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ASSET PRICES IN A SPECULATIVE MARKET

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## Abstract

The stochastic properties of prices in a speculative market are investigated. Agents in the market start with different priors, but update in a rational (i.e., Bayesian) way from realizations of payoffs on the risky asset. Convergence of the equilibrium price to the rational expectations price is investigated, as well as the asymptotic properties of two standard tests of rational expectations. The results are contrasted with stylized facts from forward markets.



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## 1 Introduction

It is well known that convergence to the truth in a model of Bayesian learning may fail for many priors. The set of priors for which beliefs diverge is topologically large (Freedman [1963]). Consequently, in a market with agents who update using Bayes' rule, the equilibrium price will not converge to the rational expectations price for many priors.

The examples leading to nonconvergence of beliefs, however, are complicated (Freedman and Diaconis [1986]). In contrast, beliefs will converge almost surely (under the true probability measure) in standard problems, such as when the volatility of a normally distributed random variable is unknown and beliefs are drawn from the family of inverted gamma distributions.

In other situations, convergence of beliefs fails either because learning is sub-optimal (as in most of the Least Squares learning models; see, e.g., Fourgeaud, Gourieroux and Pradel [1986], Marcet and Sargent [1989]), or because learning is active (i.e., in bandit problems; see, e.g., Easley and Kiefer [1988], Feldman and McLennan [1989], El-Gamal and Sundaram [1992]).

Nevertheless, I provide a very simple example of failure of equilibrium prices to converge to the rational expectations price in a market where agents trade a forward contract on an underlying asset whose price is distributed as the square of a mean-zero normal random variable. Investors do not know the mean of the underlying price, but start with differing beliefs drawn from the family of inverted gamma distributions. They update rationally, i.e., using Bayes' rule.

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\*Division of Humanities and Social Sciences 288-77, California Institute of Technology, Pasadena, CA 91125. Phone (818) 356-4028. I am very grateful to Mahmoud El-Gamal for insisting that I think differently about asset pricing. Whereas Mahmoud is not to blame for any mistake, he certainly is to be credited for the inspiration.

The main feature of this model is the difference in beliefs among agents. It is the only reason why agents trade, i.e., they have no hedging motive. In other words, the equilibrium price of the forward market is purely speculative: it reflects the distribution of beliefs in the economy.

The absence of common beliefs is the cause for the failure of forward prices to converge to the rational expectations price. Indeed, if investors had common priors, drawn from the same distribution, convergence in probability (under the true probability measure) would have ensued. Nonconvergence obtains because investors do not learn uniformly.

Unfortunately, I also show that, even if the parameters are constrained so that convergence in probability follows, standard statistics will not converge to their rational expectations equivalent. Both the average prediction error and the average prediction error multiplied by the forward rate, scaled by the square root of the sample size, converge weakly to a random variable that (i) has higher variance than if the economy had been at its rational expectations equilibrium from the outset, and that (ii) is non-normal. The differences in beliefs is not crucial here: with homogeneous beliefs, a similar result follows. Consequently, convergence to rational expectations does not justify the use of properties of statistics that were derived under rational expectations.

The aforementioned statistics diverge, moreover, when there is a possibility that the economy does not converge to rational expectations. I show that both statistics diverge at a rate that is proportional to the square root of the sample size. Suitably scaled, the first statistic (the average prediction error) converges to a mean-zero random variable, whereas the second statistic (the average prediction error times the forward rate) converges to a mean-positive random variable.

The remainder of the paper is organized as follows. The next section describes my economy of speculating agents. Section 3 deals with the first question: will rational expectations be attained? Section 4 investigates the asymptotics of two popular statistics (testing unbiasedness and predictability, respectively) when prices do converge to their rational expectations equivalent. Section 5 does the same when convergence fails. The last section discusses the relevance of the theoretical findings of this paper for the behavior of forward interest and foreign exchange rates. It suggests that speculation-based theories may be useful complements to standard theories of risk premia.

## 2 Description of the Economy

Consider a repetition of one-period economies with a speculative market. The only link between economies at adjacent dates are the beliefs: agents living in the previous period only pass their updated beliefs on to the next generation. Each date, indexed  $t = 1, 2, 3, \dots$ , a forward market is held. The payoff on the forward contract depends on

the subsequent period's value of a random variable,  $x_{t+1}$ . The forward rate, denoted  $p_t$ , clears the market. If an agent takes a long position of one forward contract, the payoff on her position will be  $x_{t+1} - p_t$ . If she shorts one unit, her payoff will be  $p_t - x_{t+1}$ .

