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RANGE CONVEXITY AND AMBIGUITY AVERSE PREFERENCES.

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Abstract

We show that range convexity of beliefs, a 'technical' condition that appears naturally in axiomatizations of preferences in a Savage-like framework, imposes some unexpected restrictions when modelling ambiguity averse preferences. That is, when it is added to a mild condition, range convexity makes the preferences collapse to subjective expected utility as soon as they satisfy structural conditions that are typically used to characterize ambiguity aversion.

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Range Convexity and Ambiguity Averse Preferences.*

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Introduction

The mathematical representations of preferences that we obtain in decision-theoretic models typically enjoy both 'empirically relevant' and 'technical' properties. Separability is an obvious example of a property belonging to the former group, while continuity of preferences, or monotone continuity for probabilities belong to the latter.[1] Technical properties are thus named because, taken in isolation, they have small or null empirical content. But it is not correct to then infer that they are harmless. For, when joined with empirically relevant properties, technical conditions may substantially alter their empirical content. For instance, it is well known that for probabilities, monotone continuity modifies the empirical content of finite additivity, as most eloquently stressed by de Finetti (1970) and Savage (1954). Analogously, Krantz, Luce, Suppes, and Tversky (1971)[Sect. 9.1] and Wakker (1988) observe that continuity adds significant empirical content to (coordinatewise) separability in problems of additive conjoint measurement. Another technical property that is commonly used is 'range convexity' (see Section 1 for a definition) of the function representing the decision maker's beliefs, that we call her *willingness to bet*. Such property follows naturally in axiomatizations that rely on Savage's (1954) rich state space construction. In this paper we show that also range convexity is less harmless than usually thought.

We argue that range convexity of the willingness to bet imposes surprisingly strong restrictions for preferences which reflect *ambiguity aversion*, the attitude displayed by the subjects in the famous thought experiment of Ellsberg (1961) (and in many others that followed). Roughly, we show that if a decision maker's willingness to bet satisfies a

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[1] A probability satisfies monotone continuity if for every increasing sequence of events $\{A_n\}_{n=1}^{\infty}$ converging to $A = \cup_n A_n$, $P(A_n)$ converges to $P(A)$.

range convexity assumption and a very mild condition — which is compatible with the presence of substantial ambiguity aversion — then if her preferences are ambiguity averse (in the sense of Ghirardato and Marinacci (1997)), they are very close to satisfying the *subjective expected utility* (SEU) model. Indeed, if her willingness to bet is also convex (a condition typically associated with ambiguity aversion), then her preferences *do* satisfy SEU.

These results hold for a very general class of preferences (the biseparable preferences introduced in Ghirardato and Marinacci (2000)) which includes the two most popular decision models with ambiguity aversion, the *Choquet expected utility* (CEU) model of Schmeidler (1989), and the *maxmin expected utility* (MEU) model of Gilboa and Schmeidler (1989). In particular, they apply to the axiomatizations of CEU preferences in a purely subjective framework of Sarin and Wakker (1992) and Gilboa (1987), which describe decision makers whose willingness to bet satisfies some range convexity conditions.

To show that range convexity is the driving force behind the mentioned results, we also look at ambiguity averse preferences in the setting of Anscombe and Aumann (1963), as exemplified by the models of Schmeidler (1989) and Gilboa and Schmeidler (1989). We show that similar results can be proved, but only by making the mild condition in the previous results much stronger and much less plausible.

The conclusion that we draw from this exercise is that one should exert caution in assuming range convexity of beliefs, as the latter adds substantial empirical content to seemingly weak (empirically relevant) conditions. For the specific purpose of modelling ambiguity averse decision makers, our results show that range convexity might deprive CEU and MEU preferences of much of their additional predictive power. We therefore conclude that building on Savage's rich state space approach might not be the most appropriate way to axiomatize ambiguity averse preferences. It seems that models which rely on a rich outcome space — like the mentioned models in the Anscombe-Aumann setting, the CEU models of Wakker (1989), Nakamura (1990) and Chew and Karni (1994), or the MEU model of Casadesus-Masanell, Klibanoff, and Ozdenoren (1998) — might be more suited to exploit the additional predictive power due to the presence of ambiguity attitudes.

The paper proceeds as follows: Section 1 introduces the required definitions and a key result. In Section 2 we warm up by showing a straightforward consequence of range convexity for CEU preferences like those described by Sarin and Wakker (1992). Sections 3 and 4 contain the main results in a Savage setting. Section 5 concludes by showing that the results lose their bite in the Anscombe-Aumann setting. Finally the Appendices contain some basic definitions (capacities and Choquet integrals), a review of the CEU and MEU models, and all the proofs.

1 Preliminaries and Notation

Here we introduce decision settings, preference models, some terminology, and a useful result.

1.1 Decision Settings

The two decision settings that we use in the paper are the one developed by Anscombe and Aumann (1963) and the one developed by Savage (1954).

In the simpler Savage setting, the objects of choice are 'acts', delivering a consequence for each state of the world. More precisely, there is a state space (S, Σ), where Σ is a σ-algebra of subsets of S, and a set $\mathcal{X}$ of *consequences* (alternatively, prizes), equipped with a σ-algebra $\mathcal{U}$ containing all the singletons. The set of acts $\mathcal{F}$ is the class of all the measurable functions from S into $\mathcal{X}$. *Simple acts* are those with a finite range. As customary, we abuse notation and identify $x \in \mathcal{X}$ with the *constant* act yielding x for every state $s \in S$.

In the Anscombe-Aumann setting, the objects of choice are also state-contingent acts, but the consequences are simple (objective) lotteries on the set of prizes $\mathcal{X}$. More precisely, let $\mathcal{P}$ be the set of all the simple (finite-ranged) lotteries on the σ-algebra $\mathcal{U}$. The set of acts is the set $\mathcal{F}$ of all the measurable functions from S into $\mathcal{P}$. As customary, we abuse notation and identify the elements $x \in \mathcal{X}$ with the degenerate lotteries in $\mathcal{P}$. Similarly, we use p to denote the *constant* act mapping every state $s \in S$ into the same point $p \in \mathcal{P}$.[2]

For any pair $p, q \in \mathcal{P}$ (in particular a pair of sure prizes in $\mathcal{X}$) and an event $A \in \Sigma$, we denote by $p\,A\,q$ the binary act which yields p if A obtains and q otherwise, and by $p\,\alpha\,q$ the lottery that yields p with probability α and q with probability $(1 - \alpha)$.

1.2 Preference Models

In general, we assume that a decision maker's choice behavior is described by a weak order (a complete and transitive binary relation) $\succcurlyeq$ on $\mathcal{F}$, and $\succ$ and $\sim$ respectively denote the asymmetric and symmetric component of $\succcurlyeq$. A weak order $\succcurlyeq$ is *nontrivial* if there are $\overline{x}, \underline{x} \in \mathcal{X}$ such that $\overline{x} \succ \underline{x}$. For any nontrivial weak order $\succcurlyeq$, we say that an event $A \in \Sigma$ is *essential for* $\succcurlyeq$ if $\overline{x} \succ \overline{x}\,A\,\underline{x} \succ \underline{x}$, for some $\overline{x} \succ \underline{x}$.

