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## THE FIRST USE OF A COMBINED VALUE AUCTION FOR TRANSPORTATION SERVICES

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Sears, Roebuck and Co. is one of the largest procurers of trucking services in the world through its wholly-owned subsidiary, Sears Logistics Services (SLS). SLS controls supply chain elements that originate at the vendor (manufacturer) through distribution centers to retail stores, and from vendor to distribution centers to cross dock facilities. This case examines a major change in the method Sears used in contracting for truckload carrier services for this supply chain. It provides a pioneering example of complex business to business e-Commerce.

In the early 90's, SLS sought to lower its truckload carrier costs by consolidating trucking service acquisition to allow truckload carriers to better deploy their assets and share the savings with SLS. Two innovations offered the promise of better asset deployment, which could result in shared savings with SLS if implemented through a properly designed auction:

- (i) use of three-year contracts that included surge demand contingencies; and
- (ii) simultaneous letting of contracts on a large number of lanes through a process that solicited single offers for multiple lanes – thereby allowing carriers to coordinate SLS business with other business and reduce related empty or low value movements.

A standard one-sided procurement auction, increasingly referred to as a reverse auction, is the process of choice to implement the first innovation. To implement the second innovation, a combined value procurement auction is necessary. In a combined value auction, an order can be comprised of multiple items. This allows a trader to express their combined value for the group rather than forcing them to cobble together individual trades and so face the risk of missing links in a desired chain.

SLS had contracted the consulting firm of Jos. Swanson & Co. (JS&Co) to advise how to implement the desired consolidation of trucking. JS&Co identified as promising the combined value trading technology being developed within the California Institute of Technology<sup>1</sup> (Caltech) by the founders of Net Exchange (NEX).<sup>2</sup> SLS, JS&Co, and NEX formed a team to implement the desired consolidation using combined value auctions. The initial auction would involve 854 lanes with a service cost of approximately \$190 million per year. The combined value sequential auction that was implemented reduced this cost to \$165 million per year, a 13% savings<sup>3</sup>. Much of what follows will describe this first auction. We also supply summary information on subsequent combined value auctions that were performed by or for SLS.

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<sup>1</sup> See Banks et al. (1989). At the time, the only other work on combined value auctions that we know of was Smith et al. (1982).

<sup>2</sup> Information concerning NEX can be found at <http://www.nex.com>

<sup>3</sup> All of the cost savings in this paper were estimated based on: (1) actual realized rates on existing lanes with a history, (2) estimated rates on existing lanes without available rate data, and (3) state to state matrix rates on new lanes.

## **Lead Up to a Combined Value Auction**

SLS put together a strategy which, it believed, would allow it to garner significant savings in its logistics costs and which it also believed would encourage carriers to participate actively in the consolidation process. Under the leadership of some imaginative and forward-looking people in SLS, and with the important help of JS&Co, SLS had begun to experiment with procurement auctions. By allowing carriers to offer single classes of transportation service to SLS through single-round sealed bid auctions, significant cost-savings were being achieved. The idea was to extend this approach to the larger consolidation effort. SLS and JS&Co recognized that the trucking firms might rebel at having a sizable piece of their regular and profitable business put up for competitive auction. In fact it was highly plausible that some of the more reliable firms might refuse to participate and that, therefore, there might not even be sufficient capacity among the rest to service the needs of SLS.

A strategy was formulated to identify a small number of "partners" who would be given exclusive rights to bid in the auction and who would be helped in their planning and participation. Limiting the number of partners provided additional incentive for participation due to the opportunity. A significant amount of effort was spent in this process so that SLS would know it could rely on the carriers who were selected from the auction process to provide the promised services.<sup>4</sup> This also provided an additional level of confidence for the carriers, since they knew that the other participants were "peer" carriers. In the end, 14 national and large regional carriers were qualified and participated.<sup>5</sup>

At this time, however, it was not known precisely how an auction of this magnitude would be organized. What was to be auctioned? That is, what would it mean if a carrier won? Would they have to deliver anything Sears requested for the term of the contract? Would they only have to handle only a fixed maximum weekly set of loads, leaving SLS to find others to handle excess shipments? Should SLS auction one lane at a time; and, if so, in what order and how fast? Or should they do them all at the same time? If so, how could they possibly coordinate all the bidding?<sup>6</sup> Let us turn to how SLS chose to answer these questions.