$x_t$  ( $t = 1, 2, 3, \dots$ ) is a real random variable with a distribution that depends on the value of a parameter  $\theta$ ,  $\theta^*$ . Agents do not know  $\theta^*$ . The first generation (those that live at date 0), however, have beliefs about  $\theta$ , speculate in the forward market, observe the payoffs, update their beliefs in a Bayesian way, and pass them on to the next generation. There are a countably infinite number of agents, indexed by  $j = 1, 2, 3, \dots$ . All behave competitively (they take prices as given). At date  $t$ , only the agents indexed  $j = 1, \dots, t$  participate in the forward market. The subsequent trading round, the agent with  $j = t + 1$  enters, after updating her beliefs using all observations from the outset on  $(x_1, x_2, \dots, x_{t+1})$ .

$\theta$  takes values in a parameter space  $\Theta$  (a subset of  $R$ ), with corresponding Borel  $\sigma$ -algebra  $\mathcal{F}(\Theta)$ . Agent  $j$ 's initial beliefs about  $\theta$  can be summarized by a measure  $\lambda_{j0}$  on  $(\Theta, \mathcal{F}(\Theta))$ .  $\lambda_{j0} \in \Pi(\Theta)$ , the set of probability measures on  $\Theta$ . The  $x_t$ s,  $t = 1, 2, 3, \dots$ , are i.i.d. random variables that live in an outcome space  $\mathcal{X}$ , with corresponding Borel  $\sigma$ -algebra  $\mathcal{F}(\mathcal{X})$ . All agents agree on the probability measure over  $\mathcal{X}$  that generates  $x_t$ , given a parameter value  $\theta$ . Let  $P_{\theta,1}$  denote this probability measure. We shall work with the product space  $\Theta \times \mathcal{X} \times \mathcal{X} \dots \times \mathcal{X}$  ( $t$  replicae of  $\mathcal{X}$  generate this product), with corresponding product  $\sigma$ -algebra  $\mathcal{F}(\Theta \times \mathcal{X} \times \mathcal{X} \dots \times \mathcal{X})$ , and probability measures  $Q_{jt}$ , where

$$\begin{aligned} Q_{jt}(d(\theta, x_1, x_2, \dots, x_t)) &= \lambda_{j0}(d\theta)P_{\theta,t}(d(x_1, x_2, \dots, x_t)) \\ &= \lambda_{j0}(d\theta)P_{\theta,1}(dx_1)P_{\theta,1}(dx_2)\dots P_{\theta,1}(dx_t). \end{aligned}$$

We are interested, in particular, in  $j$ 's conditional beliefs about  $\theta$ , given a  $t$ -period history,  $x_1, x_2, \dots, x_t$ , namely,  $Q_{jt}(A|x_1, x_2, \dots, x_t)$ , for any  $A \in \mathcal{F}(\Theta)$ . This conditional belief is formed using the rules of conditional probability (Bayes' rule), and passed on to member  $j$  of the next generation. For clarity, let  $\lambda_{jt}(A)$  denote this conditional belief. Let  $f(x_t|\theta)$  be the density of  $P_{\theta,1}$  with respect to the Lebesgue measure. As mentioned before, beliefs are updated using Bayes' rule. This means:

$$\begin{aligned} \lambda_{jt}(A) &= \frac{\int_A f(x_t|\theta)\lambda_{j,t-1}(d\theta)}{\int_{\Theta} f(x_t|\theta)\lambda_{j,t-1}(d\theta)} \\ &= \mathcal{B}(x_t, \lambda_{j,t-1})(A), \end{aligned}$$

for any  $A \in \mathcal{F}(\Theta)$ .