To define the basic preference model that we use in the paper we need to introduce some terminology and notation. As customary, a *representation* of $\succcurlyeq$ is a function V :

[2] We stick here to the traditional interpretation of the elements of $\mathcal{P}$ as lotteries. Of course the Anscombe-Aumann framework is consistent with any consequence space having a vector space structure (for instance, it could be a convex subset of $\mathbb{R}^n$, where we interpret consequences as bundles of goods).

$\mathcal{F} \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$,

$$f \succcurlyeq g \Longleftrightarrow V(f) \geq V(g). \tag{1}$$

Denote by $B(\Sigma)$ the set of all real-valued Σ-measurable simple functions. A functional $I : B(\Sigma) \to \mathbb{R}$ is *normalized* if $I(1_S) = 1$ and $I(1_\emptyset) = 0$.[3] It is *monotone* if, for all $\phi, \psi \in A$, $\phi \geq \psi$ implies $I(\phi) \geq I(\psi)$.

Definition 1 *Let $\succcurlyeq$ be a binary relation on $\mathcal{F}$. We say that a representation $V : \mathcal{F} \to \mathbb{R}$ of $\succcurlyeq$ is* canonical *if there exists a normalized and monotone functional $I_V : B(\Sigma) \to \mathbb{R}$ such that, if we let $u(x) \equiv V(x)$ for every $x \in \mathcal{X}$, $V(f) = I(u(f))$ for all $f \in \mathcal{F}$ and for all consequences $x \succcurlyeq y$ and all events A,*

$$V(x\, A\, y) = u(x)\, \rho_V(A) + u(y)\, (1 - \rho_V(A)), \tag{2}$$

where $\rho_V \equiv I_V(1_A)$ for all $A \in \Sigma$.

A relation $\succcurlyeq$ on $\mathcal{F}$ is called a biseparable ordering *if it admits a canonical representation that, if $\succcurlyeq$ has at least one essential event, is unique up to a positive affine transformation.*

Given a biseparable ordering $\succcurlyeq$, we call u its *canonical utility index*. Suppose that $\succcurlyeq$ is also nontrivial. It is easy to see that ρ_V is independent of the specific canonical representation V — so that we can denote it ρ — and that because of the monotonicity of I_V, ρ is a capacity (see Appendix A for a definition).

Moreover, for any $\underline{x}, \overline{x} \in \mathcal{X}$ such that $\overline{x} \succ \underline{x}$, define the 'likelihood' relation $\succcurlyeq^*$ on Σ as follows:[4] For every $A, B \in \Sigma$, let

$$A \succcurlyeq^* B \Longleftrightarrow \overline{x}\, A\, \underline{x} \succcurlyeq \overline{x}\, B\, \underline{x}. \tag{3}$$

That is, $A \succcurlyeq^* B$ iff the decision maker prefers to bet 'on' A rather than to bet 'on' B (with the same payoffs). By definition, $A \succcurlyeq^* B$ iff $\rho(A) \geq \rho(B)$. Therefore, we call ρ the decision maker's *willingness to bet*.

In the Anscombe-Aumann setting, we use a slightly smaller class of preferences, as we assume that they are also *affine* on the set $\mathcal{P}$ of the lotteries on the final prizes $\mathcal{X}$.

Definition 2 *Let $\succcurlyeq$ be a binary relation on the set $\mathcal{F}$ in an Anscombe-Aumann setting. We say that $\succcurlyeq$ is a* constant affine *(shortly,* c-affine*) biseparable ordering if $\succcurlyeq$ is biseparable, and its canonical representation V also satisfies the following property (called* constant affinity*): For every $p, q \in \mathcal{P}$ and every $\alpha \in [0, 1]$,*

$$V(p\, \alpha\, q) = \alpha V(p) + (1 - \alpha) V(q). \tag{4}$$

[3] For every $A \in \Sigma$, 1_A is the characteristic function of A.

[4] For biseparable orderings, the choice of $\underline{x}$ and $\overline{x}$ are inconsequential for $\succcurlyeq^*$, since these preferences satisfy a payoff independence condition (Savage (1954)'s P4 axiom).

Since it basically only restricts the decision maker's choices among bets (binary acts), the class of biseparable (or c-affine biseparable) orderings is very large, and it contains most of the decision models with state-independent utility that have been studied by decision theorists.[5] In particular, it contains CEU (hence SEU) and α-MEU orderings.[6]

1.3 Ambiguity Aversion

In Ghirardato and Marinacci (1997), we propose a behavioral notion of absolute ambiguity aversion, and we show that for biseparable preferences it is quite generally characterized as follows. Let

$$\mathcal{D}(\succcurlyeq) \equiv \left\{ P \in \Delta : \int_S u(f(s))\, P(ds) \geq V(f) \text{ for all } f \in \mathcal{F} \right\}.$$

A biseparable ordering $\succcurlyeq$ is ambiguity averse iff $\mathcal{D}(\succcurlyeq) \neq \emptyset$. In light of this result, here we define *ambiguity aversion* as having $\mathcal{D}(\succcurlyeq) \neq \emptyset$. In particular, this implies that a CEU ordering with capacity ρ is ambiguity averse if and only if $Core(\rho) \neq \emptyset$, i.e., ρ is balanced. In fact, in such a case $\mathcal{D}(\succcurlyeq) = Core(\rho)$. On the other hand in the case of a MEU ordering with set of priors $\mathcal{C}$, $\mathcal{D}(\succcurlyeq) = \mathcal{C}$. Hence all MEU orderings are ambiguity averse, whereas all 0-MEU orderings are ambiguity loving.

Schmeidler (1989) proposes a stronger property of ambiguity aversion in the Anscombe-Aumann setting. In the CEU case such property is characterized by *convexity* of ρ. As for MEU (resp. 0-MEU) orderings, they are ambiguity averse (resp. loving) also in this stronger sense.

In the same paper, we also propose a behavioral notion of unambiguous event, and we show that the set of unambiguous events of an ambiguity averse (or loving) biseparable preference is characterized as follows: An event is *unambiguous* for a preference $\succcurlyeq$ with willingness to bet ρ iff

$$A \in \Pi(\succcurlyeq) \equiv \{B \in \Sigma : \rho(A) + \rho(A^c) = 1\}.$$

Again, this result applies in particular to CEU and MEU preferences, where ρ respectively represents the beliefs and lower envelope of the set of priors (see Eq. (17) in App. B).

[5] In Ghirardato and Marinacci (2000) we provide an axiomatic characterization of biseparable orderings in both settings. However, there we call them *invariant* biseparable orderings, and the c-affine biseparable preferences satisfy a condition a bit stronger than (4).