## **Combined Value Design**

If the carriers were going to be able to create savings for themselves, thus creating value to be shared between themselves and SLS, they would need to be assured of business on a relatively riskless basis, and they would need to be able to coordinate their offers across multiple lanes. Justifying the first part of this statement is obvious and was satisfied by SLS choosing to auction three-year contracts with contingencies for surge and

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<sup>4</sup> It is not obvious that significantly different outcomes would have occurred in the auction and beyond if the qualification bars had been lowered and the penalties for non-compliance raised. Some effort and costs to redress non-performance would have arisen but less costs would have been incurred in the qualification process.

<sup>5</sup> The manner in which SLS "qualifies" transportation service bidders is proprietary. To this date, it remains one of the most complex preparation efforts in business to business commerce. It is important to note that all candidates learn the names of the competitors that have been qualified.

<sup>6</sup> It is interesting to note that the same issues would later arise in the design and operation of the FCC auctions of the electro-magnetic spectrum. In the SLS case there was little sentiment in favor of single-lane auctions, as it had made use of that approach much earlier.

slack demand. Justifying the second part of this statement is perhaps less obvious. The efficient use of assets, through the reduction of empty mile movements, requires carriers to solve a fairly complex minimization problem, coordinating not only the SLS shipments but also the shipments they expect or have under contract to others on lanes SLS may not even be auctioning. What a firm is willing to supply services for, on a specific Chicago to Los Angeles route, depends critically on what they have committed in the LA to Chicago direction (which may itself involve multiple lanes). If an SLS lane (say St. Louis to Chicago) is part of that return cartage then the amount the trucking firm is willing to accept for the Chi-LA lane depends on how much they will be paid for the St. Louis-Chicago route. That is, the amount they are willing to accept to do both the Chi-LA and the St. Louis-Chicago lanes depends on the combined value. It will generally be less than the sum of the individual parts.<sup>7</sup>

How can one design the auction to reveal and take advantage of these combined value opportunities? Late in 1992, Joe Swanson asked this question of John Ledyard whose research had involved, among other things, creating auctions that provide an answer<sup>8</sup>. Called combined value auctions, they differ primarily in only one crucial aspect from traditional auctions. They allow participants to make an offer of a single dollar amount for a collection of items. In the SLS case, a carrier would be able to say "I ask \$1 million for the Chicago-LA lane *and* the St. Louis-Chicago route" meaning "I am willing to service the two lanes for a fee of at least \$1 million if and only if I can service both lanes". Once this new type of procurement offer is considered, the rest of the design follows. First, the winners are determined by accepting the offers that minimize the total cost of procuring the services when one and only one carrier per lane is allowed.<sup>9</sup> Second, all winning offers will be paid at what was asked.

A critical, if seemingly innocuous, part of the auction design is the stopping rule. The stopping rule for an auction is absolutely crucial to its performance, both in the final cost of acquisition and in the time to completion, because it affects the incentives and the information of the bidders. One option is to let everyone submit as many offers as they wish, up to a specified time. At that time, winners are determined and the auction is over. This is just a sealed-bid procurement auction using combined value bids. The problem with this is it requires bidders to consider all contingencies and to evaluate all of the business implications of winning each subset of lanes, it encourages submission of all possible bids by all bidders, and it has been shown to result in a higher final cost of procurement. The experimental work at Caltech had shown this conclusively. We chose, instead, to use a sequential auction in which bidding would proceed in rounds. At the end of each round, provisional winners would be announced. Going into the next round, all of the provisional winning offers from the previous round were held by the auctioneer and carriers submitted new bids against that set.<sup>10</sup> Laboratory test runs had revealed that this

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<sup>7</sup> We have provided two examples in the appendix that illustrate how combined value in transportation services occurs. In Appendix A we provide a simple 3 lane stylized example. In Appendix B, we provide parts of the test-bed example we used to demonstrate to SLS and the truckers how a combined value auction would work.