Assume that preferences and endowments each date are such that the demand for forward contracts can be described by a function that depends only on an agent's beliefs

and the forward quote. (Agents observe the equilibrium forward quote when determining their demand.) Let  $D$  denote this demand function.

$$D : \Pi(\Theta) \times R \rightarrow R : (\lambda, p) \rightarrow D(\lambda, p). \quad (1)$$

Each period, the forward market will clear at a rate that sets the average demand equal to zero. Assuming competitive behavior, this means:

$$\frac{1}{t} \sum_{j=1}^t D(\lambda_{jt}, p_t) = 0. \quad (2)$$

To summarize, each period  $t$ , agent  $j$  ( $j = 1, \dots, t$ ) speculates in a forward market, depending on her beliefs  $\lambda_{jt}$  and the equilibrium quote  $p_t$ . After the forward market closes,  $x_{t+1}$  is revealed and forward contracts are settled (if  $j$  demanded  $D(\lambda_{jt}, p_t)$  contracts, she will receive  $D(\lambda_{jt}, p_t)(x_{t+1} - p_t)$  units of consumption). From  $x_{t+1}$ , beliefs are updated to  $\lambda_{j,t+1}$ , using Bayes' rule, and the result is passed on to the individual in the next generation corresponding to  $j$ . Agent  $t + 1$  then enters the forward market for the first time, with beliefs  $\lambda_{t+1,t+1}$ , updated from the initial observation on.

The rational expectations forward rate is defined to be the rate that clears the market assuming agents know  $\theta^*$ , i.e., their beliefs put unit mass on the true  $\theta$ . Let  $\delta_{\theta^*}$  denote such beliefs. Hence, the rational expectations forward quote, denoted  $p^{re}$ , solves:

$$\frac{1}{t} \sum_{j=1}^t D(\delta_{\theta^*}, p^{re}) = 0. \quad (3)$$

Of course, because we have made the assumption that demand depends only on beliefs and the forward rate, volume in the forward market will be zero when the economy is at its rational expectations equilibrium (not an uncommon feature of rational expectations models).

Consider now the following parametrization. Let  $x_t$  be the square of a normally distributed random variable, with mean zero and variance  $\theta^2$ . Let the priors on  $\theta$  be inverted gamma-2 distribution with parameters  $v_j$  and  $\nu_j$  ( $v_j > 0$ ;  $\nu_j \in \{3, 4, 5, \dots\}$ ). Let  $D(\lambda_{jt}, p_t) = \lambda_{jt}[\int_{\mathcal{X}} x f(x|\theta) dx] - p_t$ . Functional notation will be used throughout; this conveniently indicates the measures over which expectations are taken; e.g.,  $\lambda_{jt}[g(\theta)] = \int_{\Theta} g(\theta) \lambda_{jt}(d\theta)$ . The following utility function justifies the demand function:

$$\lambda_{jt}[\int_{\mathcal{X}} (D(x - p_t) - \frac{1}{2} D^2) f(x|\theta) dx],$$

This is a utility function characterized by risk neutrality and quadratic adjustment costs. Under the above assumptions,

$$\lambda_{j0}[\int_{\mathcal{X}} x f(x|\theta) dx] = v_j \frac{\nu_j}{\nu_j - 2},$$



where  $v_j$  and  $\nu_j$  parametrize agent  $j$ 's initial beliefs about  $\theta$ , and

$$\lambda_{jt}[\int_{\mathcal{X}} xf(x|\theta)dx] = \frac{\nu_j}{\nu_j + t - 2}v_j + \frac{t}{\nu_j + t - 2}w_t,$$

where  $w_t = \frac{1}{t} \sum_{\tau=1}^t x_\tau$ . Fix beliefs as follows. Agent  $j$  has  $\nu_j = j + 2$ .  $v_j$  is set to a common value,  $v$ . Consequently,

$$D(\lambda_{jt}, p_t) = \frac{j+2}{j+t}v + \frac{t}{j+t}w_t - p_t, \quad (4)$$

$$p_t = \frac{1}{t} \sum_{j=1}^t \left( \frac{j+2}{j+t}v + \frac{t}{j+t}w_t \right), \quad (5)$$

and

$$p^{re} = \theta^{*2}. \quad (6)$$

We shall consider two cases. In the first one, the value of  $v$  is fixed at  $\theta^{*2}$ . In the second,  $v$  is drawn from a distribution over  $(0, \infty)$ , with expected value equal to  $\theta^{*2}$  and finite variance. Let  $\mu$  denote the probability measure generated by this distribution. Define:

$$M_{\theta,t}(d(v, x_1, x_2, \dots, x_t)) = \mu(dv)P_{\theta,t}(d(x_1, x_2, \dots, x_t)). \quad (7)$$

### 3 Convergence to the rational expectations equilibrium

Let us first investigate whether equilibrium prices converge to their rational expectations equivalent. In order to do so, we establish the following result.

