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Abstract

This note provides an explanation for the 'declining-price anomaly' in sequential second price auctions. We illustrate how the average winning bids of risk neutral agents bidding for objects with valuations drawn from independent, identical distributions are lower in later auctions than in earlier auctions. When the objects are not identical we determine the optimal order in which they should be auctioned.

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A NOTE ON SEQUENTIAL AUCTIONS

Dan Bernhardt and David Scoones

This note explores multi-object, sequential, private-value auctions. We seek to capture phenomena such as bidding by a firm for construction contracts, oil drilling concessions, or workers, where because of constraints imposed by its physical resources or the number of its job openings, each bidder is limited in the number of objects it can acquire.

Ashenfelter [1989], Ashenfelter and Genesove [1992], and McAfee and Vincent [1993] document a puzzling 'declining-price anomaly': in sequential auctions, mean sale prices for identical objects fall in later auctions by $\frac{1}{4}\%$ to $1\frac{1}{2}\%$[1]. McAfee and Vincent observe that these findings are difficult to reconcile with accepted theory. Weber [1983] shows that in sequential auctions of identical objects with risk neutral bidders who hold independent private values, the expected sale prices in each auction are identical. Further, if there is affiliation in values, then expected prices should rise in later auctions because early auctions release information, thereby reducing winner's curse concerns. McAfee and Vincent show that risk aversion explanations are problematic. Pure strategy equilibria with declining prices require *non-decreasing* absolute risk aversion, something that does not seem to reasonably characterize individuals' attitudes toward risk.

In this paper, we first consider a simple variant in which each bidder's valuations are identically distributed across the objects to be auctioned but are *not* perfectly correlated. Even though bidders are *risk neutral*, mean sale prices fall. Second, we determine which object should be auctioned first when bidder valuations are not identically distributed across objects.

The intuition for why prices fall in the sequential auction of independently distributed, stochastically equivalent objects is most easily understood by comparison with Weber's [1983] model of a second price sequential auction of identical items. Weber shows that the symmetric equilibrium strategies call for agents to bid less than their valuations in earlier auctions to account for the option value of participating in later auctions; the profit they expect to earn were they to participate in subsequent auctions. Because the objects are identical, bidders with higher valuations in the first round have higher option values, so they discount

their early bids by a *greater* amount than low valuation bidders. All gains to waiting are arbitraged away, and the expected sale price in the two auctions is the same.

In the present model, when submitting their bids in the first auction, bidders recognize that if they do not win, then all bidders in the second auction expect the same profit from bidding on the second object. Since this option value is the same for each first auction participant, each discounts its bid in the first auction by the *same* amount.

Consequently, relative to when objects are exactly identical, when objects are stochastically identical, bidders with high valuations in the first round discount their bids by less, and those with low valuations discount by more. The key to the declining price result is that when there are sufficiently many bidders, it is usually the bidders with high valuations who determine its price; the reduced (relative to Weber's model) discount of these bidders is what leads expected prices to fall in later auctions.

We then show that sellers of objects with different distributions of buyer valuations should first auction objects that feature the greatest variation in buyer valuations. For instance, if the distribution of valuations for one object is degenerate, it should be auctioned last. Then bidders do not discount their bids in the first auction, and the identity of the winner of the first object has no effect on the sale price in the second auction. Were the auction order reversed, then the bids for the sure object would be discounted by the expected profits from participating in the second auction, and the 'wrong' bidder may win the first auction. Selling the object with the less dispersed buyer valuations last minimizes both the discount of bids from true valuations in the first auction and the cost of having fewer bidders in the second.

The Model.

There are $N \geq 3$ bidders who may bid for two objects that are to be auctioned sequentially in second price, sealed bid, private-value auctions. There is no entry fee, and resale is prohibited. Further, each bidder can win only one object. Bidders do not know their own valuations of the second object until after the first auction has taken place[2]. We initially assume that each agent's valuations are drawn from identical and independent distributions.

Let $h_j(.)$ be the continuous density from which bidders draw their (bounded, positive) valuations of object j. Let v_{ij} be i's valuation of object j and let π_{ij} be i's expected profit in auction j (which may be conditioned on v_{i1}, v_{i2}). Let b_{ij} be the bid i submits in auction j. Define $f_j(n)$ to be the first order statistic in auction j when there are n bidders and let $s_j(n)$ be the second order statistic.