<sup>8</sup> That work, best described in Banks et al. (1989), was done mostly for NASA through the Jet Propulsion Lab. Although combined value auctions had not yet been used in practice, Ledyard and his team had extensive laboratory experience demonstrating the potential power of combined value bidding over traditional methods.

<sup>9</sup> Suppose there are three bids for routes 1 and 2 where bid one is \$10 for route 1, bid two is \$35 for route 2 and bid three is \$40 for routes 1 and 2. The winning bid(s) will be bid three since \$40 is less than \$45. A mathematical formulation is given in Appendix C.

<sup>10</sup> This requirement is extremely important. Without it bidders face no penalty from, say, randomly bidding or from bidding to attack an opponent. It imposes a commitment on the bidders - each bid can be viewed as

sequential process led to a significant increase in cost savings because it allowed the firms to concentrate their efforts on those lanes that gave them a cost-advantage. In a situation where little is known by anyone about the true opportunities for cost savings from combined value offers, it is important to have a sequential auction. The stopping rule used was of the form – if total acquisition cost did not decline by at least  $x\%$  from the previous round, then the just-completed round is declared to have been the final round

## Selling Combined Value to SLS

The first step was to convince SLS that such an auction was feasible (that is, one could actually run such a thing). NEX created a *test-bed*<sup>11</sup> environment and then ran combined value auctions in that scaled down world in the Caltech Economics Laboratory. The runs were successful.

In the meantime, the NEX team had to deal with scaling up the auction part of the test-bed so that 854 lanes could be handled. This is a, potentially huge, combinatoric problem<sup>12</sup> that, even with today's technology, is daunting. It was almost overwhelming in 1993. Nevertheless, starting from standard algorithms with some front-end sorting and culling of orders, we were able to create an algorithm that easily handled this scope of problem.

Once we had designed an acceptable CVA it was necessary to explain it to the SLS team and get their approval. Here the experimental test bed became important as a demonstration tool. We took the *test-bed* to SLS so they could actually participate in a CVA and, hopefully, see that it would be possible for trucking firms to understand and how savings would occur. The SLS team bought into the concept.<sup>14</sup> The final step was to

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a contract proposal to which the bidder will adhere if the auctioneer accepts it. Allowing withdrawal of provisionally winning bids extends the auction and creates bad incentives.

<sup>11</sup> A test-bed in this case is a scaled down version of the items to be auctioned and the incentive structure of the participants. It is similar to using scale models and wind tunnels in aircraft and automotive design.

<sup>12</sup> Each bidder could theoretically submit ( $2$  raised to the  $854^{\text{th}}$  power) a huge number - possibly more than the number of stars in the universe. There were, however, practical limits on the number of bids submitted. In particular, in the actual auction bids were submitted via a spreadsheet on floppy disks. At the time this limited what a bidder could do. The maximum number of bids submitted was 4595 and no bidder ever complained that they could not submit enough bids. The algorithm had no trouble finding the optimal solution in less than an hour. (Today that would be less than a minute.) There are economic reasons for the small number of submitted bids but that is a subject for another paper.

<sup>13</sup> The FCC faced this same choice in the design of the auctions of the electromagnetic spectrum. They, and their consultants, did not rise to the occasion as SLS did. Although it is controversial, particularly among the consultants who made lots of money on the FCC design, the next statements are, I believe, true. The FCC and its consultants were unwilling to consider the SLS experience to be relevant even though they were informed about it. Doomed to relive history as a result, the FCC ultimately chose to auction everything off simultaneously but to not allow bidders to submit combined value bids - potentially forgoing very significant revenues.

