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EQUILIBRIUM EQUIVALENCE WITH J CANDIDATES AND N VOTERS

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Equilibrium Equivalence with J Candidates and N Voters

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Abstract

In this paper, we examine the incentives facing candidates in the spatial voting model. We assume that voters' types are independent, but allow for nonidentical distributions across voters. Examining candidate positional equilibria as a function of voter behavior, we find that what we term p-symmetric strict p-local equilibria when candidates maximize expected plurality are also strict p-local equilibris when candidates maximize probability of victory. This result holds for arbitrary numbers of candidates and voters. We also show that, for generic type distributions, interior p-asymmetric equilibria under maximization of expected vote share are not equilibria under maximization of probability of victory.

1 Introduction

The question, "What are political candidates' goals?" is an inherently empirical concern. However, the question's importance is a theoretical matter. Politicians may have any of several objectives when running for elected office, but which one characterizes reality is an important matter only if the different objectives lead to different behaviors in equilibrium. Unfortunately, this is the case. This paper examines two such objective functions which have been used in the theoretical literature on voting over the previous 50 years: expected vote share and probability of victory. Our main question is straight-forward. When are these objective functions equivalent? That is to say, when are predictions generated by examination of one of these objective functions valid for the other?

Several authors have written on this question. Aranson, Hinich, and Ordeshook [1] examine several candidate objective functions. The results they obtain are far from heartening. In particular, the authors generally find no powerful equivalence results.

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Hinich [6] states a claim that, in 2 candidate elections without abstention, expected vote share and probability of victory yield identical best response functions. Ledyard [9] makes a similar claim for two candidate elections with abstention. Ledyard's claim was argued at the limit, and thus it is not clear that it is true. Finally, Patty [12] examines both Hinich's and Ledyard's claims and proves that the best response functions generated by expected plurality, expected vote share, and probability of victory are identical in two candidate elections, with or without abstention, with arbitrary numbers of voters. However, Patty assumes that voters' types are independently and identically distributed across voters and provides an example with two candidates showing that best response equivalence does not hold with nonidentically distributed voter types.

In this paper, we extend the study of candidate objective functions to the question of equilibrium equivalence. Best response equivalence is essentially a decision-theoretic concern, as it is defined to hold regardless of the opponents' strategies. Equilibrium equivalence, on the other hand, is a game-theoretic concern. Two objective functions are said to be equilibrium equivalent if the sets of Nash equilibria under the two objective functions are identical. In order to study probabilistic voting models in as general a fashion as possible, we characterize candidate positions by the resulting voter behavior rather than by a specific policy space. Thus, one may consider our method as examining a game in which candidates are taking the voter behavior as given (i.e. they are backward inducing along the extensive form game tree).

We prove our results in what we term p-space, the N-fold Cartesian product of the voters' J-dimensional simplexes. It is with respect to this space that we define p-symmetry, which essentially amounts to all voters mixing with equal probability between each of the J candidates. By examining the game in p-space, we are able to provide results which apply to a very general class of probabilistic voting models. In addition, the notion of p-neighborhoods is a weaker version of locality than neighborhoods in a policy space whenever the average behavior of each voter is a continuous function of the policies proposed by each candidate. This type of continuity generally holds in most probabilistic voting models in the literature. Nevertheless, we do not impose any such restriction. An additional advantage of our framework is that our results can be applied to either traditional probabilistic models of choice (see, for example, Luce [10] or Coughlin [2])or models of incomplete information in which voters optimally choose based on privately known preferences (see, for example, Ledyard [9] or McKelvey and Patty [11]).

Our first result is that p-symmetric strict p-local equilibria under maximization of expected plurality and maximization of probability of victory are identical whenever voters' types are independently distributed. In addition, we prove that, asymmetric interior critical points generically do not coincide under the two objective functions. That is, an asymmetric interior equilibrium under one objective function is generically not an equilibrium under the other objective function.

