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TRANSACTION PRICES WHEN INSIDERS TRADE PORTFOLIOS

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Abstract

Statistical properties of transaction prices are investigated in the context of a multi–asset extension of Kyle [1985]. Under the restriction that market makers cannot condition prices on volume in other markets, Kyle's model is shown to be consistent with well-documented lack of predictability of individual asset prices, positive autocorrelation of index returns, and low cross-sectional covariance. The covariance estimator of Cohen, e.a. [1983] provides the right estimates of the "true" covariance. However, Kyle's model cannot explain the asymmetry and rank deficiency of the matrix of first–order autocovariances. Asymmetry obtains when the insider limits his strategies to trading a set of pre–determined portfolios. If these portfolios are well-diversified, the matrix of first–order autocovariances is asymptotically rank–deficient. If the insider uses only one portfolio (as when "timing the market"), its asymptotic rank equals one, conform to the empirical results in Gibbons and Ferson [1985].

Keywords: Transaction Prices, Asymmetric Information, Asset Return Predictability, Well–Diversified Portfolios.

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1 Introduction

In financial theory, the set of strategies that investors can follow (their *strategy space*) is often restricted. Short sale restrictions on investment constitute the single most important type of constraint. Other types are only rarely considered, however. This paper investigates the effects of restrictions on the combinations of securities that investors choose to buy or sell. In particular, it analyzes the effects on transaction prices if insiders (better–informed investors) limit their strategies to trading a set of pre–determined portfolios. The paper does not try to justify such a limitation in the insiders' strategy space. Insiders might optimize over particular strategies because they are simple to execute or because the execution of the fully optimal, unrestricted strategy is prohibitively costly. In the model of this paper, the insider can implement his strategies easily through, for instance, trading index futures contracts. Such trading is substantially less expensive than trading each of the underlying securities.

The paper argues that restrictions in insiders' strategy spaces are necessary to fully explain some of the well–documented intertemporal characteristics of asset returns within a multivariate extension of Kyle [1985]'s insider trading model. Chan [1991] has recently acknowledged that Kyle's model can explain some of the intertemporal characteristics of stock prices. This is fortunate, because the usual explanation, based on nonsynchronous trading, is only partially satisfactory (Lo and MacKinlay [1991]). Chan [1991] shows that Kyle's model explains standard features such as impredictability of individual asset returns, positive autocorrelation of index returns and low return covariances. Nevertheless, it cannot explain the well–documented asymmetry (Lo and MacKinlay [1990]) and rank deficiency (Gibbons and Ferson [1985]) of the matrix of first–order autocovariances. This paper proves that, if insiders limit their strategies to trading pre–determined portfolios, asymmetry and rank deficiency follow.

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As in Chan [1991], but unlike in Caballé and Krishnan [1990], market makers' quotes can depend only on volume in their own market. In contrast to Chan [1991], however, an insider exploits this fact. Market makers do take into account the presence of this cross–market arbitrageur when setting prices, hence they do not loose on average. The strategic behavior of the insider destroys the asymmetry in the matrix of first–order autocovariances which is obtained in Chan [1991]. It is shown, furthermore, that rank deficiency is incompatible with the presence of an insider who contemplates an unrestricted strategy space. Nevertheless, if insiders only trade pre–determined portfolios, asymmetry follows. If these portfolios are asymptotically well–diversified, then the matrix of first–order autocovariances is rank–deficient.

If the insider trades only one well–diversified portfolio, and this portfolio is the market, then she is effectively "timing the market". This means that, if market makers face an insider who has superior capabilities as far as market timing is concerned, they will set quotes such that transaction prices have an asymmetric and (asymptotically) rank–deficient autocovariance matrix.

The reader might wonder why a market microstructure model (*in casu*, Kyle's model) is used to explain phenomena that have been observed on a daily (Gibbons and Ferson [1985]) or weekly (Lo and MacKinlay [1990]) level. It is true that Kyle's model makes very specific assumptions about the role of the agents, their information and the sequencing of moves. Nevertheless, it captures the essential features of trade under asymmetric information. Insiders take the initiative to submit informed orders, which are subsequently aggregated with orders from liquidity, or "noise" traders. These features will reappear with varying degree in more complicated models, such as when private information is not revealed for more than one trading round (Kyle [1985] also analyzes this case; recently, Back [1991] provided a more rigorous treatment).

In a recent paper, Hollifield [1991], rank restrictions on the matrix of first–order autocovariances are obtained in a different way, by imposing a dynamic factor structure on the endowment processes in a representative investor asset pricing model. There is no asymmetric information in his model. Ferson and Gibbons [1985] also claim to have derived rank restrictions on the matrix of first–order autocovariances, but their pricing model is linear in securities' "betas", and, consequently, is based on the mean–variance efficiency of a portfolio (or combination of portfolios). From Roll [1977], we know that mean–variance efficiency does not restrict the data whatsoever. In particular, the fact that there exists an (unkown) portfolio that is mean–variance efficient, and, therefore, can represent the vector of mean returns, cannot possibly restrict the matrix of first–order autocovariances.

