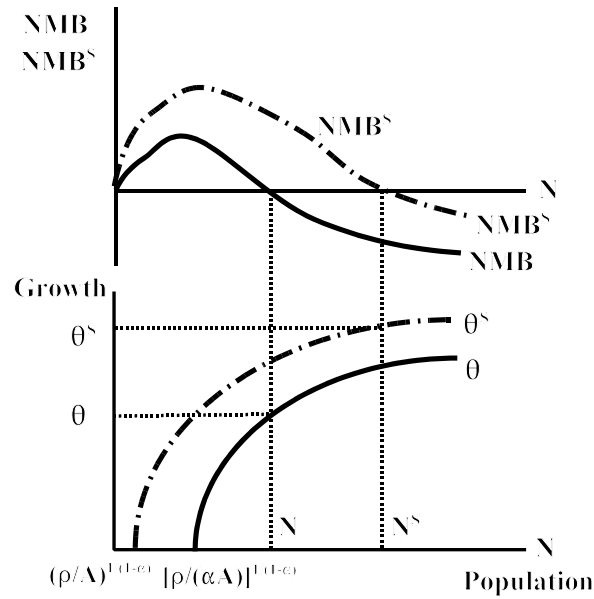


Figure 3.2. Decentralized Equilibrium vs. Social Optimum

In summary, we have: (i) the social welfare and city growth achieved under a decentralized equilibrium is lower than those under social optimum, and (ii) in a decentralized equilibrium, the city is under-populated relative to the social optimum. This latter result is in sharp contrast with the conventional textbook proposition where as a result of a negative traffic congestion externality, equilibrium cities are over-

populated (cf. Kanemoto, 1980, and Fujita, 1989). For an entirely different reason, Abdel-Rahman (1990) finds under-populated cities in a decentralized equilibrium. He argues that individuals fail to account for the benefit from the reduction of the city infrastructure fixed cost spread over residents from an incremental increase in population.

### 3.3. *Housing Dynamics and Zoning*

In the previous subsections, the housing stock was regarded as part of general capital with investment in housing a perfect substitute for the final consumption good. By construction, the relative price of housing (in units of the final consumption good) is unity. In this subsection, we follow the model developed by Lin, Mai and Wang (2001) to allow for a determination and full characterization of intertemporal housing prices within the generalized  $AK$ -framework.

In particular, we modify the model in Section 3.2 by:

- (i) interpreting the consumption good as household consumption, the capital stock as housing capital and the production activity as household production;
- (ii) decomposing housing capital as the product of housing quality (denoted  $Q$ , which can grow unboundedly) and housing quantity (denoted  $q$ , which is bounded above), i.e.,  $K/N = Qq$ ;
- (iii) allowing for the non-productive use of housing capital to augment leisure and enhance utility;<sup>9</sup>
- (iv) considering variable heights of buildings under a zoning policy such that the population density is uniform across locations within the monocentric city;
- (v) eliminating the scale effect by using the “average” housing stock to measure positive spillovers (especially, in the presence of congestion, the use of aggregate housing stock is difficult to justify);
- (vi) focusing on a closed city with migration only within the city and with the city population growing at an exogenous rate  $n$ .

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<sup>9</sup>E.g., gardens.

Letting  $\zeta$  denote the fraction of housing capital devoting to the productive activity, the fraction  $1-\zeta$  of housing capital is then allocated to non-productive home entertainment. Consider the following felicity function of the Cobb-Douglas form satisfying the constant elasticity of intertemporal substitution property:  $u(C/N, (1-\zeta)(K/N)) = \{[(C/N)^\xi((1-\zeta)(K/N))^{1-\xi}]^{1-\sigma^{-1}} - 1\} / (1 - \sigma^{-1})$ , where  $\xi \in (0, 1)$ . That is, housing capital augments leisure time, contributing to greater utility. Next, the household production technology is given by:  $Y = A(\zeta K/N)^\alpha (\bar{K}/N)^{1-\alpha}$ . Moreover, we specify a plausible floor area ratio (FAR) schedule:  $\Psi = \bar{\Psi} - \psi_0 \psi(z)$ , where  $\psi_z > 0$ ,  $\psi_{zz} < 0$ ,  $\psi_0 > 0$ , and  $\bar{\Psi} > \psi(b)$ , for all  $z \in [0, b]$ . As observed in practice, this zoning restriction permits higher buildings closer to the CBD. Under this FAR schedule, the population density at distance  $z$  from the CBD is  $2\pi z/[q(z, t)/\Psi(z)]$ , and, by construction, the population identity requires:  $2\pi z/[q(z, t)/\Psi(z)] = N(t)/b(t)$ . Furthermore, since a greater FAR can cause heavier traffic congestion, we postulate that the unit transportation cost is an increasing function in the FAR:  $\tau = \tau(\Psi)$ .