These results are motivated by the results of several previous papers in probabilistic voting models of candidate competition. For instance, Coughlin and Nitzan [3], [4] examine local Nash equilibria for two candidate elections under a probabilistic voting model. Similarly, McKelvey and Patty [11] examine a model of strategic probabilistic voting with an arbitrary number of candidates seeking to maximize expected margin of victory. They prove the existence of a p-symmetric strict Nash equilibrium at the point that maximizes the sum of the voters' utility functions whenever the number of voters is large enough. Our first result, Theorem 4, implies that the p-symmetric equilibrium characterized by McKelvey and Patty is also an equilibrium under maximization of probability of victory.

McKelvey and Patty show that the point which maximizes the sum of voters' utilities is a local equilibrium which "becomes" global as the number of voters grows without bound. The logic is that voters become approximately indifferent to the policies chosen by the candidates, implying that candidates are not able to alter the strategies of the voters very much when the number of voters is large. In addition, McKelvey and Patty show that this form of asymptotic indifference will occur in a large class of probabilistic voting models. Thus, the applications of p-local equilibria may be more general than appears at first glance.

2 The Model

Let $\mathcal{J}$, with $|\mathcal{J}| = J$, denote the set of candidates and $\mathcal{N}$, with $|\mathcal{N}| = N$, denote the set of voters. Each candidate simultaneously chooses a point in some policy space X. We denote the space of all J-dimensional vectors of policy proposals by Y.

We will write the action of voter i, given $y \in Y$, as $s_i = \sigma_i(y)$, and denote the number of votes received by candidate j by $v_j = |\{i \in \mathcal{N} | s_i = j\}|$. We will write s for the vector of s_i for all voters i. We will denote the probability that voter i votes for candidate j at $y \in Y$ by $p_{ij}(y)$ and the vector of all $p_{ij}(y)$, for some candidate j and all voters i, by $p_j(y)$. We make no assumptions concerning p_i except that it maps Y into the J-dimensional simplex. We will say that *p represents a voting strategy profile.*

For any $s \in S$, let $W(s) \in \{j \in \mathcal{J} | v_j \geq \max_{l \in \mathcal{J}} v_l\}$ denote the winning candidate at s. In the case of a tie, $W(s)$ is assumed to be determined by a fair lottery between all eligible candidates.[1]

When considering the probability of victory, let k_J^* denote the minimum number of votes with which a candidate can tie for victory.[2] For any $x \in Y$, let $G_j(i, y)$ denote the probability candidate j wins, conditional on voter i voting for j ($s_i = j$) and let $H_j(i, x)$ denote the conditional probability that candidate j wins, conditional on $s_i \neq j$.

[1]That is, we will reduce the cases of ties into sets of winners. This is possible because we assume that any tie-breaking lottery is fair, and hence independent of the identities of the candidates tied for victory.

[2]This number is of course well-defined and equal to the smallest integer greater than or equal to $\frac{N}{K}$.

3 Equilibrium

We use the notation from Section 2 to express the probability of victory for candidate j, given a candidate strategy profile x, as a sum over the voters. This sum is given in the following lemma.

Lemma 1 *Given a policy profile, $y \in Y$, the probability of victory by candidate j is given by*

$$R_j(x) = \frac{1}{N}\sum_i^N [p_{ij}(x)G_j(i,x) + (1 - p_{ij}(x))H_j(i,x)]. \quad (3.1)$$

Proof: Consider any voter i and any candidate j. From the definition of conditional probabilities and the assumption of independence,

$$\begin{aligned} R_j(x) &= \Pr[s_i = j \ \cap \ W(s) = j] + \Pr[s_i \neq j \ \cap \ W(s) = j] \\ &= \Pr[s_i = j]\Pr[W(s) = j | s_i = j] + \Pr[s_i \neq j]\Pr[W(s) = j | s_i \neq j] \\ &= p_{ij}G_j(i,x) + (1 - p_{ij})H_j(i,x). \end{aligned}$$

The result then follows immediately by summing over i. ∎

We now define p-symmetry and p-locality.