The paper is organized as follows. The next section presents the basic model and explains how it generates all but two of the stylized facts of the intertemporal behavior of asset prices. Section 3 introduces restrictions on the insider's strategies that explain asymmetries and rank deficiencies in serial correlations of returns. Section 4 concludes.

2 A Multivariate Market Microstructure Model

In a multivariate extension of Kyle [1985], we consider n synchronous asset markets where market makers compete in terms of prices after observing the net order flow submitted by traders in each market. Let $p_t \in R^n$ denote the vector of equilibrium prices quoted by the market makers in period t. These quotes can be made dependent only on the volume in each market separately. Let $z_t \in R^n$ denote the vector of net noise trades in the dealer markets in period t. Let $y_t \in R^n$ denote the net trades from an informed trader. The market makers observe the aggregate order flow, $w_t \equiv z_t + y_t$. After the dealer market closes, the true value of the asset is revealed, perhaps through a second clearing in the Walrasian style. Let $x_t \in R^n$ denote the vector of true values at the end of period t.

Conditioning on x_{t-1}, let x_t be normally distributed with mean x_{t-1} and variance-covariance Σ and let z_t be normally distributed with mean 0 and variance-covariance Z. We shall assume that the net order flow from noise traders in a market is uncorrelated with the corresponding true value of the asset. Market makers compete in a Bertrand fashion after observing the aggregate order flow.

The informed trader observes a signal $\theta_t \in R^n$ before orders are submitted in the dealership market. The signal is normally distributed with mean x_{t-1} and variance-covariance matrix Ξ. It is correlated with end-of-period values, as follows:

$$E[(\theta_t - x_{t-1})(x_t - x_{t-1})'] = \Sigma.$$

Consequently, we can write:

$$\theta_t = x_t + \epsilon_t,$$

with $E[\epsilon_t | x_t] = 0$. Assume that the signals are uncorrelated with the order flows from noise traders, z_t. Assume also that the informed trader does not observe noise trades when submitting her orders. This assumption can be relaxed, at the cost of a substantial increase in the complexity of the equilibrium pricing rule.

In a straightforward extension of Kyle [1985], one can show the existence of the following equilibrium.

Lemma 1 *In equilibrium, there exists a unique linear pricing rule, namely:*

$$p_t = x_{t-1} + \Lambda w_t, \tag{1}$$

and

$$y_t = B(\theta_t - x_{t-1}). \tag{2}$$

$\Lambda = \mathrm{diag}(\lambda_i)$, *where*

$$\lambda_i = \frac{1}{2}\sqrt{\frac{\alpha_i}{\zeta_{ii}}}, \tag{3}$$

where ζ_{ii} *denotes the* i*th diagonal element of* Z, *and*

$$\alpha_i = \sum_{l=1}^{n}\sum_{k=1}^{n} \sigma_{il}\sigma_{ik}\xi^*_{kl},$$

where ξ^*_{kl} *is element* k, l *of* Ξ^{-1}, *and*

$$B = \frac{1}{2}\Lambda^{-1}\Sigma\Xi^{-1}.$$

The proof is in the appendix. Notice that the slope coefficient in the market makers' pricing function will be positive. To show that α_i is positive, remember that $\alpha_i = \Sigma_i'\Xi^{-1}\Sigma_i$, where Σ_i denotes the ith column of Σ. This expression must be positive by the positive–definiteness of Ξ^{-1}.

We are interested in the statistical properties of transaction prices, i.e., of the time series $\{p_t - p_{t-1}\}_{t=1}^{\infty}$. Obviously, $E[p_t - p_{t-1}] = 0$. Since $p_t - p_{t-1} = (p_t - x_{t-1}) + (x_{t-1} - p_{t-1})$, the following is easily derived (see the appendix).

Proposition 1

$$E[(p_{t+1} - p_t)(p_t - p_{t-1})'] = \frac{1}{4}\Sigma\Xi^{-1}\Sigma' - \Lambda Z \Lambda'.$$

It follows that the matrix of first–order autocovariances of transaction prices will be symmetric. Although less straightforward to show, but nevertheless intuitive, Proposition 1 implies that transaction prices will not be individually autocorrelated. In other words, the entries on the diagonal of the matrix of first–order autocovariances are all zero. We collect these observations in a corollary.