Denoting by  $p(z, t)$  the relative price of housing investment (in units of the final consumption good) at location  $z$  in time  $t$ , the representative household's intertemporal optimization problem can then be specified as:

$$\max_{C/N} U = \int_0^\infty \frac{\{(C/N)^\xi [(1-\zeta)(K/N)]^{1-\xi}\}^{1-\sigma^{-1}} - 1}{1 - \sigma^{-1}} e^{-\rho t} dt \quad (3.22)$$

$$s.t. \quad \left(\frac{\dot{K}}{N}\right) = \frac{1}{p(z, t)} \left\{ A \left[ \frac{K}{N}(z, t) \right]^\alpha \left[ \frac{\bar{K}}{N}(z, t) \right]^{1-\alpha} - [1 + \tau(\Psi(z))z] \frac{C}{N}(z, t) \right\} \quad (3.23)$$

The tradeoff between consumption and leisure is governed by,

$$\frac{1-\xi}{\xi} \frac{C/N}{(1-\zeta)K/N} = \frac{\alpha A \zeta^{\alpha-1}}{1 + \tau(\Psi)z} \quad (3.24)$$

which yields,

$$\chi = \frac{\xi}{1-\xi} \frac{\alpha A (1-\zeta) \zeta^{\alpha-1}}{1+\tau(\Psi)z} \quad (3.25)$$

The modified golden rule for consumption and productive time allocation are:

$$\left(\frac{\dot{C}}{N}\right) = \left(\frac{C}{N}\right) \sigma \left( \frac{1}{p} \alpha A \zeta^{\alpha-1} - \rho \right) \equiv \left(\frac{C}{N}\right) \sigma \Theta(\zeta, p) \quad (3.26)$$

$$\dot{\zeta} = -\frac{\zeta}{1-\alpha+\zeta/(1-\zeta)} \left[ \Theta(\zeta, p) - A \zeta^\alpha + \frac{\xi}{1-\xi} \alpha A (1-\zeta) \zeta^{\alpha-1} \right] \equiv -\frac{\zeta}{1-\alpha+\zeta/(1-\zeta)} \mathbf{X}(\zeta, p) \quad (3.27)$$

where it is clearly seen that  $\Theta_\zeta < 0$ ,  $\Theta_p < 0$ ,  $X_\zeta < 0$ , and  $X_p < 0$  and that, utilizing (3.23), (3.25) and (3.26),

$$\dot{\chi} = \chi \mathbf{X}(\zeta, p) \quad (3.28)$$

The evolution of the relative price of housing satisfies:

$$\dot{p} = \frac{(1-\alpha)(1-\xi+\sigma^{-1}\xi)+\sigma^{-1}\zeta/(1-\zeta)}{1-\alpha+\zeta/(1-\zeta)} \mathbf{X}(\zeta, p) - \sigma^{-1} \Theta(\zeta, p) \quad (3.29)$$

which is a function of  $\zeta$  and  $p$  alone.

Since households are free to relocate, locational equilibrium implies that at any point in time,

$$u\left[\frac{C}{N}(z, t), (1-\zeta(z, t))\frac{K}{N}(z, t)\right] = u\left[\frac{C}{N}(b, t), (1-\zeta(b, t))\frac{K}{N}(b, t)\right] \quad \forall z \in [0, b(t)] \quad (3.30)$$

Along a BGP, consumption, housing quality, housing capital and output all growth at a common rate  $\theta =$

$\Theta(\zeta, p)$ , which can be written as (recalling (3.26)):

$$\theta = \frac{1}{p} \alpha A \zeta^{\alpha-1} - \rho \quad (3.31)$$

Moreover, along a BGP,  $\dot{\chi} = 0$  and from (3.27) and (3.28),

$$\theta = A \zeta^\alpha - \frac{\xi}{1-\xi} \alpha A (1-\zeta) \zeta^{\alpha-1} \quad (3.32)$$

Normalizing the housing price on the urban fringe to  $p(b) = \bar{p}$  and utilizing the assumption of a uniform distribution of population, the population identity, the definition of housing capital as well as equations (3.25) and (3.26), we can derive the BGP value of housing prices at each location:

$$p(z) = \bar{p} + \frac{\xi}{1-\xi} \frac{\alpha A}{\theta} [B(b,z) - 1] \zeta^{\alpha-1}(z) [1 - \zeta(z)] \quad (3.33)$$

where  $B(b,z) \equiv \frac{K_0(b)}{K_0(z)} = \frac{\Psi(b)b}{\Psi(z)z} \frac{q_0(b)}{q_0(z)}$ , measuring the ratio of the initial capital stocks between  $b$  and  $z$ , and satisfying  $B(z;b) > B(b;b) = 1$ . It can be easily verified that  $B$  is decreasing in  $\Psi_0$  and increasing in

$\bar{\Psi}$ . By the standard “land abundance” argument, we expect that for a given population, an increase in urban fringe drives down housing prices, which is guaranteed by postulating  $dB(z;b)/db < 0$ .

Equations (3.31)-(3.33) jointly determine the BGP values of  $(\theta, \zeta, p)$ . Straightforward comparative-static analysis suggests:

- (i) an increase in urban fringe ( $b$ ), under a given population, encourages the productive use of housing capital, lowers housing prices at all locations and leads to a permanently higher city growth rate;
- (ii) an increase in the unit transportation cost ( $\tau$ ) enlarges the ratio of housing stock at the border to that at any arbitrary location and raises housing prices at all locations, thus reducing the productive use of housing capital and the rate of growth of the city;
- (iii) a loose zoning restriction on the FAR by relaxing it more than proportionately for locations near the city center (higher  $\Psi_0$ ) encourages the productive use of housing capital, lowers equilibrium housing prices and fosters economic growth; however, a uniformly loose zoning restriction on the FAR (higher  $\bar{\Psi}$ ) generates reverse outcomes.