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take this test-bed to the trucking firms. It would serve both as a sales device to get them eager to participate in the ultimate auction and as a training device so their teams could learn how to participate successfully.<sup>15</sup> JS&Co did this with the firms that had been pre-qualified. It was a time intensive process.<sup>16</sup>

## The Auction Results

The auction lasted through 5 rounds with about 1 month between each round. The length of time of the overall process is one of the only complaints the bidders had. In retrospect there seems to be no reason why the time between rounds couldn't have been significantly reduced. The data are interesting and instructive.

	Round 1	Round 2	Round 3	Round 4	Round 5 (final round)
# of bids submitted	3383	4409	4595	3691	4589
# of packages submitted	2374	1698	2273	1803	1721
# of packages in the winning allocation	650	637	577	595	575
Acquisition cost (\$000) if stopped	\$187,149	\$179,288	\$172,744	\$168,337	\$165,371
% drop from previous		4.5	4.1	2.4	1.8

There was significant use of the multi-lane offer capability of the combined value auction. A fair number of package bids won indicating significant savings from this feature. And, contrary to the expectations of some that combinatoric calculations would be a problem, the time to calculate winning bids was generally in the 15-30 minute range, once some modifications had been made to the software.

One fact not obvious from these statistics is that many bidders wanted to be able to submit a bid like "I will (supply lanes A, B, and C for \$100) or (supply lanes D, E, and F for \$120) but not both". Although this feature, called XOR, is now standard on combined value auctions, the SLS auction did not allow such bids.<sup>17</sup> Nevertheless, the bidders found a clever way to do this. They would submit two overlapping bids using a small,

<sup>15</sup> The Swanson team did most of this sales and training using the materials developed by the Caltech team.

<sup>16</sup> Today, creative use of the Web and interactive software can easily cut the time involved in teaching and training. For example, at Caltech we have been able, by using the web, to cut the time devoted to explaining the process to less than an hour.

<sup>17</sup> We did allow, however, an alternate procedure in the experiments we ran which allowed bidders to put a cap on the total commitments they were making with all of their bids. See appendix C and the use of a capacity constraint.

inexpensive route, say, G. They would submit "I bid \$101 for A, B, C, G" and "I bid \$121 for D, E F, G". Obviously both couldn't win.<sup>18</sup>

All participants were contacted 2-3 weeks after the final results were distributed and they were asked for their thoughts on the process and outcome. The carriers overall reaction to this type of auction was favorable. They particularly liked the format and level of detail in the traffic information provided. They were reasonably happy with their outcomes. Each carrier had lost one or two lanes they were disappointed to lose, but they recognized it as a reality of the business and the process.

Most of the negative issues related to execution items. With respect to the length of the process, most thought it should have gone fewer rounds.<sup>19</sup> Whether they had enough time to respond varied by carrier. Some looked at every lane in every round and wanted more time (these were the national carriers). Others were more focused on what they were bidding and would have been happy with less turnaround time. The level of sophistication in analysis varied greatly among carriers. It involved detailed modeling for some; others worked out their strategies using pencil and paper.<sup>20</sup>

Some bidders thought that the process was purely price driven but shouldn't have been. They felt there was no consideration given to the service capabilities of the carriers<sup>21</sup> and that there was no advantage to being an incumbent on a route.<sup>22</sup>

## The Market Test

Following up on the success of this first SLS procurement auction, SLS bought software and hardware from NEX with the capability of running similar auctions. We have data from 5 auctions run by SLS during 1995 and 1996 with the help of JS&Co.