Corollary 1

$$E[(p_{it+1} - p_{it})(p_{jt} - p_{jt-1})] = E[(p_{jt+1} - p_{jt})(p_{it} - p_{it-1})], \quad i, j = 1, ..., n,$$

$$E[(p_{it+1} - p_{it})(p_{it} - p_{it-1})] = 0, \quad i = 1, ..., n.$$

Proof: see the appendix. The conclusions of this Corollary agree only in part with those of Chan [1991]. In particular, Chan obtains asymmetric cross-autocovariances when the quality of his signals are unequal across securities (p. 18). In contrast, symmetry is obtained here even when the signal-to-noise ratios are unequal. The difference stems from the presence of a *strategic* cross-market arbitrageur, who exploits the market makers' inability to incorporate information from the order flow of other securities in his pricing.

Of additional interest is the variance-covariance matrix of changes in transaction prices, $p_t - p_{t-1}$, and its relationship to the variance-covariance matrix of changes in the "true values", $x_t - x_{t-1}$. Since the latter could be associated with clearing prices in a traditional Walrasian market, they could be considered to be restricted by a traditional asset pricing model such as the Capital Asset Pricing Model. The transaction prices, however, will not be restricted in the same way, but we can learn about the variance-covariance matrix of the "true values" (the main ingredient of an asset pricing model) from the variance-covariance matrix of transaction prices. The following proposition provides the relationship between the two.

Proposition 2

$$\begin{aligned}\Sigma = \\ & E[(p_t - p_{t-1})(p_t - p_{t-1})'] + E[(p_{t+1} - p_t)(p_t - p_{t-1})'] \\ & + E[(p_t - p_{t-1})(p_{t+1} - p_t)'].\end{aligned}$$

Proof: see the appendix. The relationship between the variance-covariance matrix of the transaction prices and that of the "true values" is identical to the one in Cohen, e.a. [1983] between the variance-covariance matrix of the observed prices and that of the underlying value process. They, however, derive this relationship from an *exogenously specified* process for transaction prices (their eqn. (2)). Obviously, the estimator of the variance-covariance matrix in their paper is the right estimator of the variance-covariance matrix of the "true values" in the present model, Σ.

Because of the low signal-to-noise ratio in asset return data, empirical research often focuses on the returns of portfolios rather than individual assets, in order to extract more precise information from the data. Consequently, we should investigate the implications of the foregoing for portfolios. Consider a set of m portfolios ($m < n$). Each can be characterized by a vector $q_l \in R^n$ of weights (i.e., $q_l'e = 1$, where e denotes the vector of ones). Let Q be the $n \times m$ matrix having the vectors q_l, $l = 1, ..., m$, as its columns. We define positively-weighted portfolios as those for which $q_l > 0$.

Consider the sequence of changes in the transaction prices of the m portfolios, $\{Q'(p_t -$

$p_{t-1})\}_{t=1}^{\infty}$. From Proposition 1, we obtain the following. Define:

$$\alpha_{ij} = \sum_{k=1}^{n} \sum_{l=1}^{n} \sigma_{ik}\sigma_{lj}\xi_{kl}^{*}.$$

Corollary 2 *If*

$$\frac{\alpha_{ij}}{\sqrt{\alpha_i \alpha_j}} - \frac{\zeta_{ij}}{\sqrt{\zeta_{ii}\zeta_{jj}}} > 0,$$

then the dealership market returns on positively–weighted portfolios will be positively autocorrelated.

Again, the proof is in the appendix. Notice that we obtain positive autocorrelation of index (positively weighted portfolios) returns without reference to nontrading. Lo and MacKinlay [1991] have investigated the impact of nontrading and concluded that this phenomenon is insufficient to explain the degree of positive autocorrelation in index returns. Corollary 2 provides an alternative explanation. The sufficient condition for positive index autocorrelation in Corollary 2 is satisfied, for instance, for positive α_{ij} (this essentially requires positive correlation between "true values" and between signals) and zero ζ_{ij} (uncorrelated noise trades). Positive autocorrelation of index returns is precisely what has been observed in the data (see, e.g., Lo and MacKinlay [1990]).

Two further remarks should be made. First, if the condition of Corollary 2 is not satisfied, index returns might not be positively correlated. For instance, it can be shown that if Σ and Ξ are diagonal (uncorrelated "true values" and signals) whereas $Z \geq 0$ (i.e., $\zeta_{ij} \geq 0$, all i, j), then index returns will be negatively autocorrelated. Second, the expression in the condition of Corollary 2 is the cross–autocovariance between asset i's and asset j's prices. Notice that this cross–autocovariance is not always positive, unlike in Chan [1991] (p. 17). The presence of a strategic cross–market arbitrageur upsets his result.