Finally, we turn to examining the stability of the dynamical system summarized by (3.27) and (3.29) in  $(\zeta, p)$  space, because of the recursive nature of the model. By totally differentiating the system

and evaluating the Jacobian at the BGP values, we can compute its determinant and the trace:

$$\text{determinant} = -\frac{\alpha A p \Theta_p \sigma^{-1} \zeta^\alpha}{1-\alpha+\zeta/(1-\zeta)} \left[ 1 + \frac{\xi}{1-\xi} \frac{1-\zeta}{\zeta} \left( 1-\alpha + \frac{\zeta}{1-\zeta} \right) \right] > 0 \quad (3.34)$$

$$\text{trace} = -\frac{1}{1-\alpha+\zeta/(1-\zeta)} \left[ p \Theta_p (1-\alpha)(1-\xi)(\sigma^{-1}-1) + \zeta X_\zeta \right] > 0 \quad (3.35)$$

Notice that in this 2 x 2 system,  $\zeta$  and  $p$  are jump variables that can be adjusted instantaneously. Since the determinant is positive, the two characteristic roots must have the same sign. This together with positive trace implies both roots are positive. Similar to the ad hoc one-sector urban growth model presented in Section 3.1, the BGP equilibrium is locally determinate in this one-sector economy, contrasting with findings in Romer (1986) and Rebelo (1991) where the economy must jump instantaneously to the BGP. Therefore, household relocation and locational equilibrium in an optimizing setting act as stabilizing forces, enabling saddle-path stability of the BGP equilibrium in the city economy. One may wonder whether this stability property continues to hold in a fully specified two-sector framework with heterogeneous non-housing and housing capitals, to which we now turn.

### 3.4. Two-Sector Endogenous Urban Growth and Stability

The materials presented in this subsection extends the original work by Anas, Palivos and Wang (1995) and Chang (1999). Other contributors in this area include Turnovsky and Okuyama (1994), Black and Henderson (1999), Li (2002), and Rossi-Hansberg and Wright (2003), to name but a few. Consider a city economy featuring a spatial structure as delineated in Section 3.2 above. However, there are now two productive sectors: a final (composite) consumption - (non-housing) investment good sector, and a housing investment sector. The focus herein is on the interactions between the two sectors and their consequences for the growth dynamics of the underlying city economy.

Assume that labor, with endowment normalized to one, is inelastically supplied to production of the final good (at a fraction  $\zeta$ ) and housing investment (at a fraction  $1-\zeta$ ) and with housing capital

yielding no direct consumption value. Housing capital ( $H$ ) can be combined with non-housing capital ( $K$ ) to produce the final good. Both reproducible capitals generate sector-specific positive spillovers and both production technologies exhibit constant social returns, taking the Cobb-Douglas form. Specifically, denoting  $x_{ij}$  as the fraction of factor  $i$  allocated to sector  $j$ , the output of sector  $j$  is then given by:

$$Y_j = A_j (x_{Kj} K/N)^{\alpha_{Kj} - \varepsilon_{Kj}} (x_{Hj} H/N)^{\alpha_{Hj} - \varepsilon_{Hj}} (\bar{x}_{Kj} \bar{K}/N)^{\varepsilon_{Kj}} (\bar{x}_{Hj} \bar{K}/N)^{\varepsilon_{Hj}} \quad (3.36)$$

where  $\alpha_{ij}, \varepsilon_{ij} \in (0, 1)$ ,  $i = K, H$  and  $j = 1, 2$ . While  $\alpha_{ij} - \varepsilon_{ij}$  measures the private output elasticity of factor  $i$  in sector  $j$ ,  $\alpha_{ij}$  represents the corresponding social output elasticity with  $\varepsilon_{ij}$  indicating the degree of sector-specific externality. By constant social returns, we have:  $\alpha_{Kj} + \alpha_{Hj} = 1$ . Further, we assume that the two technologies are different in the sense of both private and social returns, i.e., the matrices  $[\alpha_{ij} - \varepsilon_{ij}]$  and  $[\alpha_{ij}]$  are non-singular.

The representative agent's optimization is therefore given by:

$$\max_{C/N} U = \int_0^\infty \frac{(C/N)^{1-\sigma^{-1}} - 1}{1 - \sigma^{-1}} e^{-\rho t} dt$$

subject to the evolution equations for per capita capital and housing:

$$\left(\frac{\dot{K}}{N}\right) = A_1 (x_{K1} \frac{K}{N})^{\alpha_{K1} - \varepsilon_{K1}} (x_{H1} \frac{H}{N})^{\alpha_{H1} - \varepsilon_{H1}} (\bar{x}_{K1} \frac{\bar{K}}{N})^{\varepsilon_{K1}} (\bar{x}_{H1} \frac{\bar{K}}{N})^{\varepsilon_{H1}} - n \frac{K}{N} - \left[1 + \tau \left(\frac{m}{\pi}\right)^{1/2} N^{1/2}\right] \frac{C}{N} \quad (3.37)$$

$$\left(\frac{\dot{H}}{N}\right) = A_2 (x_{K2} \frac{K}{N})^{\alpha_{K2} - \varepsilon_{K2}} (x_{H2} \frac{H}{N})^{\alpha_{H2} - \varepsilon_{H2}} (\bar{x}_{K2} \frac{\bar{K}}{N})^{\varepsilon_{K2}} (\bar{x}_{H2} \frac{\bar{K}}{N})^{\varepsilon_{H2}} - n \frac{H}{N} \quad (3.38)$$

the factor reallocation constraints,  $x_{i1} + x_{i2} = 1$  ( $i = K, H$ ), and the given values of the initial stocks,