Definition 2 *Given a voting strategy profile represented by p, a policy profile $y \in Y$ is p-symmetric if, for all $i \in \mathcal{N}$ and all $j, k \in J$,*

$$p_{ij}(y) = p_{ik}(y).$$

Any policy profile which is not p-symmetric is referred to as p-asymmetric.

Definition 3 *For some real number ε, two policy profiles, $x, y \in Y$, are ε-p-local if, for each $i \in \mathcal{N}$ and each $j \in \mathcal{J}$, $|p_{ij}(x) - p_{ij}(y)| < \varepsilon$.*

Let U denote a vector of payoff functions in the candidate game, let $x \in Y$ be a candidate policy profile, and let x'_j be any unilateral deviation by candidate j from x. Then x is a *strict p-local equilibrium under U* if there exists $\varepsilon^* > 0$ satisfying the following. For all $j \in \mathcal{J}$ and for all x'_j which are ε^*-p-local to x,

$$U_j(x) < U_j(x'_j.$$

We can now prove our main result. A p-symmetric strategy profile by the candidates is a strict p-local equilibrium under maximization of expected vote share if and only if it is a strict p-local equilibrium under maximization of probability of victory.

Theorem 4 *An interior p-symmetric strategy profile x^* is a strict p-local equilibrium under maximization of expected vote share if and only if x^* is a strict p-local equilibrium under probability of victory maximization.*

Proof: ($\Rightarrow$) Suppose that x^* is an interior p-symmetric strategy profile for the candidates such that x^* is a strict p-local Nash equilibrium given maximization of expected vote share. That is, given x_{-j}, each candidate j is maximizing

$$\frac{1}{N}\sum_{i=1}^{N} p_{ij}(x). \tag{3.2}$$

Now consider the probability of victory for any candidate j. Note that $G_j(i, x^*) = G_k(l, x^*)$ for all $i, l \in \mathcal{N}$ and $j, k \in \mathcal{J}$ at any p-symmetric candidate strategy profile x^*. Then

$$R_j(x) = \frac{1}{N}\sum_{i}^{N}[p_{ij}(x)G_j(i,x) + (1 - p_{ij}(x))H_j(i,x)].$$

We prove the result by showing that there exists no p-local unilateral deviation in p that increases a candidate's probability of victory. We argue using a Taylor series approach. Taking first derivatives of $R_j(x)$, we obtain

$$\frac{\partial R_j}{\partial p_{ij}(x)} = \frac{1}{N}\left[G_j(i,x) + H_j(i,x) + \sum_{k\neq i} p_{kj}(x)\frac{\partial G_j(k,x)}{\partial p_{ij}(x)} + (1 - p_{kj}(x))\frac{\partial H_j(k,x)}{\partial p_{ij}(x)}\right]. \tag{3.3}$$

Notice that the first term on the RHS of Equation (3.3) is weakly greater than zero because it is simply a conditional probability. The second term on the RHS of Equation (3.3) is strictly greater than zero as well. This follows because we can express $G_j(k, x)$ as the following for any $i \neq k$:

$$\begin{aligned} G_j(k,x) &= p_{ij}\Pr\{s \in S | j \in W(s), s_i = s_k = j\} \\ &\quad +(1 - p_{ij})\Pr\{s \in S | j \in W(s), s_k = j, s_i \neq j\}. \end{aligned} \tag{3.4}$$

Note that for all $s \in S$, simple plurality rule implies that

$$(j \in W(s), s_k = j, s_i \neq j) \Rightarrow (j \in W(s), s_k = s_i = j),$$

which implies that

$$\Pr\{v \in V | j \in W(s), s_k = s_i = j\} > \Pr\{v \in V | j \in W(s), s_k = j, s_i \neq j\},$$

and since

$$\frac{\partial G_j(k,x)}{\partial p_{ij}} = \Pr\{s \in S | j \in W(s), s_k = s_i = j\} - \Pr\{s \in S | j \in W(s) s_k = j, s_i \neq j\},$$

then

$$\frac{\partial G_j(k,x)}{\partial p_{ij}} > 0$$

for all i, j, k, and x. Similar logic applies for the partial derivatives of $H_j(i,x)$, for all i, j, and x.[3]