[6] See Appendix B for the definitions. Here we just remark that by α-MEU we mean those orderings which are given by α times the minimum expected utility plus $(1-\alpha)$ times the maximum expected utility. As customary, we call a 1-MEU ordering just a MEU ordering.

1.4 Range Convexity and a Useful Result

A capacity ρ is *convex-ranged on* $\Lambda \subseteq \Sigma$ if for every $A \in \Lambda$ and $x \in [0, \rho(A)]$, $y \in [\rho(A), 1]$ there exist $B, C \in \Lambda$ with $B \subseteq A \subseteq C$, such that $\rho(B) = x$ and $\rho(C) = y$.[7] In particular, when $\Lambda = \Sigma$ we just say that ρ is *convex-ranged*.

Remark 1 When ρ is a probability measure, range convexity as defined above is equivalent to the following notion: For every $A \in \Sigma$ and every $\alpha \in [0, \rho(A)]$, there is $B \in \Sigma$ such that $B \subseteq A$ and $\rho(B) = \alpha$. However, the equivalence fails for general capacities, as the following example (suggested to us by Itzhak Gilboa) illustrates: Let $S = [0, 1] \cup [2, 3]$, and λ be the Lebesgue measure. Define a capacity ρ as follows: For every $A \subseteq S$, write $A = A_1 \cup A_2$, where $A_1 = A \cap [0, 1]$ and $A_2 = A \cap [2, 3]$, and let

$$\rho(A) = \begin{cases} (\lambda(A_1) + \lambda(A_2))/2 & A_1 \neq \emptyset,\ A_2 \neq \emptyset, \\ \lambda(A_1)/4 & A_2 = \emptyset, \\ \lambda(A_2)/4 & A_1 = \emptyset. \end{cases}$$

It is easy to see that for every $0 < \alpha < \rho(A)$ there exists $B \subseteq A$ such that $\rho(B) = \alpha$. However, consider $A = [0, 1]$: $\rho(A) = 1/4$, but there is no set $C \supseteq A$ such that $\rho(C) = 1/3$. $\diamond$

The following result about the uniqueness of probability measures in the presence of range convexity is introduced and proved in Ghirardato and Marinacci (1998, Theorem 1), where we study its consequences for SEU preferences.

Lemma 1 *Let P_1 and P_2 be probability measures on (S, Σ), with P_1 convex-ranged. Suppose that there is $A \in \Sigma$ such that $P_i(A) \in (0, 1)$ for $i = 1, 2$, and such that for all $B \in \Sigma$,*

$$P_1(A) = P_1(B) \Longleftrightarrow P_2(A) = P_2(B). \tag{5}$$

Then $P_1 = P_2$.

2 Prologue: Complement Symmetry

Consider a nontrivial CEU ordering $\succcurlyeq$ on $\mathcal{F}$. As mentioned above, the capacity ρ is a numerical representation of the decision maker's 'likelihood' relation $\succcurlyeq^*$ on Σ. Consider now the alternative 'likelihood' relation $\succcurlyeq_*$ defined by

$$A \succcurlyeq_* B \Longleftrightarrow \overline{x}\, B^c\, \underline{x} \succcurlyeq \overline{x}\, A^c\, \underline{x}. \tag{6}$$

That is, $A \succcurlyeq_* B$ if the decision maker prefers betting 'against' B to betting against A. If ρ is a probability measure (i.e., $\succcurlyeq$ is a SEU preference), then $\succcurlyeq^* = \succcurlyeq_*$. That is,

[7] This is weaker than the notion of range convexity for capacities commonly used in the literature (see, e.g., Gilboa (1987, p. 69).

one obtains an identical likelihood relation if instead of considering bets 'on' events, one considers bets 'against' events. However, this is not necessarily true if ρ is not additive (as observed, for example, by Gilboa (1989)).

In a comment on Sarin and Wakker (1992)'s axiomatization of CEU orderings in a Savage setting, Nehring (1994) argues that Sarin and Wakker's interpretation of one of their axioms (P4) is compelling only for those CEU preferences $\succcurlyeq$ for which $\succcurlyeq^* = \succcurlyeq_*$. This is tantamount to imposing the following condition, that he dubs 'complement symmetry' (CS): For all $A, B \in \Sigma$,

$$\overline{x}\, A\, \underline{x} \succcurlyeq \overline{x}\, B\, \underline{x} \Longleftrightarrow \overline{x}\, B^c\, \underline{x} \succcurlyeq \overline{x}\, A^c\, \underline{x}. \tag{7}$$

Nehring claims (p. 936) that "[...] a CEU preference that is representable by a capacity ρ is complement symmetric if and only if ρ is symmetric [...]",[8] a quite restrictive class of capacities that, as Nehring observes (*loc. cit.*), "rules out all Ellsberg-type phenomena", and hence is of limited practical interest.

This claim is not generally true: The following example shows that there are CEU preferences which are complement symmetric, but do not induce a symmetric capacity.

Example 1 On an arbitrary state space (S, Σ), given a probability measure P on Σ and constant $0 < k < 1$, consider the capacity $\hat{\rho}$ defined as follows: For every $A \in \Sigma$,

$$\hat{\rho}(A) = \begin{cases} kP(A) & A \neq S \\ 1 & A = S. \end{cases}$$

A CEU ordering represented by $\hat{\rho}$ is complement symmetric, but $\hat{\rho}$ is not symmetric. For another example, observe that any CEU ordering represented by a 'distortion' $\rho = g(P)$, where $g : [0,1] \to [0,1]$ is strictly increasing and such that $g(0) = 0$ and $g(1) = 1$, is complement symmetric. However, its capacity ρ may not be symmetric (one only needs that there be some $\alpha \in [0,1]$ for which $g(\alpha) \neq 1 - g(1-\alpha)$). △

However, Nehring's claim that CS implies symmetry is true for a subset of the CEU orderings, to which the orderings axiomatized by Sarin and Wakker (1992) also belong: The ones whose capacity is convex-ranged and symmetric on a class of events containing S and closed w.r.t. complements.[9] Indeed, we will presently show that for CEU orderings in this class, Nehring's claim can be considerably strengthened: a preference is ambiguity averse and satisfies a weakening of CS *only if* it is SEU. The weaker version of CS is the following:

Definition 3 *We say that a nontrivial weak order satisfies* weak complement symmetry (WCS) *if for every $A, B \in \Sigma$,*

$$\overline{x}\, A\, \underline{x} \sim \overline{x}\, B\, \underline{x} \Longleftrightarrow \overline{x}\, B^c\, \underline{x} \sim \overline{x}\, A^c\, \underline{x}. \tag{8}$$

[8] That is, for every $A \in \Sigma$, $\rho(A) + \rho(A^c) = 1$ (see App. A).

[9] Sarin and Wakker's axiomatization implies that there is a sub-σ-algebra Σ^{ua} of Σ on which the capacity ρ is convex-ranged and additive.