#### Theorem 1

$$p_t - \theta^{*2} \rightarrow \ln\left(\frac{e}{2}\right)(v - \theta^{*2})$$

in  $M_{\theta^*, \infty}$ .

(All proofs are in the Appendix.) Notice that in this Theorem, convergence is analyzed under  $M_{\theta^*, \infty}$ , i.e.,  $v$  is considered to be random.

Several comments can be made about Theorem 1. First, since  $p^{re} = \theta^{*2}$  (Equation (6)), the theorem effectively is an example of failure of convergence of prices to the rational expectations price. Nonconvergence does not follow from learning *per se*, but

from the differences in beliefs. As a matter of fact, if agents start with homogeneous beliefs (i.e., all agents draw the same  $v$  and  $\nu$ ), then convergence will follow. What keeps prices from converging to the rational expectations price is that in my economy, while agents are all expected to eventually learn the truth, their beliefs do not converge uniformly. I conjecture that uniform convergence of beliefs is a necessary conditions for convergence of equilibrium prices to their rational expectations equivalent.

Second, the result is not an artefact of certain agents entering the market after some time. These agents enter the market after updating their beliefs with observations from the first trading round on (in financial markets, public disclosure of prices means that new participants can start at an equal footing with experienced agents). The entry behavior is postulated merely to keep the mathematics simple.

Third, the nonconvergence result embedded in Theorem 1 is not based on a complicated argument, as are the examples in Diaconis and Freedman [1986]. As already mentioned, each individual is expected to learn the truth. Nonconvergence merely follows from differences in learning speed.

Fourth, nonconvergence does not follow from irrational behavior. All agents are rational: they do know the laws of probability and use them to update their beliefs. This is in contrast to the Least Squares learning literature, where agents do not follow optimal updating rules (although the latter might be too hard to derive, and, hence, agents could be excused for updating suboptimally).

In Theorem 1, the value of  $v$  that is drawn before the market starts is not necessarily equal to  $\theta^{*2}$ . If we fix  $v$ , however, at  $\theta^{*2}$ , then  $p_t - \theta^{*2} \rightarrow 0$  in  $P_{\theta^*, \infty}$ , and, because  $p^{re} = \theta^{*2}$  (Equation (6)), equilibrium prices do converge in probability to the rational expectations price. We write this result as a corollary.

**Corollary 1.1** *Let  $v = \theta^{*2}$ . Then:*

$$p_t \rightarrow p^{re}$$

*in  $P_{\theta^*, \infty}$ .*

## 4 Properties of statistics when convergence to rational expectations holds

Let us now investigate the asymptotic properties of two popular tests of rational expectations (market efficiency). The first test examines whether the average prediction error ( $p_t - x_{t+1}$ ) equals zero. I shall refer to it as the *unbiasedness test*. The second test explores whether the prediction error is correlated with the past forward rate, i.e.,

whether the average prediction error multiplied by the forward rate  $((p_t - x_{t+1})p_t)$  equals zero. I shall refer to this one as the *predictability test*. The unbiasedness test involves the average prediction error, namely,

$$\frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}). \quad (8)$$

The predictability test uses the average prediction error multiplied by the forward rate, i.e.,

$$\frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})p_t. \quad (9)$$

To test unbiasedness and predictability, the quantities in (8) and (9) are scaled. The scaling factor is chosen such that, if prices indeed were formed by rational expectations from time 1 on, the corresponding statistics would converge to a standard normal random variable.

What if beliefs do not correspond to rational expectations from the initial trading round on? What if, in addition, beliefs initially differ, so that, at least during early rounds, prices are the consequence of speculating agents, as in my model? In this section, we shall investigate these questions for the case when equilibrium prices eventually do converge to the rational expectations price. In my model, this is obtained by fixing  $v$  to be equal to  $\theta^{*2}$  (Corollary 1.1). The next section will deal with the asymptotics of both statistics when prices do not necessarily converge, i.e.,  $v$  is drawn from some (nontrivial) measure  $\mu$ .