What about off–diagonal elements in the matrix of autocovariances of positively weighted portfolios? In order to avoid trivial results, let us assume that Q, the matrix of portfolio weights, is of full rank. In addition, let us assume that Σ has nonzero entries everywhere ($\Sigma \neq 0$). This can be relaxed: one basically needs a minimum number of nonzero off–diagonal elements. Nevertheless, the assumption is not irrealistic from an equilibrium asset pricing point of view. Σ represents the variance–covariance matrix of the "true values" of the assets in the economy, x_t. They assume meaning in the neoclassical context if we identify them with the clearing prices of a Walrasian market (as opposed to p_t, which is the vector of transaction prices in the dealership market). In the absence of arbitrage opportunities, the elements of x_t should satisfy (see, e.g., Chamberlain and Rothschild [1983]):

$$x_{it} = E[\rho_{t+1} p_{it+1}],$$

for some random variables ρ_{t+1}. p_{it+1} is the payoff from selling asset i in the subsequent dealership market. Since ρ_{t+1} is common to all asset values, one can expect a minimum level of correlation.

These assumptions lead to the following corollary.

Corollary 3 *The matrix of first–order autocovariances of changes in transaction prices of portfolios is symmetric and full–rank.*

Proof: see appendix. The conclusions of Corollary 3 contrast singularly with the empirical results. As the evidence in Lo and MacKinlay [1990] shows, the matrix of first–order autocovariances of index returns is asymmetric. Moreover, from earlier studies (Gibbons and Ferson [1985]), we know that it is rank–deficient. In other words, while the model is capable of explaining some of the empirical findings, such as positive autocorrelation of positively weighted portfolio returns, it cannot explain (1) asymmetry, (2) rank deficiency. Rather than abandoning the model altogether, let us investigate next whether restrictions on the strategy space of the insider can explain the empirical findings. In particular, assume that the insider decides to optimize over strategies that are easy to execute. In the context of securities trading, there is an obvious restriction one can impose: the insider's actions are limited to the trading of a number of pre–determined portfolios.

3 Restrictions on the Insider's Strategies

Before, the insider's orders, y_t, were an unrestricted vector in R^n. Assume now that the insider limits his actions to trading K portfolios ($K < n$). Let $\hat{q}_l$ be the vector of weights corresponding to the lth portfolio ($l = 1, ..., K$). Construct an $n \times K$ matrix $\hat{Q}$ by incorporating the $\hat{q}_l$s as columns. The insider's actions are limited to trading the K portfolios if y_t lies in the span of $\hat{Q}$, i.e.,

$$y_t = \hat{Q}\eta_t, \tag{4}$$

for some $\eta_t \in R^K$. Given the (diagonal) matrix of slope coefficients of market makers' pricing functions Λ, the optimal η_t is shown in the appendix to be the following.

Lemma 2

$$\eta_t = B^{\hat{Q}}(\theta_t - x_{t-1}),$$

where

$$B^{\hat{Q}} = \frac{1}{2}(\hat{Q}'\Lambda\hat{Q})^{-1}\hat{Q}'\Sigma\Xi^{-1}.$$

Lemma 2 does not provide an expression for Λ, but this can be derived as in equation (3). Λ has the slope coefficients of the market makers' pricing functions, the λ_is, on its diagonal. Because of the restriction on the insider's trading strategy, λ_i will not necessarily be positive. No λ_i can be negative, however (it means that the insider faces a downward sloping supply curve for some security). But some can be zero. We will ignore this complication, and assume that the necessary restrictions on Σ, Ξ and Z are satisfied for all λ_is to be strictly positive.

From Lemma 2, it is straightforward to prove the following proposition.

Proposition 3

$$\begin{aligned} &E[(p_{t+1} - p_t)(p_t - p_{t-1})'] = \\ &\quad \frac{1}{2}\Sigma\Xi^{-1}\Sigma'\hat{Q}(\hat{Q}'\Lambda\hat{Q})^{-1}\hat{Q}'\Lambda' - \Lambda Z \Lambda' \\ &\quad - \frac{1}{4}\Lambda\hat{Q}(\hat{Q}'\Lambda\hat{Q})^{-1}\hat{Q}'\Sigma\Xi^{-1}\Sigma'\hat{Q}(\hat{Q}'\Lambda\hat{Q})^{-1}\hat{Q}'\Lambda'. \end{aligned}$$

Proof: see the appendix. The second and third terms on the right–hand side of the expression in this proposition are clearly symmetric. The first one is generally not. Indeed, assume:

$$\hat{Q} = \begin{bmatrix} \frac{1}{n} \\ \cdots \\ \frac{1}{n} \end{bmatrix}.$$

In words, the insider trades one equally–weighted portfolio. Let σ^*_{ij} denote element i,j of $\Sigma\Xi^{-1}\Sigma'$. Element i,j of $\Sigma\Xi^{-1}\Sigma'\hat{Q}(\hat{Q}'\Lambda\hat{Q})^{-1}\hat{Q}'\Lambda'$ equals:

$$\frac{1}{n^2}\lambda_j \sum_{k=1}^{n} \sigma^*_{ik},$$

whereas element j,i equals:

$$\frac{1}{n^2}\lambda_i \sum_{k=1}^{n} \sigma^*_{jk}.$$

These elements will in general not be equal (despite the fact that $\sigma^*_{ij} = \sigma^*_{ji}$). Consequently, the matrix of first–order autocovariances is generally asymmetric. We state this as a corollary.