Condition (8) can be reworded as follows: Given $A \in \Sigma$, let $[A]$ denote the 'likelihood indifference class' passing through A; that is, using $\sim^*$ to denote the symmetric component of $\succcurlyeq^*$, $[A] \equiv \{B \in \Sigma : B \sim^* A\}$. Then Eq. (8) says that for every $A \in \Sigma$, $B \in [A]$ if and only if $B^c \in [A^c]$.

We can now state the announced result:

Proposition 1 *Consider a nontrivial CEU ordering $\succcurlyeq$ represented by a capacity ρ, which is convex-ranged on a class $\Lambda \subseteq \Sigma$ containing S and closed with respect to complements. Then:*

(i) $\succcurlyeq$ *satisfies WCS* if and only if *it satisfies CS;*

(ii) $\succcurlyeq$ *satisfies WCS and it induces a ρ symmetric on* Λ if and only if ρ *is symmetric (on Σ);*

(iii) $\succcurlyeq$ *satisfies WCS and it induces a ρ balanced and symmetric on* Λ if and only if $\succcurlyeq$ *is a SEU ordering.*

It will be observed that the capacity $\hat{\rho}$ used in the example above is not convex-ranged. Also, we remark an interesting corollary to the Proposition: Consider a preference which satisfies the assumptions of the Proposition and is represented by a distortion $\rho = g(P)$, as described above. Then ρ embodies ambiguity aversion *if and only if* it is a probability measure.

3 A Complement Symmetric Event

An obvious objection to Proposition 1 is that CS is too strong a property to require of ambiguity averse preferences. Indeed, requiring complement symmetry robs them of one of their most interesting features (see also Sarin and Wakker (1998)). In fact, it is quite clear that the presence of ambiguity might make one's evaluation of bets on different events depend on whether one stands to win or to lose contingently on those events: If the decision maker perceives more ambiguity about event B than about A, he might well prefer the bet on A over the bet on B, *and* the bet on A^c over the bet on B^c. This is exactly what complement symmetry rules out. Because of point (i) of Proposition 1, under range convexity, WCS is subject to similar criticism.

However, we can use Lemma 1 to show that, in the presence of range convexity, one draws similarly strong conclusions even if CS only holds for a single event. The following definition formalizes our main requirement.

Definition 4 *Given a nontrivial weak order $\succcurlyeq$, we say that an event $A \in \Sigma$ is* complement symmetric *if for all $B \in \Sigma$ and some $\overline{x} \succ \underline{x}$,*

$$\overline{x}\, A\, \underline{x} \succ \overline{x}\, B\, \underline{x} \implies \overline{x}\, B^c\, \underline{x} \succ \overline{x}\, A^c\, \underline{x} \tag{9}$$

$$\overline{x}\, A\, \underline{x} \sim \overline{x}\, B\, \underline{x} \implies \overline{x}\, B^c\, \underline{x} \sim \overline{x}\, A^c\, \underline{x}. \tag{10}$$

In words, an essential event A is complement symmetric if the decision maker prefers (resp. is indifferent) betting 'on' A over betting 'on' B, then he prefers (resp. is indifferent) betting 'against' B over betting 'against' A.

We assume that our preference has (at least) *one* essential and complement symmetric event. This is significantly weaker than CS, that requires that *all* events be complement symmetric. Moreover, notice that here the symmetry of betting behavior is required only 'below' A. That is, we do not require that $B \succ^* A$ imply $A^c \succ^* B^c$.

It is important to stress that we do not attach any normative value to the existence of an essential complement symmetric event. Our perspective is completely positive. Indeed, we remark that this condition is extremely weak, and, differently from CS, it is compatible with high levels of ambiguity aversion. The following example illustrates this point:

Example 2 Consider the classical Ellsberg 3-color urn, containing 30 red balls and 60 blue or yellow balls. The state space of the possible ball extractions is $S = \{r, b, y\}$. Consider a CEU decision maker with beliefs represented by the capacity ρ defined as follows: $\rho(r) = 1/3$, $\rho(b, y) = 2/3$, $\rho(b) = \rho(y) = 1/6$ and $\rho(r, y) = \rho(r, b) = 1/2$. This is clearly an ambiguity averse decision maker, displaying the preferences traditionally observed in this problem. If we consider $A = \{b\}$ and $B = \emptyset$ we see that his preference does not satisfy CS. However, $A = \{r\}$ is an essential complement symmetric event for these beliefs. △

To have anyway a better feel of the interpretation of this assumption, observe that the capacity $\rho(A)$ can be conceptually seen as the synthesis of two different types of considerations: One is a 'pure likelihood' judgement on the plausibility of A happening, the other is an 'ambiguity' factor, that modifies the pure likelihood judgement to take into account the ambiguity perceived about A. Assuming, as it seems reasonable, that the ambiguity perceived about A is the same as that perceived about A^c, the number $\rho(A)/\rho(A^c)$ is then an estimate of the 'pure' odds associated with A. Then, A is an essential complement symmetric event if $\rho(A) > \rho(B)$ (resp. $\rho(A) = \rho(B)$) implies

$$\frac{\rho(A)}{\rho(A^c)} > \frac{\rho(B)}{\rho(B^c)} \quad \left(\text{resp. } \frac{\rho(A)}{\rho(A^c)} = \frac{\rho(B)}{\rho(B^c)}\right).$$

That is, all the events which are as likely as A have identical 'pure' odds, and all those which are less likely than A have lower 'pure' odds.

In the main result of this section we show that under the assumptions on the willingness to bet ρ used in Proposition 1, additivity of the willingness to bet ρ almost follows from the assumption made above.

Theorem 1 *Consider a nontrivial biseparable ordering $\succcurlyeq$ whose willingness to bet ρ is convex-ranged and symmetric on a class $\Lambda \subseteq \Sigma$ containing S and closed under complementation. Then:*

(i) If $\succcurlyeq$ has an essential complement symmetric event, then Core(ρ) is at most a singleton;

(ii) $\succcurlyeq$ has an essential complement symmetric event and ρ is exact if and only if *ρ is additive.*

Thus, when there is an essential complement symmetric event, the willingness to bet can at most have one probability measure P in its core. Moreover, if it is also exact, then ρ must coincide with such P (which exists, since exact capacities are balanced). In particular, this will be the case if, as is often assumed, ρ is convex.