Evaluating the first order Taylor series approximation of $R_j(x)$ at x^*,

$$R_j(x) \approx R_j(x^*) + \nabla R_j(x^*)^T \cdot (p_j(x) - p_j^*),$$

where $p_j^* = (\frac{1}{J}, \ldots, \frac{1}{J})$. By the fact that x^* is a strict p-local Nash equilibrium under maximization of expected vote share and continuity of $p_{ij}(x)$ for all i, j, and x, there exists a neighborhood $B(x^*)$ such that for all $x \in B(x^*)$, $(p_j(x) - p_j^*) \cdot \mathbf{1} < 0$, where $\mathbf{1}$ represents a column vector of 1s. Finally, it follows from the p-symmetry of x^* that $\nabla R_j(x^*)$ is a scalar multiple of $\mathbf{1}$. Therefore, for all $x \in B(x^*)$,

$$R_j(x) - R_j(x^*) < 0,$$

which implies that a p-local unilateral deviation by one candidate from x^* strictly decreases her payoff. It follows, then, that if x^* is a p-symmetric strict p-local Nash equilibrium under maximization of expected vote, then x^* is also a p-symmetric strict p-local Nash equilibrium under maximization of probability of victory. The converse follows similarly. ∎

Theorem 5 *Let $x \in Int(X)$ be an asymmetric Nash equilibrium under maximization of expected vote share. Then, the set of type distributions, F, for which x is a Nash equilibrium under maximization of probability of victory possesses Lebesgue measure zero.*

Proof: In order to establish the theorem, it is sufficient to prove that, for any direction $\omega \in X$, satisfaction of the first order conditions for Nash equilibrium under expected vote share imply that the first order conditions for Nash equilibrium under maximization of probability of victory generically do not hold.

Let x be an asymmetric Nash equilibrium under maximization of expected vote and let D denote the space of possible p. Note that, for finite N, D is compact and convex.

Now choose, for each $j \in \mathcal{J}$, any vector $\mathrm{d}p(j;x)$ such that

$$\sum_{i=1}^{N} \frac{\partial p_i(j;x)}{\partial \alpha} = 0, \tag{3.5}$$

such that, for each $j \in \mathcal{J}$, there exists some $i \in \mathcal{N}$ such that $\frac{\partial p_i(j;x)}{\partial \alpha} \neq 0$, and for all $j \in \mathcal{J}$,

$$\sum_{j=1}^{J} \frac{\partial p_i(j;x)}{\partial \alpha} = 0. \tag{3.6}$$

[3] To see this, simply replace $s_k = j$ with $s_k \neq j$ in the above argument.

These are simply the first order conditions for a Nash equilibrium under maximization of expected vote share, a condition restricting our attention to non-*i.i.d.* type distributions, and the requirement that $\sum_{j=1}^{J} p_i(j;x) = 1$ for all $i \in \mathcal{N}$ and $x \in X$.

The first order conditions for a Nash equilibrium under maximization of probability of victory are simply

$$\left.\frac{\partial R_j}{\partial p} \cdot \frac{\partial p}{\partial \alpha}\right]_{\alpha=0} = 0. \tag{3.7}$$

By equation 3.5, the space of $\frac{\partial R_j}{\partial p}$ satisfying equation 3.7 is spanned by an $N-1$ dimensional subspace of D.[4] However, the range of $\frac{\partial R_j}{\partial p} : D \to [0,1]^N$ is N-dimensional. Thus, the dimensionality of the space of solutions is strictly less than that of the space of possible vectors. It follows that the set of vectors satisfying equation 3.7 possesses Lebesgue measure zero. ∎

4 Extensions and Examples

In this section we discuss the tightness of our assumptions. That is, how much more can we obtain beyond Theorem 4? We discuss the possibility of a general equivalence result in some detail, and touch upon several other possible extensions to our results, including global equilibrium results and relaxing the assumption of independence.