Corollary 4 *In general,*

$$E[(p_{it+1} - p_{it})(p_{jt} - p_{jt-1})] \neq E[(p_{jt+1} - p_{jt})(p_{it} - p_{it-1})],\ \ i,j = 1,...,n,\ i \neq j.$$

The matrix of first–order autocovariances will still be full–rank, because of the presence of $\Lambda Z \Lambda'$. In other words, the restriction on insider's strategies that we imposed is not able to explain the rank deficiency of the matrix of first–order autocovariances. One obtains rank deficiency, however, by restricting the insider's strategies even more, and by analyzing the outcome as securities are added to the economy ($n \rightarrow \infty$).

Assume that the insider trades only it one asymptotically well–diversified portfolio. (The subsequent analysis can easily be changed to accomodate trade in more than one portfolio.) An asymptotically well–diversified portfolio is defined as follows. Consider the sequence of economies indexed by the number of assets, n. Consider the corresponding sequence of weights of the insider's portfolio (represented by the vector $\hat{Q}^n$), $\hat{Q}^n_i$. The portfolio is asymptotically well–diversified if:

$$\hat{Q}^n_i = O(\frac{1}{n}),$$

i.e.,

$$\lim_{n \rightarrow \infty} n\hat{Q}^n_i = C,$$

where $|C| < \infty$. This is satisfied, for instance, when $\hat{Q}^n$ is an equally–weighted portfolio ($\hat{Q}^n_i = \frac{1}{n}$).

The insider's strategies are restricted to trades in one asymptotically well–diversified portfolio. If this portfolio is the market portfolio, the insider's actions correspond to *timing the market*: the insider uses his superior information to predict the future course of the market portfolio and to invest accordingly.

A sequence of matrices will be considered to be asymptotically rank–deficient if it converges to a rank–deficient matrix. Mathematically, if A^n is $n \times n$, then the sequence $\{A^n\}_{n=1}^{\infty}$ is asymptotically rank–deficient if $A^n \rightarrow A^*$, with $\text{rank}(A^*) = K < \infty$. (Convergence of matrices can be defined in different ways; it suffices that the 2–norm, i.e., the largest eigenvalue, of the difference $A^n - A^*$ converge to zero.) We are now able to prove the following.

Proposition 4 *If the insider trades only in one asymptotically well–diversified portfolio, then the matrix of first–order autocovariances of changes in the transaction prices of the n securities will asymptote to a rank–one matrix.*

The proof is in the appendix. It consists of two parts. First, it is shown that the slope coefficients of market makers' pricing strategies are $O(\frac{1}{n})$. Second, $\Lambda^n Z^n \Lambda^n$ (the superscripts indicate that these matrices change as securities are added to the economy) is shown to converge to 0. The latter convergence is very fast. Indeed, $\Lambda^n Z^n \Lambda^n = o(\frac{1}{n})$, i.e., if ζ^{*n}_{ij} denotes element i, j of $\Lambda^n Z^n \Lambda^n$, then:

$$\lim_{n \rightarrow \infty} n\zeta^{*n}_{ij} = 0. \tag{5}$$

In words, if the insider restricts his trades to one well–diversified portfolio, then the matrix of first–order autocovariances of transaction prices in very large economies is of rank one. This is exactly what Gibbons and Ferson [1985] find.

Notice the speed of convergence in equation (5): the matrix of first–order autocovariances converges at the rate n to a rank–deficient matrix. In other words, in sufficiently large but finite economies, this matrix will be *near* rank–deficient. Because of the sampling error, statistical tests of the rank of the first–order autocovariance matrix (such as the one in Bossaerts [1988], which is based on canonical correlations) will not be able to distinguish clearly between a near rank–deficient and a rank–deficient matrix.

4 Conclusion

This paper has shown how restrictions on the insider's strategies can explain some of the stylized facts about transaction prices. In particular, if the insider trades only predetermined, well–diversified portfolios, the matrix of first–order autocovariances is asymmetric and (asymptotically) rank–deficient. Without such restrictions, this matrix would be symmetric and full–rank.

One wonders whether other changes to Kyle's insider trading model would produce a similar outcome. To the best of this author's knowledge, restrictions on the strategy space of the insider are the only way to explain asymmetry and rank deficiency. If there are multiple insiders, each specializing in a (potentially overlapping) subset of markets, the matrix of first–order autocovariances of transaction prices will be block–symmetric and full–rank. If each specializes in just one market, asymmetry will result (as in Chan [1983], where there are no *cross–market* arbitrageurs), but rank deficiency will not.