Theorem 1 does not yet enable us to conclude that the preference in question must be a SEU ordering. Indeed, in Ghirardato and Marinacci (1997) we show that there are biseparable orderings whose ρ is a probability measure which are *strictly* ambiguity averse, hence non-SEU. However, more can be said if we limit our attention to a subset of the biseparable preferences that contains both the CEU and MEU models as particular cases. These are the biseparable $\succcurlyeq$ such that their canonical representation V satisfies the following condition: For all $f \in \mathcal{F}$,

$$V(f) \geq \int_S u(f(s))\, \rho(ds), \tag{11}$$

where ρ is the willingness to bet associated with V, and the integral is taken in the sense of Choquet. We then obtain:

Corollary 1 *Consider a relation $\succcurlyeq$ satisfying the assumptions of Theorem 1. Suppose moreover that $\succcurlyeq$ is ambiguity averse, it satisfies (11) and it has an exact ρ. Then $\succcurlyeq$ has an essential complement symmetric event* if and only if *$\succcurlyeq$ has a SEU representation.*

Remark 2 The assumptions that $\succcurlyeq$ is ambiguity averse and it satisfies (11) are only used in the corollary to guarantee that $\succcurlyeq$ is SEU whenever ρ is additive. Thus, the same result can be obtained for any class of biseparable orderings with the latter property. ◇

We now apply this corollary to CEU and MEU preferences to obtain the following immediate:

Corollary 2 *Consider a nontrivial biseparable ordering $\succcurlyeq$ whose willingness to bet ρ is convex-ranged and symmetric on a class $\Lambda \subseteq \Sigma$ containing S and closed under complementation. Then the following hold:*

(i) $\succcurlyeq$ is a CEU ordering with exact ρ and an essential complement symmetric event if and only if *$\succcurlyeq$ is a SEU ordering;*

(ii) $\succcurlyeq$ is a MEU ordering with an essential complement symmetric event if and only if *$\succcurlyeq$ is a SEU ordering.*

This result shows that the range convexity of ρ, rather than CS or WCS, is what drives the strong conclusions we obtain in Theorem 1. Even in the presence of the significantly milder condition that there be a single essential complement symmetric event, range convexity reduces MEU preferences to SEU, and it almost reduces ambiguity averse CEU preferences to SEU (the exception being those ambiguity averse CEU preferences whose ρ is balanced but not exact).

In particular, Corollary 2 allows us to reformulate Nehring's critique of the axiomatization of Sarin and Wakker (1992) as follows: The range convexity of beliefs that plays such a crucial role in that axiomatization makes the scope of the preferences they describe fairly narrow. Differently from Nehring, what motivates this point is not the fact that the interpretation of the axioms may implicitly be based on the validity of CS. Rather, our motivation is the observation that the existence of a complement symmetric event is a very mild condition which is likely to be often satisfied. Clearly, a similar critique could be made of an axiomatization of MEU preferences constructed along the same lines as Sarin and Wakker (1992), since that would likely yield a ρ satisfying the conditions of Corollary 2.

4 Without Symmetry on Λ

One could of course raise doubts about the role that the symmetry of ρ on Λ plays in the results in the previous section. For example, it is material to obtaining the symmetry of ρ on Σ in Theorem 1. We now show that similar conclusions can be obtained even if there is no class Λ on which ρ is symmetric.

We reinforce range convexity by assuming that it holds over the whole Σ, and we add the requirement that the essential complement symmetric event A be also unambiguous (i.e., $A \in \Pi(\succcurlyeq)$). For example, event A in Example 2 also satisfies this stronger condition. We then obtain the following result for the CEU case:

Proposition 2 *Consider a nontrivial CEU ordering $\succcurlyeq$ whose willingness to bet ρ is convex and convex-ranged. Then $\succcurlyeq$ has an essential, complement symmetric and unambiguous event* if and only if *$\succcurlyeq$ is a SEU ordering.*

This result shows that the CEU axiomatization of Gilboa (1987) has a similar limitation as Sarin and Wakker's: Because of the range convexity of ρ, the existence of an unambiguous complement symmetric event and a little structure on the preferences,[10] are tantamount to satisfying all of Savage's axioms.

Similarly to what we did for earlier results, Proposition 2 can be proved more generally for biseparable orderings; we have only stated the CEU version because we deem it to be the most interesting one. (For instance, the condition that ρ is convex-ranged on Σ

[10] Wakker (1997) provides a characterization of concavity of ρ for CEU orderings in the Savage setting.

seems too strong for MEU preferences, where $\rho = \underline{P}$.) However, there is another result that symmetrically seems to be more interesting for α-MEU orderings than for CEU orderings, even though it could also be generalized to biseparable preferences (assuming ambiguity aversion and using $\mathcal{D}(\succcurlyeq)$ in place of $\mathcal{C}$).

Proposition 3 *Consider a nontrivial α-MEU ordering $\succcurlyeq$ such that $\alpha \neq 1/2$ and at least one $P \in \mathcal{C}$ is convex-ranged. Then $\succcurlyeq$ has an essential, complement symmetric and unambiguous event* if and only if *$\succcurlyeq$ is a SEU ordering.*

Remark 3 The proposition is false if $\alpha = 1/2$. In fact, it is easy to see that for any such preference $\Pi(\succcurlyeq) = \Sigma$, and every event is complement symmetric. Hence, the assumptions have no bite. This does imply that a 1/2-MEU preference is ambiguity averse iff ρ is additive (see Lemma 2 in Appendix C). However, ρ could be additive without $\succcurlyeq$ being SEU. $\diamond$

While no axiomatizations of MEU (or α-MEU) preferences with this range convexity property are available, it seems fairly likely that any axiomatization which uses a rich-state framework would deliver at least one $P \in \mathcal{C}$ convex-ranged. Proposition 3 shows that such axiomatization would be narrowly more general than Savage's.

5 Range Convexity and the Anscombe-Aumann Setting

In the Anscombe-Aumann setting, the range convexity of the decision maker's beliefs is implicitly built into the framework, by assuming that all the simple lotteries on $\mathcal{X}$ are available. It is therefore natural to conjecture that results similar to those presented in the previous two sections could be proved for c-affine biseparable preferences, in particular for the CEU and MEU models of Schmeidler (1989) and Gilboa and Schmeidler (1989).[11] However, that conjecture is easily disproved by looking at the following extension of Example 2.

Example 3 Consider the Ellsberg urn described in Example 2, and imagine that an independent randomizing device (a 'roulette wheel') is available, which can be used to make simple lotteries on $\mathcal{X}$ as payoffs. In this Anscombe-Aumann setting, consider a CEU ordering $\succcurlyeq$ whose beliefs are given by the capacity ρ in Example 2. As explained there, $\succcurlyeq$ *does* have an essential complement symmetric event, and ρ is convex. But $\succcurlyeq$ is clearly not a SEU ordering. $\triangle$

The reason for the failure of Proposition 2 in this example is that the capacity ρ is not convex-ranged on Σ.

[11] This is what happens in the SEU case, where the same results that hold in the Savage setting (with convex-ranged beliefs) hold in the Anscombe-Aumann setting (see Ghirardato and Marinacci (1998)).