$K(0)/N(0)$  and  $H(0)/N(0)$ . To ease the complexity of notation, let  $x_j \equiv x_{Kj}$  and hence  $x_{Hj} = 1 - x_j$ .

Denote the nominal shadow price of factor  $i$  as  $W_i$  and the nominal shadow price of sector  $j$  output as  $P_j$ . Following the techniques developed by Bond, Wang and Yip (1996), we can combine the first-order conditions (with respect to  $C/N$  and  $x_j$ ) and the Euler equations (with respect to  $K/N$  and  $H/N$ ) to



yield:

$$c^{-\sigma} = P_1 \quad (3.39)$$

$$W_i = P_j(\alpha_{ij} - \varepsilon_{ij})A_j(x_{Kj}K/N)^{\alpha_{Kj}}(x_{Hj}H/N)^{\alpha_{Hj}}/(x_{ij}i/N) \quad (3.40)$$

$$\frac{\dot{P}_1}{P_1} = (\rho + n) - \frac{W_K(P)}{P_1} \quad \text{and} \quad \frac{\dot{P}_2}{P_2} = (\rho + n) - \frac{W_H(P)}{P_2} \quad (3.41)$$

for  $i = K, H$  and  $j = 1, 2$ , where  $P = (P_1, P_2)$ .

We now transform the endogenous variables into stationary forms by dividing each of them by their growth factors:  $k \equiv (K/N)/e^{\theta t}$ ,  $h \equiv (H/N)/e^{\theta t}$ ,  $y_j \equiv Y_j/e^{\theta t}$ ,  $w_i \equiv W_i e^{\sigma \theta t}$ , and  $p_j \equiv P_j e^{\sigma \theta t}$  (see Mulligan and Sala-i-Martin, 1993, and Benhabib, Meng and Nishimura, 2000, for a discussion of this time-elimination method for solving recursive optimal growth models). We can therefore obtain the  $4 \times 4$  transformed system in  $(k, h, p_1, p_2)$  as follows:

$$\dot{k} = y_1 - (n + \theta)k - [1 + \tau(m/\pi)^{1/2}N^{1/2}]p_1^{-1/\sigma} \quad (3.42)$$

$$\dot{h} = y_2 - (n + \theta)h \quad (3.43)$$

$$\dot{p}_1 = (\rho + n + \sigma \theta)p_1 - w_K(p) \quad (3.44)$$

$$\dot{p}_2 = (\rho + n + \sigma \theta)p_2 - w_H(p) \quad (3.45)$$

where  $p = (p_1, p_2)$ .

To characterize this dynamical system, we follow Bond, Wang and Yip (1996) and utilize the duality properties of two-sector dynamic general equilibrium models. In equilibrium, competitive profit yields the following Samuelsonian relationship:

$$p_j = \left( \frac{w_K}{\alpha_{Kj} - \varepsilon_{Kj}} \right)^{\alpha_{Kj}} \left( \frac{w_H}{\alpha_{Hj} - \varepsilon_{Hj}} \right)^{\alpha_{Hj}}, \quad j = 1, 2 \quad (3.46)$$

The unit factor input coefficients are:

$$a_{ij} = \frac{(\alpha_{ij} - \varepsilon_{ij})p_j}{w_i}, \quad i = K, H, \quad j = 1, 2 \quad (3.47)$$

For convenience, let  $[a_{ij}]$  denote the input coefficient matrix. Using full employment, we have:

$$[a_{ij}] \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} k \\ h \end{bmatrix} \quad (3.48)$$

Further, based on the duality relationships, the cost shares coefficients can be written as:

$$\hat{a}_{ij} = a_{ij} \left( \frac{\alpha_{ij}}{\alpha_{ij} - \varepsilon_{ij}} \right), \quad i = K, H, \quad j = 1, 2 \quad (3.49)$$

Accordingly, we can denote by  $[\hat{a}_{ij}]$  the cost share coefficient matrix. The Samuelsonian relationship

(3.46) therefore implies:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = [\hat{a}_{ij}]' \cdot \begin{bmatrix} w_K \\ w_H \end{bmatrix} \quad (3.50)$$