The next step would be to develop the methodology to distinguish between the two explanations for rank–deficient matrices of first–order autocovariances. This paper provided one based on asymmetric information. As mentioned in the Introduction, Hollifield [1991], however, also obtained rank restrictions, within a representative investor asset pricing model, by imposing a dynamic factor structure on the endowment processes. Which of the two explanations is more likely to have generated the rank deficiencies that we observe in return data? The answer to this question is left for future research.

Appendix

Proof of Lemma 1:

On the one hand, the insider correctly posits the pricing strategy $p_t = x_{t-1} + \Lambda w_t$. Consequently, he maximizes the following expression with respect to y_t:

$$\begin{aligned} E[y_t'(x_t - p_t)|\theta_t] &= E[y_t'(x_t - x_{t-1} - \Lambda w_t)|\theta_t] \\ &= E[y_t'(x_t - x_{t-1} - \Lambda z_t - \Lambda y_t)|\theta_t]. \end{aligned}$$

The first–order conditions are:

$$E[x_t|\theta_t] - x_{t-1} = 2\Lambda y_t.$$

$E[x_t|\theta_t] = x_{t-1} + \Sigma\Xi^{-1}(\theta_t - x_{t-1})$, hence:

$$y_t = \frac{1}{2}\Lambda^{-1}\Sigma\Xi^{-1}(\theta_t - x_{t-1}).$$

On the other hand, the market makers compete in a Bertrand fashion. As in Kyle [1985], the normality assumption will imply:

$$\lambda_i = \frac{E[(x_{it} - x_{it-1})w_{it}]}{E[w_{it}^2]}.$$

But

$$E[(x_{it} - x_{it-1})w_{it}] = \frac{1}{2}\frac{\alpha_i}{\lambda_i},$$

where $\alpha_i = \sum_{l=1}^{n}\sum_{k=1}^{n}\sigma_{il}\sigma_{ik}\xi_{kl}^*$, and

$$E[w_{it}^2] = \zeta_{ii} + \frac{1}{4}\frac{\alpha_i}{\lambda_i^2}.$$

Consequently,

$$\lambda_i = \frac{\frac{1}{2}\frac{\alpha_i}{\lambda_i}}{\zeta_{ii} + \frac{1}{4}\frac{\alpha_i}{\lambda_i^2}},$$

a quadratic equation in λ_i, with as positive root:

$$\lambda_i = \frac{1}{2}\sqrt{\frac{\alpha_i}{\zeta_{ii}}}.$$

□

Proof of Proposition 1:

$$\begin{aligned}
&E[(p_{t+1} - p_t)(p_t - p_{t-1})'] \\
&\quad = E[(p_{t+1} - x_t + x_t - p_t)(p_t - x_{t-1} + x_{t-1} - p_{t-1})'] \\
&\quad = E[(x_t - p_t)(p_t - x_{t-1})'] \\
&\quad = E[(x_t - x_{t-1} - \Lambda w_t)(x_{t-1} + \Lambda w_t - x_{t-1})'] \\
&\quad = \Sigma B'\Lambda' - \Lambda Z \Lambda' - \Lambda B \Xi B' \Lambda' \\
&\quad = \frac{1}{4}\Sigma \Xi^{-1} \Sigma' - \Lambda Z \Lambda'.
\end{aligned}$$

□

Proof of Corollary 1: The first result is obvious by inspection. As to the second result,

$$\begin{aligned}
&E[(p_{it+1} - p_{it})(p_{it} - p_{it-1})] \\
&\quad = \frac{1}{4}\sum_{l=1}^{n}\sum_{k=1}^{n} \sigma_{il}\sigma_{ik}\xi^{*}_{kl} - \lambda_i^2 \zeta_{ii} \\
&\quad = \frac{1}{4}\alpha_i - \frac{1}{4}\frac{\alpha_i}{\zeta_{ii}}\zeta_{ii} \\
&\quad = 0.
\end{aligned}$$

□

Proof of Proposition 2:

$$\begin{aligned}
&E[(p_t - p_{t-1})(p_t - p_{t-1})'] \\
&\quad = E[(p_t - x_{t-1} + x_{t-1} - p_{t-1})(p_t - x_{t-1} + x_{t-1} - p_{t-1})'] \\
&\quad = \Lambda B \Xi B' \Lambda' + \Lambda Z \Lambda' + \Sigma - \Sigma B' \Lambda' + \Lambda Z \Lambda' \\
&\quad\quad + \Lambda B \Xi B' \Lambda' - \Lambda B \Sigma' \\
&\quad = \Sigma - \frac{1}{2}\Sigma \Xi^{-1} \Sigma' + 2\Lambda Z \Lambda' \\
&\quad = \Sigma - 2E[(p_{t+1} - p_t)(p_t - p_{t-1})'] \\
&\quad = \Sigma - E[(p_{t+1} - p_t)(p_t - p_{t-1})'] - E[(p_t - p_{t-1})(p_{t+1} - p_t)'].
\end{aligned}$$