4.1 Best Response Equivalence

One might hope for a general equivalence result, a result which states that, in elections with arbitrary numbers of candidates and independent voter types, the best response functions generated by maximization of expected vote share and maximization of probability of victory are identical. Unfortunately, this is not the case. Indeed, it is not even the case that such equivalence is obtained asymptotically. This is shown in Patty [12] through a replicated three voter example. An open question is the following, however. Is there any objective function which is computationally simpler than probability of victory which yields an equivalent best response function?

4.2 Other Directions

There are several other directions in which our results might be extended. First among these are the question of equivalence among global Nash equilibria under the two objective functions and the relaxation of the independence assumption.

[4] That is, given that Equations 3.7 and 3.5 are simultaneously satisfied, $(p_1, \ldots, p_{N-1})$ uniquely determine p_N.

Theorem 4 gives sufficient conditions only for p-local equilibria. The proof of Theorem 4 does not examine boundary conditions or second-order conditions. It turns out that in many settings the maximum probability of victory for a candidate k does not fall in the interior of the N-fold product of $J-1$-dimensional simplices (i.e. the p-space). As Patty [12] points out, this leads to the failure of general best response equivalence when voters' types are nonidentically distributed.

Another obvious extension of our results would allow for type distributions in which the realizations of voter types exhibit dependence across voters. The arguments for Theorem 4 do not neccessarily work in such environments for several reasons. The first of these is that we are no longer assured that G_j and H_j are nonnegative functions (in the usual vector sense of nonnegativity). This is because the vote of one voter may affect the vote of another, so that increasing one voter's probability of voting for candidate j may decrease another voter's probability of voting for j. The second reason is that the notion of p-locality becomes less sensible in such an environment. In particular, p-locality is defined with respect to rectangles in p-space. Such a definition of locality is not necessarily the most appropriate definition when independence fails to hold. Simply put, a failure of independence may imply that $R_j(p)$ is no longer linear with respect to each p_{ij}.

On a positive note, however, the basic conclusion of Theorem 4 should continue to hold even in the absence of independence. The logic is that independence encompasses every possible vector of p_{ij} for any candidate j. Thus, so long as individual behavior can be characterized as multinomial processes of some type, Theorem 4 should remain true. The major difficulty is that it may mean nothing without imposing some sort of additional structure on the nature of the dependence (such as the measure of vectors of votes cast is absolutely continuous with respect to the product measure of votes cast by each voter), as the set of policy profiles which are p-local to any p-symmetric policy profile may indeed be empty.

5 Conclusions

In this paper, we have provided general p-local equilibrium equivalence results for different candidate objectives in probabilistic voting models with independently distributed voter types. Our first result, Theorem 4, states that p-symmetric strict p-local equilibria are equivalent under the two objective functions, regardless of the number of candidates. The second result, Theorem 5, states that asymmetric equilibria are generically not equivalent under the objective functions. That is, with near certainty, a p-asymmetric equilibrium under one objective function is not an equilibrium under the other. That this applies to two candidate contests may be somewhat surprising, but recall that our definition of p-symmetry may be satisfied by candidate strategy profiles which are asymmetric with respect to candidate actions.

This result implies that the equilibrium found in McKelvey and Patty [11], for instance, is invariant to the authors' choice of objective functions. This follows because McKelvey and Patty show that, as the number of voters increases, the amount any candidate can change any given voters' likelihood of voting for him or her vanishes. Hence, the neighborhood of potential p vectors available to any candidate is shrinking, such that eventually the p-symmetric strict local equilibrium becomes a global equilibrium.

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