We are now prepared to analyze the dynamics of the system (3.42)-(3.45). Denote output, factor input and factor price vectors as:  $y = (y_1, y_2)$ ,  $\ell = (k, h)$ , where  $w = (w_K, w_H)$ . Further denote  $\mathbf{I}$  as the  $2 \times 2$  identity matrix. Utilizing (3.47) and (3.49), we can derive the Jacobian of this  $4 \times 4$  dynamical system in  $(k, h, p_1, p_2)$  evaluated at their BGP values as:

$$J = \begin{bmatrix} \left[ \frac{\partial y}{\partial \ell} \right] - (n + \theta)\mathbf{I} & \left[ \frac{\partial y}{\partial p} \right] - [\rho_{ij}] \\ \mathbf{0} & (\rho + n + \sigma\theta)\mathbf{I} - \left[ \frac{\partial w}{\partial p} \right] \end{bmatrix} = \begin{bmatrix} [([a_{ij}])^{-1} - (n + \theta)\mathbf{I}] & \left[ \frac{\partial y}{\partial p} \right] - [\rho_{ij}] \\ \mathbf{0} & [(\rho + n + \sigma\theta)\mathbf{I} - ([\hat{a}_{ij}])^{-1}] \end{bmatrix} \quad (3.51)$$

where  $[\rho_{ij}] = \begin{bmatrix} -(1/\sigma)[1 + \tau(m/\pi)^{1/2}N^{1/2}]p_1^{-(1/\sigma)-1} & 0 \\ 0 & 0 \end{bmatrix}$ . Since  $p_1$  and  $p_2$  are both jump variables

whose values may be adjusted instantaneously at any point in time, the  $4 \times 4$  dynamical system is locally indeterminate (i.e., a sink) if there are less than two roots with positive real parts.

From (3.42), (3.43) and (3.48), we have:

$$\underline{Q} = \ell - [a_{ij}]y = \ell - [a_{ij}] \begin{bmatrix} (n+\theta)k + [1 + \tau(m/\pi)^{1/2} N^{1/2}]p_1^{-\frac{1}{\sigma}} \\ (n+\theta)h \end{bmatrix}$$

or, by rearranging,

$$[\mathbf{I} - [a_{ij}](n+\theta)]\ell = [a_{ij}] \begin{bmatrix} [1 + \tau(m/\pi)^{1/2} N^{1/2}]p_1^{-\frac{1}{\sigma}} \\ 0 \end{bmatrix} = [1 + \tau(m/\pi)^{1/2} N^{1/2}]p_1^{-\frac{1}{\sigma}} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad (3.52)$$

On the other hand, (3.44), (3.45) and (3.50) together yield:

$$\underline{Q} = [\hat{a}_{ij}]'w - p = [\hat{a}_{ij}]'p(\rho + n + \sigma\theta) - p$$

or, equivalently,

$$[[\hat{a}_{ij}]'p(\rho + n + \sigma\theta) - \mathbf{I}]p = \underline{Q} \quad (3.53)$$

Thus, the  $2 \times 2$  matrix  $[(\rho + n + \sigma\theta)\mathbf{I} - (\hat{a}_{ij})']^{-1}$  in the Jacobian expression (3.51) must be singular. This implies the Jacobian itself must also be singular and hence the four transformed endogenous variables are linearly dependent along the BGP. Because of this singularity property, a quick examination of (3.51) and (3.52) suggests that the unit transportation cost ( $\tau$ ) and the spatial density parameter ( $m$ ), both entering (3.51) and (3.52) symmetrically with respect to  $(k, h)$ , will not affect the sign of the roots of  $J$ .

We can now conclude that, regardless of the transportation and spatial density parameters,

- (i) price dynamics are recursive; they have a zero root and a negative root if the final goods sector (sector 1) is more capital intensive in the social sense, i.e.,  $\alpha_{11}/\alpha_{12} > \alpha_{21}/\alpha_{22}$ ;
- (ii) quantity dynamics feature at least one negative root if the final goods sector (sector 1) is more

- housing intensive in the private sense, i.e.,  $(\alpha_{11} - \epsilon_{11})/(\alpha_{12} - \epsilon_{12}) < (\alpha_{21} - \epsilon_{21})/(\alpha_{22} - \epsilon_{22})$ ;
- (iii) when the final goods sector (sector 1) is more capital intensive in the social sense and more housing intensive in the private sense, the Jacobian  $J$  has one zero root and at least two negative roots – in this case, dynamic indeterminacy arises where the BGP equilibrium is a sink.

These corroborate with findings in Bond-Wang-Yip (1996) once we (i) reinterpret Marshallian externalities in sector-specific factors as distortionary factor taxation/subsidy and (ii) establish a formal link between private (social) factor intensity rankings and physical (value) factor intensity rankings.

Summarizing, our results point to the possibility of dynamic indeterminacy in this generalized two-sector endogenous urban growth model, regardless of the transportation cost and spatial density parameters. An immediate economic implication is that urban growth and housing price dynamics in transition to a long-run balanced growth equilibrium may be very different across cities. Along these lines, it may be interesting to allow unbalanced development between the final good sector and the housing sector in a way similar to Kongsamut, Rebelo and Xie (2001). Such an extension is interesting as it may allow a thorough examination of housing dynamics with long-run trends in housing prices relative to the price of consumption good.