The last equality follows from the symmetry of the matrix of first–order autocovariances.
□

Proof of Corollary 2: The diagonal elements of the matrix of first–order autocovariances of transaction prices equal 0 (see Corollary 1). The off–diagonal elements can be written as follows.

$$\begin{aligned}
&E[(p_{it+1} - p_{it})(p_{jt} - p_{jt-1})] \\
&\quad = \frac{1}{4}\sum_{k=1}^{n}\sum_{l=1}^{n} \sigma_{ik}\sigma_{lj}\xi^{*}_{kl} - \lambda_i \lambda_j \zeta_{ij}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{k=1}^{n}\sum_{l=1}^{n}\sigma_{ik}\sigma_{lj}\xi^{*}_{kl} - \frac{1}{4}\sqrt{\alpha_i\alpha_j}\frac{\zeta_{ij}}{\sqrt{\zeta_{ii}\zeta_{jj}}} \\
&= \frac{1}{4}\sqrt{\alpha_i\alpha_j}\left(\frac{\alpha_{ij}}{\sqrt{\alpha_i\alpha_j}} - \frac{\zeta_{ij}}{\sqrt{\zeta_{ii}\zeta_{jj}}}\right).
\end{aligned}
$$

Consequently, the off-diagonal elements (and positively weighted averages of the off-diagonal elements) will be positive if

$$\frac{\alpha_{ij}}{\sqrt{\alpha_i\alpha_j}} - \frac{\zeta_{ij}}{\sqrt{\zeta_{ii}\zeta_{jj}}} > 0.$$

□

Proof of Corollary 3: Symmetry follows from the symmetry of $E[(p_{t+1}-p_t)(p_t-p_{t-1})']$. Moreover,

$$
\begin{aligned}
&\operatorname{rank}(Q'E[(p_{t+1}-p_t)(p_t-p_{t-1})']Q) \\
&\quad= \operatorname{rank}(Q'(\frac{1}{4}\Sigma\Xi^{-1}\Sigma' - \Lambda Z\Lambda')Q) \\
&\quad\leq \min\{\operatorname{rank}(Q), \operatorname{rank}(\frac{1}{4}\Sigma\Xi^{-1}\Sigma' - \Lambda Z\Lambda')\}.
\end{aligned}
$$

Barring trivial results, $\frac{1}{4}\Sigma\Xi^{-1}\Sigma' - \Lambda Z\Lambda'$, a matrix with zeros on the diagonal but nonzeros elsewhere, will be of full rank. But Q is of full rank. Hence, the matrix of first-order autocovariances of the portfolio prices will be of full rank (ignoring the cases when the above inequality is satisfied as a strict inequality). □

Proof of Lemma 2: The insider maximizes the following expression:

$$
\begin{aligned}
&E[y_t'(x_t - p_t)] = \\
&\quad E[\eta_t'\hat{Q}'(x_t - x_{t-1} - \Lambda z_t - \Lambda\hat{Q}\eta_t)].
\end{aligned}
$$

The first-order conditions are:

$$\hat{Q}'(E[x_t|\theta_t] - x_{t-1}) = 2\hat{Q}'\Lambda\hat{Q}\eta_t.$$

Since $E[x_t|\theta_t] = x_{t-1} + \Sigma\Xi^{-1}(\theta_t - x_{t-1})$,

$$\eta_t = \frac{1}{2}(\hat{Q}'\Lambda\hat{Q})^{-1}\hat{Q}'\Sigma\Xi^{-1}(\theta_t - x_{t-1}).$$

□

Proof of Proposition 3:

$$\begin{aligned}
&E[(p_{t+1}-p_t)(p_t-p_{t-1})'] \\
&= E[(p_{t+1}-x_t+x_t-p_t)(p_t-x_{t-1}+x_{t-1}-p_{t-1})'] \\
&= E[(x_t-p_t)(p_t-x_{t-1})'] \\
&= E[(x_t-x_{t-1}-\Lambda w_t)(x_{t-1}+\Lambda w_t-x_{t-1})'] \\
&= \Sigma B^{\hat{Q}\prime}\hat{Q}'\Lambda' - \Lambda Z\Lambda' - \Lambda\hat{Q}B^{\hat{Q}}\Xi B^{\hat{Q}\prime}\hat{Q}\Lambda' \\
&= \frac{1}{2}\Sigma\Xi^{-1}\Sigma'\hat{Q}(\hat{Q}'\Lambda\hat{Q})^{-1}\hat{Q}'\Lambda' - \Lambda Z\Lambda' \\
&\quad - \frac{1}{4}\Lambda\hat{Q}(\hat{Q}'\Lambda\hat{Q})^{-1}\hat{Q}'\Sigma\Xi^{-1}\Sigma'\hat{Q}(\hat{Q}'\Lambda\hat{Q})^{-1}\hat{Q}'\Lambda'.
\end{aligned}$$