	#carriers	# lanes	Acquisition cost	Savings est <sup>23</sup>	Savings %
1	12	17	\$1,200,000	\$80,000	6 2/3
2	12	35	\$15,000,000	\$3,000,000	20
3	24	135	\$49,000,000	\$5,000,000	10
4	16	190	\$10,000,000	\$2,500,000	25
5	16	159	\$27,000,000	\$2,000,000	7 1/2

<sup>18</sup> It is interesting that computer scientists now use this trick- they create a dummy route with zero value - to create algorithms to solve these combinatoric problems.

<sup>19</sup> This may be because each had a different favorite stopping round - the one they would have won more in. Or simply because in each round they had to compete further to stay alive and they saw profits slowly shrink.

<sup>20</sup> New processes providing better feedback to bidders now exist. These "next generation processes" appear to be significantly more user friendly and faster than the path-breaking but simple process used by SLS. One example of these new processes can be found in DeMartini et al. (1999).

<sup>21</sup> There was, in fact, such consideration given at the time of qualification.

<sup>22</sup> Performance capabilities were definitely part of the participant selection process. Incumbency benefits had been considered and would realistically be implicit as the carrier would have been able to develop supporting business to support those routes during its tenure of operation.

<sup>23</sup> These savings values may be somewhat inconsistent. For those lanes where established prices did not exist, SLS utilized State to State Matrix rates (rates available for "one off" type shipments).

Thus, logistics services for a total of 536 lanes were acquired for about \$102.2 million. The total savings to SLS were about 13% or about \$13.3 million. Combining these data with those from the first auction yields the conclusion that, over a three-year period, SLS saved more than \$84 million by running six combined value auctions. Truckload transportation services were acquired for 1390 lanes for a total cost of \$587 million. This became the accepted methodology of transportation services procurement for SLS. The concept is still in use today and has been fully supported by management since that time.



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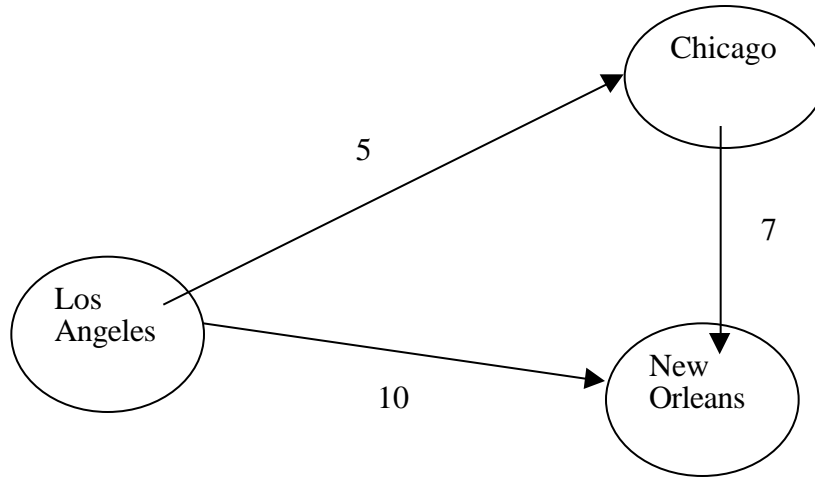
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## APPENDIX A

### A Stylized Example of Combined Value

Consider the following lane configuration and weekly load requirements of a retail company.



One way to fulfill the requirements is to buy or lease 10 trucks and allow a lot of empty back-hauls. But this would be a very costly solution. There are cheaper solutions, but one might ask a different question. Can one gain by outsourcing?

Suppose there are 3 trucking firms - each with the same costs but different customer bases. Each trucking firm has a current contract for a lane and has some (usually uncertain) revenue from the return trip.