### *3.5. Endogenous Growth in a Perfectly Competitive Economy with a System of Cities*

The urban growth models with perfect competition elaborated in the previous subsections can easily be extended to feature a system of cities. A natural way to do this is to consider a fixed, finite number of cities where there are knowledge spillovers across cities. For example, one may follow Eaton and Eckstein (1997) by assuming that the engine of growth is human capital and that the existing stocks of average human capital in all cities contribute to the rate of accumulation of human capital in the representative city of interest.<sup>10</sup> Thus, an exogenous spatial correlation matrix gives the influence of the stock of human capital in one city on the effectiveness of time spent learning in other cities, where the

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<sup>10</sup>E.g. Rochester, NY.

diagonal elements are thus the effect of aggregate human capital in a city on the effectiveness of time spent learning on its own residents. In balanced growth equilibrium, individual human capital accumulation must converge to the city average, whereas net migration flows between each pair of cities must all be zero. The latter cross-city locational equilibrium condition is reached when the positive effects of human capital accumulation less the negative congestion effects of population growth are equal in all cities.

Controlling for other factors, the city contributing more to knowledge spillovers has a relatively high stock of human capital, a relatively high level of productivity, and a relatively large population. This therefore yields a prediction that in the long-run, larger cities feature higher wages per worker, though all cities grow at a common, balanced rate. That is, there is no positive correlation between population agglomeration and economic growth across cities.

In principle, we may allow the degree of knowledge spillovers between each pair of cities (i.e., the off-diagonal elements of the spatial correlation matrix) to depend on the distance between cities in a fashion similar to Ogawa and Fujita (1980), Fujita and Ogawa (1982) and Berliant, Peng and Wang (2002). This extension may allow a complete characterization of the dynamic patterns of city formation in the process of economic development. Alternatively, one may rely on vertical integration rather than knowledge spillovers to study dynamic patterns of agglomeration in a system of cities. Peng, Thisse and Wang (2003) make an attempt at such an endeavor in a simple two-region setup.

#### **4. Urban Growth Models with an Imperfectly Competitive Market**

In Sections 2 and 3, our spatial economy features perfect competition, despite the presence of Marshallian externalities in the form of uncompensated knowledge spillovers. In this section, we turn to considering dynamic urban growth models without a perfectly competitive labor, intermediate good, or consumption good market. The first subsection introduces imperfect market structures, focusing in particular on the case of monopolistic competition. The second subsection introduces non-Walrasian

setups that incorporate market frictions into the urban growth framework.

#### *4.1. The Role of Marshallian Externalities and Imperfect Competition*

The New Economic Geography provides the type of circular causation that is amenable to dynamic modeling. There is a large literature studying the static version of the model, including Abdel-Rahman (1988), Abdel-Rahman and Fujita (1990), Krugman (1991, 1993), and Berliant and Kung (2002), to name but a few. Comparative statics exercises where exogenous variables, such as population, are presumed to grow at an exogenous rate tied to time can be found in Fujita and Krugman (1995) and Fujita and Mori (1997). For instance, in a stripped down form, growth in Henderson and Ioannides (1981) is tied exclusively to an increase in the number of cities as a response to an exogenous increase in population. Our focus is instead on the dynamic growth version of the model, as described in the introduction of this survey.

To our knowledge, the most comprehensive treatment of the New Economic Geography with dynamics is found in Fujita and Thisse (2002, chapter 11). The basic, static model with two regions, monopolistic competition, iceberg transportation cost, and a preference for consuming a variety of commodities is modified by introducing a perfectly competitive R&D sector that operates under an endogenous growth externality: the production of patents in a region per unit of time is a function of knowledge capital in the economy and the number of skilled workers in that region, while knowledge capital is a function of both individual knowledge in the region and the entire economy. Individual knowledge is a function of the total stock of patents produced up to any given time. There is a cost to migration. If a firm wants to produce a particular variety, then it must pay a fixed cost that is the cost of a patent.

The basic results of this model are reinforcement of agglomeration in equilibrium. Agglomeration and growth are perfectly correlated, in part because having an agglomerated R&D sector

causes higher growth due to the endogenous growth externality. This reinforces circular causation.<sup>11</sup>

While the behavior of such a model is very interesting and complex, the fundamentals do not connect very well with the theoretical structures we have surveyed so far. In particular, the R&D sector doesn't look like part of an endogenous growth model - see equation (3.1) above. The allocation of skilled workers across regions and the stock of total patents in the entire economy (that affects individual knowledge) completely determine patent production rather than having *consumption good output* determined by an individual agent's investment in physical or human capital with an externality component based on the total investment in the city.

The predecessors of Fujita and Thisse are related as follows. Ioannides (1994) employs a multi-monocentric city framework with monopolistic competition in the output market. Overlapping generations yield life-cycle effects. In contrast with the balance of the literature, each city is assumed to produce only one commodity, and these cities are presumed to be symmetric (though the commodity that each produces is different). Each city has public capital, and transportation cost is decreasing in this capital. There is free mobility. In a steady state equilibrium, city size is constant and the number of cities grows exponentially in population. Product variety is the engine of endogenous growth.

Walz (1996) employs a framework with two regions, free migration, and monopolistic competition in the intermediate goods sector. The fixed cost for production of intermediate goods is, like Fujita and Thisse, the cost of a patent. Various patterns can emerge in equilibrium, depending on initial parameters (to some degree exhibiting a bang-bang pattern as a function of endowments, due to free migration).