One could of course argue that the decision maker's beliefs in an Anscombe-Aumann setting are really described by ρ *and* by her beliefs on the behavior of the randomizing device, i.e., on the 'roulette wheel' events. An 'event' is then a product of an event in Σ and an event on the roulette wheel. However, in such a case the notion of complement symmetric event becomes more demanding than in Savage's setting. At the very least least, we need to find an 'event' which satisfies the following condition:

Definition 5 *Given a nontrivial weak order in an Anscombe-Aumann setting, we say that it has an* essential fully complement symmetric 'event' *if there are $\overline{x}, \underline{x} \in \mathcal{X}$ such that $\overline{x} \succ \underline{x}$ and $\alpha \in (0,1)$ such that, for every $B \in \Sigma$ and every $\beta, \beta' \in [0,1]$,*

$$\overline{x}\,\alpha\,\underline{x} \sim [(\overline{x}\,\beta\,\underline{x})\,B\,\underline{x}] \implies [(\overline{x}\,(1-\beta)\,\underline{x})\,B\,\overline{x}] \sim \overline{x}\,(1-\alpha)\,\underline{x} \tag{12}$$

$$\overline{x}\,\alpha\,\underline{x} \sim [\overline{x}\,B\,(\overline{x}\,\beta'\,\underline{x})] \implies [\underline{x}\,B\,(\overline{x}\,(1-\beta')\,\underline{x})] \sim \overline{x}\,(1-\alpha)\,\underline{x}. \tag{13}$$

The two conditions correspond to Eq. (10) in Definition 4, and they say that there is some α such that all 'events' of probability α have complements which are equally likely. Since the set of 'events' is very rich, they are significantly more demanding than requiring the existence of an $A \in \Sigma$ which satisfies (10). For example, the preference $\succcurlyeq$ in Example 3 does *not* have an essential fully complement symmetric 'event'.[12]

However, we do have the following result:

Proposition 4 *Consider a nontrivial c-affine biseparable ordering (in an Anscombe-Aumann setting) $\succcurlyeq$ with willingness to bet ρ. Then $\succcurlyeq$ has an essential fully complement symmetric 'event'* if and only if *ρ is symmetric. If, in addition, ρ is balanced, then it is additive.*

Proceeding as in Section 3, we immediately obtain the consequences for CEU and MEU orderings:

Corollary 3 *Consider a nontrivial c-affine biseparable ordering $\succcurlyeq$ with whose willingness to bet ρ. Then the following hold:*

(i) $\succcurlyeq$ *is an ambiguity averse CEU ordering with an essential fully complement symmetric 'event'* if and only if $\succcurlyeq$ *is a SEU ordering;*

(ii) $\succcurlyeq$ *is a MEU ordering with an essential fully complement symmetric event* if and only if $\succcurlyeq$ *is a SEU ordering.*

This result applies to the CEU model of Schmeidler (1989) and to the MEU model of Gilboa and Schmeidler (1989). However, we do not think that it raises any serious concerns as to the possibility of those models of generally describing ambiguity averse decision makers. The reason is that, as hinted above, in this setting the existence of a fully

[12] For instance, it is easy to see that taking $\alpha = 1/3$ does not work: Take $B = \{r, y\}$ and $\beta = 2/3$.

complement symmetric 'event' does not seem as mild an assumption as the existence of a complement symmetric $A \in \Sigma$ in the Savage setting. What magnified the power of the latter assumption in the results in the previous sections was the additional range convexity assumptions, which are naturally provided by certain axiomatic structures. In contrast, the existence of a fully complement symmetric 'event' incorporates the consequences of range convexity that are needed for the results (and this is why it does not need additional range convexity assumptions). This difference makes it a much less plausible assumption from a descriptive point of view.

Appendix A Capacities and Choquet Integrals

A *capacity* on (S, Σ) is a set-function $\rho : \Sigma \to [0, 1]$ which is monotonic (i.e., $A \subseteq B$ implies $\rho(A) \leq \rho(B)$) and normalized (i.e., $\rho(\emptyset) = 0$ and $\rho(S) = 1$). A capacity that satisfies finite additivity (i.e., $\rho(A \cup B) = \rho(A) + \rho(B)$ for all disjoint $A, B \in \Sigma$) is called a *probability measure*.[13] The *core* of a capacity $\rho : \Sigma \to [0, 1]$ is the possibly empty set of all the probability measures that setwise dominate ρ, that is,

$$Core(\rho) \equiv \{P : P \text{ is a probability measure on } (S, \Sigma),\ P(A) \geq \rho(A) \text{ for all } A \in \Sigma\}.$$

A capacity ρ is called *balanced* if $Core(\rho)$ is nonempty; it is called *exact* if it is balanced and $\rho(A) = \min_{P \in Core(\rho)} P(A)$ for all $A \in \Sigma$; it is called *convex* if for every $A, B \in \Sigma$,

$$\rho(A \cup B) \geq \rho(A) + \rho(B) - \rho(A \cap B).$$

Convex capacities are exact, and exact capacities are balanced, but the converses are not true.

Given a class $\Lambda \subseteq \Sigma$ closed with respect to complements, a capacity $\rho : \Sigma \to [0, 1]$ is called *symmetric on* Λ if $\rho(A) + \rho(A^c) = 1$ for all $A \in \Lambda$. For convenience, if $\Lambda = \Sigma$, we just say that ρ is *symmetric*.

The notion of integral used for capacities is that due to Choquet (1953). The *Choquet integral* of a a measurable function $\varphi : S \to \mathbb{R}$ w.r.t. ρ is defined as follows:

$$\int_S \varphi \, d\rho = \int_0^\infty \rho(\{s \in S : \varphi(s) \geq \alpha\}) \, d\alpha + \int_{-\infty}^0 [\rho(\{s \in S : \varphi(s) \geq \alpha\}) - 1] \, d\alpha, \tag{14}$$

where the integrals are taken in the sense of Riemann (they are well-defined because ρ is monotone). When ρ is a probability measure, Choquet and Lebesgue integrals are equal.

Appendix B CEU and MEU Orderings

B.1 CEU Orderings

In the Savage setting, a binary relation $\succcurlyeq$ on $\mathcal{F}$ is called a *CEU ordering* if there exist a utility function $u : \mathcal{X} \to \mathbb{R}$ and a capacity ρ on Σ such that the decision maker's preferences are represented by the *Choquet integral* of u with respect to ρ (see Appendix A for the definitions). That is, $\succcurlyeq$ is represented by $V : \mathcal{F} \to \mathbb{R}$ defined as follows

$$V(f) = \int_S u(f(s)) \, \rho(ds). \tag{15}$$

Axioms that characterize a CEU ordering in this setting are presented in Gilboa (1987), Wakker (1989) and Sarin and Wakker (1992). In the Anscombe-Aumann setting, $\succcurlyeq$ is

[13] Except where otherwise noted, all the probability measures in this paper are *finitely* additive.

a *CEU ordering* if the V functional satisfies (15) and is also affine on $\mathcal{P}$. Axioms that characterize a CEU ordering in this setting are presented in Schmeidler (1989). In either setting, a *SEU ordering* is a CEU ordering whose ρ is a probability measure.

Clearly, a nontrivial CEU ordering is biseparable, with canonical representation given by V (Choquet integrals are easily seen to satisfy monotonicity). In the Anscombe-Aumann setting, a CEU ordering is also c-affine biseparable.