In the second auction, each bidder submits a bid equal to its valuation, and the winner pays the second highest bid. Each agent has *ex ante* identical chances of winning. Hence, bidders expect second auction profits of

$$\pi_{i2}(N-1) = 1/(N-1)\ E\{f_2(N-1) - s_2(N-1)\}.$$

π_{i2} is the option value of not winning the first auction. This is equal to the value of participating in the second auction since a bidder can only win one object. In the first auction, standard arguments demonstrate that a bidder's equilibrium strategy is to shade its bid by the expected value of participation in the second auction, π_{i2}: Bidding below $v_{i1} - \pi_{i2}$ reduces the probability of winning in the first auction only when the gain exceeds the expected value of participation in the second auction; bidding above $v_{i1} - \pi_{i2}$ increases the probability of winning only when the expected profit is less than the expected value of participation in the second auction. Recognizing that bids must be non-negative, this reasoning implies:

$$b_{i1}(v_{i1}) = \max\{0,\ v_{i1} - \pi_{i2}\}. \tag{1}$$

Assuming the minimum valuation exceeds π_{i2}, expected sale prices are then

$$E(P_1) = E\{s_1(N)\} - 1/(N-1)\ E\{f_2(N-1) - s_2(N-1)\};$$

$$E(P_2) = E\{s_2(N-1)\}.$$

Taking differences and rearranging we find

$$E(P_1) - E(P_2) = E\{s_1(N)\} - \left[E\{\frac{1}{N-1}f_2(N-1) + \frac{N-2}{N-1}s_2(N-1)\}\right]. \tag{2}$$

That is, the difference in expected sale prices is equal to the expected second order statistic with N bidders minus a weighted average of the first and second order statistics when there are N-1 bidders.

Example 1: Suppose valuations are independently and uniformly distributed on [1,2]. In auction 2, the expected first and second order statistics are 1+(N-

1)/N and 1+(N-2)/N, respectively, so that the winner's expected profit is 1/N. In the first auction, each bidder shaves 1/[(N-1)N] from its valuation when bidding. The expected sale price in the first auction is then

$$E(P_1) = 1+(N-1)/(N+1) - 1/[(N-1)N] > E(P_2) = 1+(N-2)/N \text{ for } N > 3.^3 \quad \blacksquare$$

Example 2: Suppose valuations are independently and identically distributed and take on the value 2 with probability p and 1 with probability 1-p. Then the expected sale prices in each auction are given by

$$E(P_1) = 2[1- (1-p)^N - Np(1-p)^{N-1}] + 1[(1-p)^N - Np(1-p)^{N-1}] - p(1-p)^{N-2}$$

$$= 2 - (1-p)^N - Np(1-p)^{N-1} - p(1-p)^{N-2};$$

$$E(P_2) = 2[1 - (1-p)^{N-1} - (N-1)p(1-p)^{N-2}] + 1[(1-p)^{N-1} - (N-1)p(1-p)^{N-2}]$$

$$= 2 - (1-p)^{N-1} - (N-1)p(1-p)^{N-2},$$

so that

$$E(P_1) - E(P_2) = p(1-p)^{N-2}[(N-1)p - 1] > 0 \quad \text{if} \quad N > \frac{1+p}{p}. \quad \blacksquare$$

These examples illustrate that when there are sufficiently many bidders that the expected second highest bidder has a high valuation, then relative to when identical objects are auctioned, the average amount by which the second highest bidder discounts its first auction bid from its true values is less. The consequence is that average sale prices fall.

The above analysis assumes that a seller will accept a bid below the lowest possible valuation of any bidder in the first auction. If the seller sets a reserve price equal to the lowest possible bidder valuation, this reduces the ability of a low-valuation bidder in the first auction to shade his bid, further driving up the difference between the first auction price and the second.

There are two interesting cases to consider: (1) participation in the first auction is a precondition for participation in the second auction, so that a low-valuation bidder submits the reserve price; (2) a bidder can participate in either auction so that a low-valuation bidder does not bid in the first auction because its maximum profit there is less than the value of participating in the second auction.

Continuing Example 2, in the first scenario, where a low-valuation bidder submits the reserve price of 1, the expected price and hence profit in the second auction are unaffected by the reserve price, but the expected price in the first auction is increased by $[(1-p)^N + Np(1-p)^{N-1}]p(1-p)^{N-2}$ to:

$$E(P_1) = 2[1 - (1-p)^N - Np(1-p)^{N-1}] + 1[(1-p)^N + Np(1-p)^{N-1}] - [1 - (1-p)^N - Np(1-p)^{N-1}]p(1-p)^{N-2}$$

$$= 2 - (1-p)^N - Np(1-p)^{N-1} - [1 - (1-p)^N - Np(1-p)^{N-1}]p(1-p)^{N-2}.$$

In the second scenario, with probability $(1-p)^N$, no one participates in the first auction, so that the first object is not be sold and there are N bidders in the second auction. Consequently, the expected price in the second auction increases by $(N-1)p^2(1-p)^{2N-2}$ to

$$E(P_2) = 2 - [1-(1-p)^N][(1-p)^{N-1} + (N-1)p(1-p)^{N-2}] - (1-p)^N[(1-p)^N + Np(1-p)^{N-1}].$$