Baldwin and Forslid (2000) employ a two region model with monopolistic competition in the

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<sup>11</sup>Both the workings of the model and the notation are very complicated. It seems important for future work in this literature to emphasize simplicity in order to keep the models tractable.

consumption good sector and migration proportional to the wage differential between regions.<sup>12</sup>

Increasing returns in the consumption good sector are generated by a one time fixed cost of physical capital that must be sunk by every firm that enters. A capital or investment sector subject to an endogenous growth externality, in that production is a function of a weighted sum of lagged total capital production in the regions, is employed under perfect competition. The results in the paper are purely numerical. Aside from studying the effect of integration of the two regions on equilibrium, the usual correlation between agglomeration and growth is found. This framework generates a model that is simple relative to the others studied in this subsection. We think that it would be interesting to explore subgame perfect equilibria that are not steady states in this model.

#### *4.2. The Non-Walrasian Approach to Agglomeration and Growth*

The previous sections of this survey have examined the dynamics of urban growth where the accumulation of capital has occurred in markets, either perfectly or imperfectly competitive. There have been externalities that have not been priced, for example endogenous growth externalities, so the equilibrium allocations were not efficient. But the markets for private goods were complete.

Recently, further abstractions have been attempted in models to provide microfoundations for the localization and urbanization externalities that are simply assumed to take a particular reduced form in standard dynamic regional models. In order to simplify matters, it is assumed that externalities take a pure form, in that there are no markets at all, so all interaction takes place in a non-market setting. This is what we call the non-Walrasian approach. Of course, it is hoped that these pure externality models will eventually provide microfoundations for reduced forms used in other models, and in this way the non-

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<sup>12</sup>With a dynamic general-equilibrium framework, it is desirable to use an explicit migration cost, as in Fujita and Thisse (2002), rather than an *ad hoc* adjustment process (analogous to the static framework of Krugman, 1991). This allows explicit calculations on the part of consumers of the benefits and costs of migration and true migration dynamics. Although Baldwin and Forslid (2000, Footnote 4) attempt to justify such an adjustment process, we find the assumption that migration costs are quadratic in the rate of migration to be strange.



Walrasian models will be integrated with market or Walrasian models.

The idea of non-Walrasian approach to urban economics starts from Helsley and Strange (1990), where matching of heterogeneous labor leads to the formation of a system of identical cities with a and where workers with different skills achieve the same utility level in equilibrium. Abdel-Rahman and Wang (1995, 1997) go beyond this early urban labor matching framework to permit *ex post* heterogeneity and as a consequence, the formation of a core-periphery urban structure and the dispersion of wage incomes. In two recent studies, Coulson, Laing and Wang (2001) and Brueckner and Zenou (2003) examine in a urban labor market with search and matching frictions why spatial mismatch occurs in the sense that unemployment in central cities and job vacancies in suburbs coexist.

Since none of these previous papers consider urbanization dynamics and city growth, our discussion is based on unpublished work by Berliant, Reed and Wang (2003). First we shall describe the basic model that does not involve growth, and then give its extensions to various growth contexts.

This model has a focus on horizontal knowledge differentiation, where knowledge is represented by a circle of unit circumference.<sup>13</sup> The basic model has a continuum of consumers, each of whom has a knowledge type on the circle. Agents in a city meet each other pairwise and randomly, according to a Poisson process where the rate at which meetings occur for each agent is proportional to the number of unmatched agents in a city. When agents meet, they learn each other's type, and decide whether or not to produce together. Production uses the pair's time and results in the output of a homogeneous consumption commodity. If a pair decides to produce, they match, and detach after a time determined by another Poisson process. If they decide not to match, they detach immediately and search for other partners.

Consider any pair that meets. The model attempts to capture the idea that if the pair has

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<sup>13</sup>Thus, the setup contrasts sharply with the vertical knowledge differentiation model of Jovanovic and Rob (1989).

knowledge that is too close together, there are few complementarities between the pair and production is relatively low. Similarly, if the pair has knowledge that is very distant, communication problems dominate, and production is also very low. Thus, it is assumed that production as a function of the two types who meet is linearly increasing beginning with zero distance between the knowledge types of the pair as the pair becomes more diverse up to some optimal distance. Then it is linearly decreasing beyond that optimal distance. In other words, production as a function of knowledge type distance on the circle is piecewise linear and single peaked.

Felicity is simply the quantity of consumption good produced and consumed in a time period, assumed to be zero if production is not taking place. Time is continuous and infinite. Utility is the (expected) present discounted value of consumption, where the discount rate is the same for all agents (so the only difference between agents is their knowledge types). Agents choose a range of others with whom they will accept matches and production; this range will typically be the union of two intervals around the optimal match type on opposite sides of the agent. The length of these intervals is called the “knowledge spread.” In other words, the choice of agents with whom to match will be symmetric. Since only symmetric equilibria in pure strategies are considered, if one agent in a pair wants to match, so does the other. In general, there is a tradeoff in the choice of knowledge spread between the quality and quantity of matches.

In a closed city model with exogenously given population, a steady state equilibrium exists and is unique. Moreover, higher population implies a lower knowledge spread, as agents are more selective about their partners. Finally, the per capita rate of matches (which one might interpret as patents for the purposes of empirical implementation) is higher in cities with larger population.