B.2 α-MEU Orderings

In the Savage setting, a binary relation $\succcurlyeq$ on $\mathcal{F}$ is called an *α-MEU ordering* if the following hold: There are a utility function $u : \mathcal{X} \to \mathbb{R}$, a (weak*-) closed and convex set of probabilities $\mathcal{C}$ on (S, Σ), and a coefficient $\alpha \in [0,1]$ such that $\succcurlyeq$ is represented by the functional $V : \mathcal{F} \to \mathbb{R}$ given by

$$V(f) = \alpha \min_{P \in \mathcal{C}} \int_S u(f(s))\, P(ds) + (1-\alpha) \max_{P \in \mathcal{C}} \int_S u(f(s))\, P(ds). \tag{16}$$

When $\alpha = 1$, we call $\succcurlyeq$ a *MEU ordering.* Casadesus-Masanell *et al.* (1998) have recently developed an axiomatization of MEU orderings in this setting. In an Anscombe-Aumann setting, $\succcurlyeq$ is an *α-MEU ordering* if the V functional satisfies (16) and is affine on $\mathcal{P}$. An axiomatic characterization of MEU orderings in this setting is presented in Gilboa and Schmeidler (1989). In either setting, a SEU ordering is a α-MEU ordering which has $\mathcal{C} = \{P\}$.

A nontrivial α-MEU ordering is biseparable, with canonical representation V, and

$$\rho(\cdot) = \alpha\, \underline{P}(\cdot) + (1-\alpha)\, \overline{P}(\cdot), \tag{17}$$

where $\underline{P}(\cdot) \equiv \min_{P \in \mathcal{C}} P(\cdot)$ and $\overline{P}(\cdot) \equiv \max_{P \in \mathcal{C}} P(\cdot)$. The set-function $\underline{P}$ is seen to be an exact capacity. In the Anscombe-Aumann setting, a α-MEU ordering is also c-affine biseparable.

Appendix C Proofs

The following easy lemma is used in the proof of Proposition 1, and it has some independent interest.

Lemma 2 *A symmetric capacity ρ on (S, Σ) is balanced (i.e., Core$(\rho) \neq \emptyset$)* if and only if *it is a probability measure.*

Proof: Let $P \in Core(\rho)$. We have:

$$\rho(A) \leq P(A) = 1 - P(A^c) \leq 1 - \rho(A^c) = \rho(A)$$

so that $\rho(A) = P(A)$ for all $A \in \Sigma$. This implies that ρ is additive. ∎

Proof of Proposition 1: Item (i) is shown as follows. The only implication that needs proof is that WCS implies CS. Hence, suppose that $A \succ^* B$. By the range convexity of ρ on Λ (and the fact that $S \in \Lambda$) we have that there is $C \in \Lambda$ such that $C \sim^* A$ (which, by WCS, implies $A^c \sim^* B^c$), so that, using the fact that $\succcurlyeq^*$ is a weak order, we get $C \succ^* B$. Again by the range convexity assumption, we can find $D \in \Lambda$, $D \subseteq C$ such that $D \sim^* B$ (again by WCS: $D^c \sim^* B^c$). Hence $C \succ^* D$. Since $C^c \subseteq D^c$, monotonicity of ρ implies $D^c \succcurlyeq^* C^c$, or, equivalently, $B^c \succcurlyeq^* A^c$. If it were $B^c \sim^* A^c$, WCS would imply $A \sim^* B$, which is not the case. Hence $B^c \succ^* A^c$, showing that CS holds.

We next prove (ii). From the fact that $\succcurlyeq$ is a CEU ordering represented by ρ, and (8) we get that, for all $A, B \in \Sigma$

$$\rho(A) = \rho(B) \Longrightarrow \rho(A^c) = \rho(B^c).$$

In particular, since ρ is convex-ranged on Λ, there is some $B_0 \in \Lambda$, such that $\rho(B_0) = \rho(A)$, so that we obtain

$$\rho(A) + \rho(A^c) = \rho(B_0) + \rho(B_0^c) = 1,$$

where the last equality follows from the fact that ρ is symmetric on Λ. This shows that ρ is symmetric on Σ.

As for (iii), it follows immediately from the fact that ρ is symmetric and Lemma 2. ∎

Proof of Theorem 1: We start by proving (i). Employing the same argument used in showing (ii) of Theorem 1, we see that for all $B \in \Sigma$, if $\rho(B) = \rho(A)$ (where A is the essential complement symmetric event), then $\rho(B) + \rho(B^c) = 1$. Letting $\rho(A) = \alpha \in (0, 1)$, we thus have that if $P \in \mathit{Core}(\rho)$, then $\rho(B) = P(B) = \alpha$. Hence, for every $B \sim^* A$, $P(B) = \alpha$ for all $P \in \mathit{Core}(\rho)$.

We now show that, conversely, if for $B \in \Sigma$ there is $P \in \mathit{Core}(\rho)$ such that $P(B) = \alpha$, then $B \sim^* A$. For every $P \in \mathit{Core}(\rho)$ and every $C \in \Sigma$, $\rho(C) \leq P(C) \leq \bar{\rho}(C)$ (where $\bar{\rho}$ is the complementary capacity of ρ defined by: for all $C \in \Sigma$, $\bar{\rho}(C) = 1 - \rho(C^c)$). Hence $\rho(B) \leq \alpha = \rho(A)$. If equality holds, we are done. Otherwise, suppose that $\rho(B) < \alpha$. By (9), $\bar{\rho}(B) < \bar{\rho}(A) = \alpha$. But then $P(B) < \alpha$, and we get a contradiction.

We thus conclude that for every $P \in \mathit{Core}(\rho)$ and every $B \in \Sigma$,

$$P(B) = \alpha \Longrightarrow \rho(B) = \alpha. \tag{18}$$

Since ρ is convex-ranged on Λ, we can find a chain $\{C_\beta\}_{\beta \in [0,1]} \subseteq \Lambda$ such that $\rho(C_\beta) = \beta$ for all $\beta \in [0, 1]$. Let Σ_0 be the algebra generated by the chain $\{C_\beta\}$. We want to show that each $P \in \mathit{Core}(\rho)$ is strongly continuous on Σ_0. That is, for every $\varepsilon > 0$ one can find a finite partition $\{B_i\}_{i=1}^n$ of S such that $0 < P(B_i) \leq \varepsilon$ for all $i = 1, \ldots, n$.

Let $0 < \varepsilon < 1$. There exists $C_\varepsilon \in \{C_\beta\}$ such that $P(C_\varepsilon) = \varepsilon$. Since $\varepsilon < 1$, $P(C_\varepsilon^c) > 0$. If $P(C_\varepsilon^c) \leq \varepsilon$, we are done. Otherwise $\Sigma_0 \ni C_{2\varepsilon} \setminus C_\varepsilon \subseteq C_\varepsilon^c$ and $P(C_{2\varepsilon} \setminus C_\varepsilon) = \varepsilon$. Since

$P(C_\varepsilon^c) > \varepsilon$, we have $\varepsilon < 1/2$. Hence, $P(C_{2\varepsilon}^c) > 0$. If $P(C_{2\varepsilon}^c) \leq \varepsilon$, we are done. Otherwise $\mathcal{A}_0 \ni C_{3\varepsilon} \setminus C_{2\varepsilon} \subseteq C_{2\varepsilon}^c$ and $P(C_{3\varepsilon} \setminus C_{2\varepsilon}) = \varepsilon$. Proceeding in this way, we can construct a finite partition $\{B_i\}_{i=1}^n \subseteq \mathcal{A}_0$ such that $0 < P(B_i) \leq \varepsilon$ for all i, where n is the smallest positive integer such that $\varepsilon \geq 1/n$. This proves our claim that that each $P \in \mathit{Core}(\rho)$ is strongly continuous on Σ_0. Since $\Sigma_0 \subseteq \Sigma$, each $P \in \mathit{Core}(\rho)$ is strongly continuous on Σ, hence (by a classical result of (Savage 1954)) convex-ranged on Σ.