The expected price in the first auction given that there is a sale is increased to

$$E(P_1) = \left[\frac{1-(1-p)^N - Np(1-p)^{N-1}}{1-(1-p)^N}\right][2 - p(1-p)^{N-2}[1-p(1-p)^{N-1}]] + \left[\frac{Np(1-p)^{N-1}}{1-(1-p)^N}\right]1$$

$$> [1 - (1-p)^N - Np(1-p)^{N-1}][2 - p(1-p)^{N-2}] + [(1-p)^N + Np(1-p)^{N-1}]1,$$

which exceeds the expected price in the first auction were there no reserve price by $[(1-p)^N + Np(1-p)^{N-1}]p(1-p)^{N-2}$, an increase which exceeds that in the second auction.

The reserve price only binds in the first auction. Hence, introducing a reserve price equal to the lowest possible bidder valuation drives up the expected sale price in the first auction, thus magnifying the 'declining-price anomaly.'

Different Objects

Suppose the seller plans to auction two objects with different distributions of buyer valuations. Which object should the seller auction first? The answer is unambiguous if buyer valuation distributions can be ordered by the following notion of dispersion:

Definition: Buyer valuation distribution A is said to have **more dispersed** order statistics than distribution B if

A1. $E\{f_A(N-1)\} - E\{s_A(N-1)\} \geq E\{f_B(N-1)\} - E\{s_B(N-1)\}$

A2. $E\{s_A(N)\} - E\{s_A(N-1)\} \geq E\{s_B(N)\} - E\{s_B(N-1)\}$,

where f_k is the first order statistic and s_k is the second order statistic from buyer valuation distribution k, k=A,B.

For example, if the cumulative distribution of a bidder's valuations satisfies

$$H_A(x) = H_B(x/\alpha + c), \ \alpha > 1, \quad \forall x$$

(e.g. valuations are uniformly distributed and distribution A has a greater support) then distribution A has more dispersed order statistics than B. Similarly, if the valuation distribution B is degenerate, then distribution A is more dispersed than B. In many common families of bidder valuation distributions the notion of dispersion of order statistics either corresponds to a measure of variance (e.g. normal, uniform, Poisson, 2 point distributions), or is captured by one of the parameters characterizing the distribution.

Proposition: Suppose the distribution of buyer valuations for object A is more dispersed than the distribution for object B. Then the seller's expected revenues are greater if object A is auctioned first.

Proof: The difference in expected profit from auctioning object A before B is:
$E\{s_A(N)\} - [E\{f_B(N-1)\} - E\{s_B(N-1)\}]/(N-1) + E\{s_B(N-1)\} -$

$$[E\{s_B(N)\} - [E\{f_A(N-1)\} - E\{s_A(N-1)\}]/(N-1) + E\{s_A(N-1)\}]$$

$$= [E\{s_A(N)\} + E\{s_B(N-1)\}] - [E\{s_B(N)\} + E\{s_A(N-1)\}] \qquad (3)$$

$$- [E\{f_B(N-1)\} - E\{s_B(N-1)\}]/(N-1) + [E\{f_A(N-1)\} - E\{s_A(N-1)\}]/(N-1) > 0.$$

The first line of (3) is positive from A2 and A1 implies that the second line of (3) is positive. ■

Stated intuitively, when the object with the greater dispersion in buyer valuations is auctioned first, there are more bidders around to bid up the price. As well, the expected profits associated with purchasing good B are lower, so that bidders do not discount their first bids by as much. This is

easiest to see when the distribution of valuations for object B is degenerate. If B is auctioned first, each bidder discounts its valuation by its expected profit from participating in the second auction. Further, if the winner of object B had one of the two highest valuations of good A, then the sale price of object A is reduced as well. In contrast, if A is auctioned first then bidders do not discount their bids for A, since there is no profit from winning object B in the second auction.

End Notes

[1]Ashenfelter [1989] and McAfee and Vincent [1993] investigate auctions of identical cases of wine; Ashenfelter and Genesove examine auctions of (virtually identical) condominiums.

[2]In independent work, Gale and Hausch [1992] consider a two bidder version of Example 1, in which each bidder knows both of its object valuations prior to the first auction. They show that declining prices still emerge when the information structure is changed in this way. One conjectures that the result extends to more competitive auctions featuring more than two bidders.

[3]When N=3, the expected sale price in each auction is the same.

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