In an open city model, population is endogenous and determined by a per capita cost of city infrastructure that is linear in population. Entry into a city occurs until the value of an unmatched agent (or potential entrant) is equal to the infrastructure cost. Even though both the population and knowledge spread are endogenous, the same correlations between population and knowledge spread on the one hand,

and population and matches on the other, hold when exogenous variables such as the technology are changed.

In this context, there are two external effects that interact. First, there is the classical urban economic congestion externality, that agents entering the city consider only the average congestion cost, not the marginal congestion cost imposed on others in the city. This leads to cities that are overpopulated in equilibrium relative to the social optimum. Second, there is an externality in the search process, since more unmatched agents mean a higher frequency of meetings and thus higher utility for unmatched agents. An agent's decision concerning knowledge spread has an external effect on other agents, since it affects the number of unmatched agents and thus the utility of others in the city. This leads to knowledge spreads that are too large in equilibrium relative to social optimum, and cities that are too small. Overall, equilibrium knowledge spread and population can be anything relative to the social optimum.

This basic model leaves agents unchanged after they meet, match, and produce. It is extended to the growth context, where knowledge accumulates in various ways, as follows. First, it is assumed that there is an endogenous growth externality in the production function when two agents match and produce, following in the spirit of Laing, Palivos and Wang (1995). This externality lasts over the infinite horizon, and matches are thus permanent. Second, it is assumed that production of every match is multiplied by a decreasing function of the city's average knowledge spread, assuming that more specialization leads to higher growth. Third, the circumference of the knowledge circle is allowed to expand with the city's average knowledge spread, representing a larger diversity of possible knowledge when people are matching more broadly. Fourth, agents' knowledge types is allowed to change when they match. They are assumed to move closer to their partners after a match. These extensions can be combined in any way, and despite these variations, all of the basic results outlined above continue to hold.

## **5. Avenues for Future Research**

We have surveyed dynamic models of the urban economy that feature various forms of capital

accumulation and thus growth. There are several features of these models that require elaboration and improvement in order to be useful, especially to empiricists.

First, it is apparent from our survey that an almost universal feature of the models in the literature is that agglomeration and growth are perfectly correlated.<sup>14</sup> Of course, in the real world this is not true; see, for example, Tables 1 and 2 in Section 1. Eventually, in cities with very large populations, gains from agglomeration diminish and congestion costs increasing in population will overtake the benefits of agglomeration, yielding cities that do not grow forever. It seems very important to include this in the models.

Second, it is important to develop testable hypotheses that can distinguish among the various models. This is especially important for detecting market failures, so appropriate public policies can be formulated if there are large welfare losses associated with equilibrium. Which models are best capable of predicting the dynamic process of city development and decline? The comparative statics we have derived in Sections 2 and 3 of this essay should be useful for this purpose. However, there are few comparative statics available for the models in Section 4.1; most are rudimentary descriptions of either symmetric or a core-periphery equilibria. Priority should be given to developing potentially testable comparative statics.

Third, in all of the models we have discussed, there are “black boxes” - pieces of the model that are not based on microeconomic optimizing behavior. In other words, a reduced form is assumed. Often this is used to simplify the inner workings and notation, but it is often not clear that the black box can be opened and its contents found to be consistent with a simple micro model. Examples drawn from earlier sections include the aggregate production function and fixed savings rate used in the exogenous growth literature, the aggregate production function used in the endogenous growth literature, the representative consumer assumption (implying no heterogeneity in preferences), and assumptions about population

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<sup>14</sup>Notable exceptions can be found in Eaton and Eckstein (1997) and Berliant, Reed and Wang (2003).

growth or migration dynamics. In the non-Walrasian literature, there are assumptions made about the reduced forms of population arrival (both of rates and types), and the average cost of population in a city is taken as exogenous. For instance, an alternative to assuming that types enter uniformly into a city would be that the entry of types of agents depends on the unmatched types already there.

For each of these black boxes, one can ask whether they are consistent with a standard microeconomic model of location within a city, say a monocentric model, where all choices, including location, are made by consumers. The exercise in much of the literature is opening a black box to find more but smaller black boxes inside, occupying the same volume as the original black box. It would be better, of course, to simply provide the micro foundations for assumed behavior and dispose of the boxes.

The fourth and final issue that we wish to discuss is the concept of “social optimum” in open city models. These models generally have the feature that population in the city is endogenous. Only one city is considered. Social optimum is often defined to be the solution an optimization problem that has a utilitarian objective for agents in the city and standard feasibility constraints for resources in the city. Population is assumed to flow between the city and the rest of the world. How is this concept related to the standard notion of efficiency, Pareto optimality? Taken at face value, Pareto optimality is not well-defined in an open city model, since the agents present in the city and model are not well-defined. Whose welfare is accounted for? The concept of social optimum is really trying to capture efficiency in a system of (identical) cities. The number of such cities would have to be infinite, as the concept of social optimum assumes that there is no shadow cost for removing population from another city. The assumptions under which a social optimum is the same (or results in the same allocations) as a Pareto optimum in such an extended but closed model should be clarified and examined, and a theorem proved. Evidently, the theorem would involve two parts: conditions under which any Pareto optimum is a social optimum, and conditions under which any social optimum is a Pareto optimum. Models that do not satisfy the conditions of the theorem should avoid use of the concept of social optimum.

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