For all $P, P' \in \mathit{Core}(\rho)$, (18) implies that, for all $B \in \Sigma$,

$$P(B) = \alpha \Longleftrightarrow P'(B) = \alpha.$$

By Lemma 1, we thus have $P = P'$. This clearly implies that $\mathit{Core}(\rho) = \{P\}$. The proof of (ii) then follows immediately: If ρ is exact, then $\rho = P$. ∎

Proof of Corollary 1: From Theorem 1 it follows that ρ is additive, so that $\mathit{Core}(\rho) = \{\rho\}$. By ambiguity aversion $\mathcal{D}(\succcurlyeq) \neq \emptyset$, and since it is immediate that $\mathit{Core}(\rho) \supseteq \mathcal{D}(\succcurlyeq)$, we have that $\mathcal{D}(\succcurlyeq) = \{\rho\}$. Hence, using Eq. (11) we get that for every $f \in \mathcal{F}$

$$\int_S u(f(s))\, \rho(ds) = V(f),$$

which proves that $\succcurlyeq$ has a SEU representation with utility u and probability ρ. The converse is immediate. ∎

Proof of Proposition 2: Since $A \in \Pi(\succcurlyeq)$, (10) implies that if $\rho(B) = \rho(A)$, then $\rho(B) + \rho(B^c) = 1$. We proceed as in the proof of Theorem 1, to show that (18) holds for every $P \in \mathit{Core}(\rho)$ and every $B \in \Sigma$.

Also, since ρ is convex-ranged, we reason as in that proof to show that there exists a chain $\{C_\beta\}$ such that $\rho(C_\beta) = \beta$ for all $\beta \in [0,1]$. As ρ is convex, there exists a $P \in \mathit{Core}(\rho)$ such that $P(C_\beta) = \rho(C_\beta) = \beta$ for all $\beta \in [0,1]$ (see Delbaen (1974)). Let Σ_0 be the algebra generated by the chain $\{C_\beta\}$. Yet again, we follow the previous argument to show that P is strongly continuous on Σ_0, and so on Σ as well. Hence, P is convex-ranged on Σ. Hence, we know that every measure in the core is convex-ranged, and that for all $P, P' \in \mathit{Core}(\rho)$, we have

$$P(A) = \alpha \Longleftrightarrow P'(A) = \alpha$$

so that, again by Lemma 1, $P = P'$. Hence, $\mathit{Core}(\rho) = \{P\}$. Since ρ is convex, $\rho = P$. ∎

Proof of Proposition 3: We need a lemma first.

Lemma 3 *Suppose that $\succcurlyeq$ is an α-MEU preference, with $\alpha \neq 1/2$. Then for every $A \in \Pi(\succcurlyeq)$,*

$$P(A) = P'(A) \quad \text{for all } P, P' \in \mathcal{C}.$$

Proof of Lemma 3: Using the definition of ρ in Eq. (17) and the fact that $A \in \Pi(\succcurlyeq)$, we obtain

$$\alpha\,\underline{P}(A) + (1-\alpha)\,\overline{P}(A) + \alpha\,\underline{P}(A^c) + (1-\alpha)\,\overline{P}(A^c) = 1.$$

If we now use the identity $\underline{P}(A^c) = 1 - \overline{P}(A)$, we can rewrite the equation above as follows:

$$\alpha\,(\underline{P}(A) - \overline{P}(A)) = (1-\alpha)(\underline{P}(A) - \overline{P}(A)).$$

When $\alpha \neq 1/2$, this can only hold if $\underline{P}(A) = \overline{P}(A)$, i.e., $P(A) = P'(A)$ for all $P, P' \in \mathcal{C}$, concluding the proof of the lemma. ∎

Let $\gamma = \underline{P}(A)$. Let $P' \in \mathcal{C}$ and $B \in \Sigma$ be such that $P'(B) = \gamma$. Then $\underline{P}(B) \leq \gamma$. We show that, actually, $\underline{P}(B) = \gamma$. Suppose not. Then, by (9), $\underline{P}(B^c) > 1 - \gamma$, so that $\overline{P}(B) < \gamma$, which contradicts $P'(B) = \gamma$. Hence, $\underline{P}(B) = \gamma$. In turn, this and Eq. (10) implies $\min P(B^c) = 1 - \gamma$, and so $P(B) = \gamma$ for all $P \in \mathcal{C}$. Since P' was arbitrary, we conclude that for all $B \in \Sigma$, $P(B) = \gamma$ if and only if $P(B) = \gamma$ for all $P, P' \in \mathcal{C}$.

Consider now the convex-ranged measure $P^0 \in \mathcal{C}$. We have that for every $B \in \Sigma$ and every $P \in \mathcal{C}$, $P(B) = \gamma$ iff $P^0(B) = \gamma$. Lemma 1 then shows that $P = P^0$, as wanted. ∎

Proof of Proposition 4: To prove the 'only if' of the first statement we rewrite (12) and (13) in terms of the canonical representation V, having normalized u so that $u(\overline{x}) = 1$ and $u(\underline{x}) = 0$, and obtain:

$$\rho(B)\beta = \alpha \implies \beta\rho(B^c) + (1-\beta) = 1 - \alpha \tag{19}$$
$$\beta' + (1-\beta')\rho(B) = \alpha \implies \rho(B^c)(1-\beta') = 1 - \alpha. \tag{20}$$

Consider any $B \in \Sigma$. Suppose first that $\rho(B) \geq \alpha$. Then there is $\beta \in (0,1)$ such that $\rho(B)\beta = \alpha$. Applying (19) and summing the two equations together we get

$$1 = (1-\beta) + \beta(\rho(B) + \rho(B^c)),$$

which implies $\rho(B) + \rho(B^c) = 1$.

Suppose now that $\rho(B) < \alpha$. Then we can find $\beta' \in (0,1)$ such that $\beta' + (1-\beta')\rho(B) = \alpha$. Applying (20) and summing we thus obtain

$$1 = \beta' + (1-\beta')(\rho(B) + \rho(B^c)),$$

which again implies $\rho(B) + \rho(B^c) = 1$. The proof of the 'if' follows immediately. As for the second statement, it is immediate from the first one and Lemma 2. ∎

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