Firm #	Contract Route	Current loads	Revenue from return trip
1	Chicago to LA	5	X
2	NO to Chi	10	Y
3	NO to LA	10	Z

Notice that firm 1 would be happy to carry the retailer's required 5 loads from LA to Chi for any price greater than X. That is because they can take advantage of the combined value in servicing both directions on the LA to Chicago lane. If  $X < C(\text{LA to Chi})$ , the cost of shipping 5 loads from LA to Chicago, then the retailer and Firm 1 both have the potential to gain if the retailer outsources the transportation services for that route. At any price P such that  $X < P < C(\text{LA to Chi})$ , both gain.

One can continue the analysis. Firm 2 will carry the 7 loads from Chi to NO for any price greater than Y and Firm 3 will carry the 10 loads from LA to NO for a price greater than Z. Finally one should note that Firm 3 is also willing to carry the 5 from LA to Chi and the 7 from Chi to NO for a price greater than  $W = Z + C(\text{LA to Chi to NO}) - C(\text{LA to NO})$ . In this case, Firm 3 is able to reap combined value from the triple lane combination of LA to Chicago to New Orleans to Chicago.

If the retailer knew the values of  $X$ ,  $Y$ ,  $Z$ , and  $W$ , what should they do? The answer is easy. If  $X+Y+Z < W + C(\text{LA to NO})$ , then hire 1 to handle the LA to Chicago loads, hire 2 to handle the Chicago to New Orleans loads, and hire 3 to handle the LA to New Orleans loads. However, if  $X+Y+Z > W + C(\text{LA to NO})$  then hire 3 to handle both of the LA to Chicago and Chicago to New Orleans loads. The retailer does the LA to New Orleans themselves. That is, in the second case they do not outsource the whole thing.

In practice, the retailer does not know what the value of  $W$ ,  $X$ ,  $Y$ , or  $Z$ . And the optimization problem can involve 854 lanes instead of 3. Signing contracts one at a time can interfere with the retailer's ability to take advantage of the combined values available. That is why a combined value auction can provide value to both the retailer and the trucking firms.

## APPENDIX B

### The SLS Experimental Set-up

The goal in designing the experiments for the SLS auction was to provide a test-bed that exhibited the benefits of combined value in an example that looked familiar to truckers and that would enable us to demonstrate how easy and productive a combined value auction would be. Taking a route map of the United States with 854 lanes and reducing it to something manageable was the first step. We chose to focus on 7 locations and 9 lanes. The structure can be seen in the next figure.<sup>24</sup>

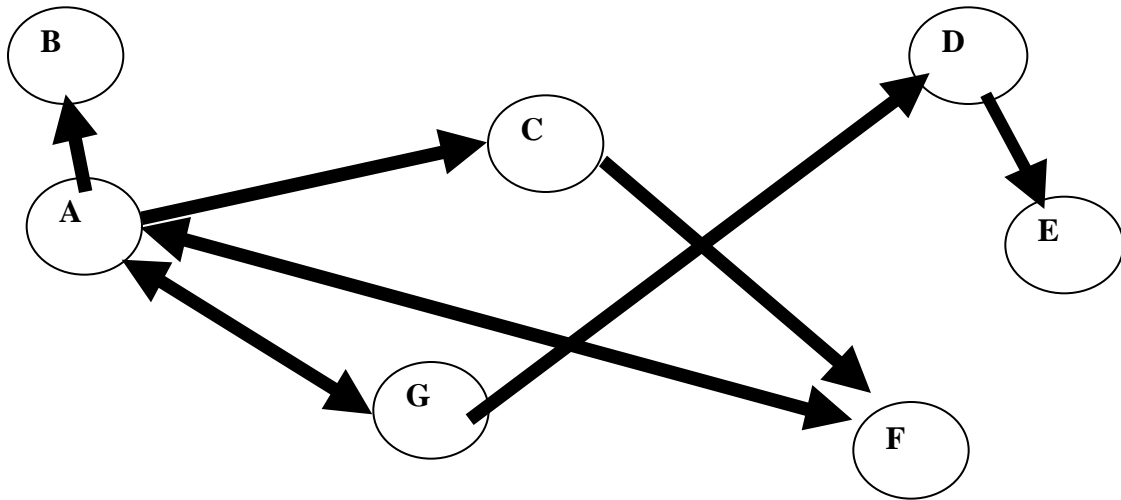


Figure B.1

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<sup>24</sup> One can use one's imagination to determine what the locations might be. But the structure was chosen so that most combined value opportunities known to exist could be represented.