□

Proof of Proposition 4:

As in the proof of Lemma 1, market makers will set prices such that

$$\lambda_i = \frac{E[(x_{it}-x_{it-1})w_{it}]}{E[w_{it}^2]}.$$

From Lemma 2:

$$E[(x_{it}-x_{it-1})w_{it}] = \frac{1}{2}\frac{\hat{\alpha}_i}{\hat{\lambda}_i},$$

where $\hat{\alpha}_i = \sum_{j=1}^n \hat{Q}_j \sum_{l=1}^n \sum_{k=1}^n \sigma_{jl}\sigma_{ik}\xi_{kl}^*$, and $\hat{\lambda}_i = \frac{\hat{Q}'\Lambda\hat{Q}}{\hat{Q}_i}$. Also,

$$E[w_{it}^2] = \zeta_{ii} + \frac{1}{4}\frac{\sum_{j=1}^n \hat{Q}_j\hat{\alpha}_j}{\hat{\lambda}_i^2}.$$

Consequently,

$$\lambda_i = \frac{\hat{\lambda}}{\hat{Q}_i}\frac{\frac{1}{2}\hat{\alpha}_i}{\zeta_{ii}\frac{\hat{\lambda}^2}{\hat{Q}_i^2} + \frac{1}{4}\sum_{j=1}^n \hat{Q}_j\hat{\alpha}_j},$$

where $\hat{\lambda} = \hat{Q}'\Lambda\hat{Q}$.

Now verify that $\lambda_i = O(\frac{1}{n})$. If so,

$$\begin{aligned}
\lim_{n\to\infty} n^2\hat{\lambda} &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n (n\hat{Q}_i)^2(n\lambda_i) \\
&= C_1, \quad |C_1| < \infty,
\end{aligned}$$

i.e., $\hat{\lambda} = O(\frac{1}{n^2})$.

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n^2}\hat{\alpha}_i \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}(n\hat{Q}_j)(\sum_{l=1}^{n}\frac{\sigma_{jl}}{n}(\sum_{k=1}^{n}\frac{\xi^*_{lk}\sigma_{ki}}{n})) \\ &= C_2, \ |C_2| < \infty, \end{aligned}$$

i.e., $\hat{\alpha}_i = O(n^2)$. Consequently,

$$\frac{\hat{\lambda}^2}{\hat{Q}_i^2} = O(\frac{1}{n^2}).$$

Next,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n^2}\sum_{j=1}^{n}\hat{Q}_j\hat{\alpha}_j \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}(n\hat{Q}_j)(\frac{\hat{\alpha}_j}{n^2}) \\ &= C_3, \ |C_3| < \infty, \end{aligned}$$

i.e., $\sum_{j=1}^{n}\hat{Q}_j\hat{\alpha}_j = O(n^2)$. It follows that

$$\begin{aligned} &\lim_{n\to\infty} n\lambda_i \\ &= \frac{n^2\hat{\lambda}}{n\hat{Q}_i}\frac{\frac{1}{2}\frac{1}{n^2}\hat{\alpha}_i}{\zeta_{ii}\frac{1}{n^2}\frac{\hat{\lambda}^2}{\hat{Q}_i^2} + \frac{1}{4}\frac{1}{n^2}\sum_{j=1}^{n}\hat{Q}_j\hat{\alpha}_j} \\ &= C_4, \ |C_4| < \infty, \end{aligned}$$

i.e., $\lambda_i = O(\frac{1}{n})$, as required.

Using these facts, it follows that the elements of $\Lambda Z \Lambda'$ will be $o(\frac{1}{n})$. As to the elements of $\Sigma B^{\hat{Q}\prime}\Lambda'$, element i, j equals

$$\frac{1}{n}\sum_{k=1}^{n}\sigma_{ik}B^{\hat{Q}}_{jk}(n\lambda_j), \tag{6}$$

where $B^{\hat{Q}}_{jk}$ is element j, k of $B^{\hat{Q}}$. The expression in (6) is $O(1)$. Finally, element i, j of $\Lambda B^{\hat{Q}}\Xi B^{\hat{Q}\prime}\Lambda'$ equals:

$$\frac{1}{n}\sum_{l=1}^{n}(n\lambda_i)B^{\hat{Q}}_{il}(\frac{1}{n}\sum_{k=1}^{n}\xi_{lk}B^{\hat{Q}}_{jk}(n\lambda_j)),$$

which is $O(1)$. From the expression in Proposition 3, it follows that the matrix of first-order autocovariances equals the difference between a matrix of rank one with elements that are $O(1)$, and a matrix of full rank whose elements are $o(\frac{1}{n})$. The result of the proposition follows immediately. □

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