To put meat on this structure we added network loads, one-way times and costs. This information is common knowledge, available and known to all.

Lane	Miles	Cost	Loads	Reserve price
AB	300	450	5	5000
AC	300	450	15	13000
AF	400	520	20	11500
AG	200	360	10	7000
CF	100	210	5	2100
DE	100	210	10	4000
FA	400	520	15	13000
GA	200	360	15	10000
GD	500	600	15	15000
FG	300	450	0	
EF	200	360	0	

There is also private information including other traffic revenue (both yours and some information on the possibilities for others), units (trucks) at one's disposal, and opportunity costs. For example, here is the information we gave Firm 1.

Lane	Your potential loads/week	Network average loads/week	Your potential revenue/load	Network average revenue/load
BA	5	5	300	325
CA	10	12	400	300
FA	5	5	500	300
AG	5	5	160	75
FC	10	7	150	100
ED	8	8	190	100
AF	5	5	500	300
DG	15	15	650	450
EF	8	10	390	350

The **number of units** in your fleet is 8 and the **profits/week** per unit not used is 10.

Finally, to help them, we calculated the combined value of several packages and reported it as "some packages that you may want to consider". For firm 1 these looked as follows.

Package you would bid for	Other revenue	Operating cost	Required units	Break-even bid
AB	BA 1500	4500	2	3000
AG GA	AG 800	10800	3	10000
GD	DG 9750	18000	3	10000
AF FA	FA 2500	20800	8	18300
AC CF	FC 1500 CA 4000	19800	6	14300
DE CF	ED 1520 EF 3120 FC 1500	15600	4	9460

## APPENDIX C

### The Simple Mathematics of A Combined Value Auction

In the experiments, a bid is a detailed specification of a package of lanes a firm is willing to service along with a total minimal amount of revenue the firm requires to receive for supplying the lanes in that package. (Lanes were supplied in an all-or-none fashion.) A bid consisted of 4 items:

1. □ The list of lanes to supply
2. □ A bid (revenue): the minimum amount needed by the firm to supply those lanes/week.
3. □ An estimated capacity utilization in whole units for each package and
4. □ A capacity constraint that specifies the total units you all willing to supply across all of your accepted bids.

The last two features of a bid allowed each firm to submit more bids than they would actually be able to service since the algorithm would not accept more than could be serviced. These last two features were not included in the actual SLS auction.

One can think of a bid as  $\langle \{(x,b,u)\}, U \rangle$  where  $x(j,k,l) = 1$  if firm  $j$  wants package  $k$  and lane  $l$  is part of that package, where  $b(j,k)$  is  $j$ 's revenue bid for package  $k$ , where  $u(j,k)$  is the unit capacity estimated by  $j$  for  $k$ , and  $U(j)$  is  $j$ 's total capacity.

Given a collection of bids from every one, the combined value auction algorithm solves

$$\begin{aligned} & \min_{j,k} b(j,k)d(j,k) \\ \text{Subject to:} \\ & \sum_{j,k} x(j,k,l)d(j,k) \leq 1 \text{ for all lanes } l, \\ & \sum_{j,k} u(j,k)d(j,k) < U(j) \text{ for all } j, \\ \text{and} \\ & d(j,k) = 0 \text{ or } 1. \end{aligned}$$

If  $d(j,k) = 1$  in the solution, then firm  $j$  is (provisionally) awarded lane  $k$  at a price of  $b(j,k)$ .

(For the SLS auction, the middle two constraints were not included.)