Let us investigate the two parts of the test statistics, namely, the averages in (8) and (9), and the scaling factors, separately. Consider first the averages. Theorem 2 determines the asymptotics in the general case, i.e., when  $v$  does not necessarily equal  $\theta^{*2}$ . Consequently, we will be able to refer to this Theorem in the next section as well.

## Theorem 2

$$\frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) \rightarrow \ln\left(\frac{e}{2}\right)(v - \theta^{*2}),$$

$$\frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})p_t \rightarrow \left(\ln\left(\frac{e}{2}\right)\right)^2(v - \theta^{*2})^2 - \theta^{*2} \ln\left(\frac{e}{2}\right)(v - \theta^{*2}),$$

both in  $M_{\theta^*, \infty}$ .

Setting  $v = \theta^{*2}$ , one obtains:

**Corollary 2.1** *Let  $v = \theta^{*2}$ . Then:*

$$\frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) \rightarrow 0,$$

$$\frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) p_t \rightarrow 0,$$

*both in  $P_{\theta^*, \infty}$ .*

Corollary 2.1 clearly indicates that the temporary speculative nature of equilibrium prices affects neither the average prediction error nor the average prediction error multiplied by the forward rate. This is good news: when the economy is known to converge to rational expectations, certain quantities have the same asymptotic behavior as if the economy were at its rational expectations equilibrium from the initial day on.

Unfortunately, the positive result in Corollary 2.1 does not carry over to the corresponding test statistics, which are obtained by scaling. Let us first explore what happens when we scale the averages in (8) and (9) by multiplying with the square root of the sample size,  $\sqrt{T}$ .

**Theorem 3** *Let  $v = \theta^{*2}$ . Then:*

$$\sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) \right) \rightarrow \sqrt{2} \theta^{*2} \left( \ln 2 \int_0^1 \frac{1}{\eta} W(\eta) d\eta - W(1) \right),$$

$$\sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) p_t \right) \rightarrow \sqrt{2} \theta^{*4} \left( \ln 2 \int_0^1 \frac{1}{\eta} W(\eta) d\eta - W(1) \right),$$

*both weakly.  $W(\eta)$  is the value of a standard Brownian motion at  $\eta$  ( $\eta \in [0, 1]$ ).*

Next, divide by the square root of the sample variances of (8) and (9). Define:

$$V_{1T} = \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})^2 \tag{10}$$

and

$$V_{2T} = \left( \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})^2 \right) \left( \frac{1}{T} \sum_{t=1}^T p_t^2 \right). \tag{11}$$

(Equation (11) is an estimator of the variance of (9) assuming homoscedasticity.) Both quantities converge in probability, as stated in the next Theorem.

**Theorem 4** *Let  $v = \theta^{*2}$ . Then:*

$$\begin{aligned} V_{1T} &\rightarrow 2\theta^{*4}, \\ V_{2T} &\rightarrow 2\theta^{*8}, \end{aligned}$$

*both in  $P_{\theta^*, \infty}$ .*

Combining Theorems 3 and 4, we obtain the asymptotics of the unbiasedness and predictability tests in an economy of speculating investors which is known to converge to rational expectations.

**Corollary 4.1** *Let  $v = \theta^{*2}$ . Then:*

$$\begin{aligned} \sqrt{T} \left( \frac{1}{T\sqrt{V_{1T}}} \sum_{i=1}^T (p_t - x_{t+1}) \right) &\rightarrow \ln 2 \int_0^1 \frac{1}{\eta} W(\eta) d\eta - W(1), \\ \sqrt{T} \left( \frac{1}{T\sqrt{V_{1T}}} \sum_{i=1}^T (p_t - x_{t+1}) p_t \right) &\rightarrow \ln 2 \int_0^1 \frac{1}{\eta} W(\eta) d\eta - W(1), \end{aligned}$$

*both weakly.  $W(\eta)$  is the value of a standard Brownian motion at  $\eta$  ( $\eta \in [0, 1]$ ).*

Comparing the asymptotics of Corollary 4.1 with the corresponding result when the economy is at its rational expectations equilibrium from the beginning on (both statistics then converge to  $W(1)$ ), we see that the difference equals  $\ln 2 \int_0^1 \frac{1}{\eta} W(\eta) d\eta$  in both cases. This random variable is a functional of a standard Brownian motion over the interval  $[0, 1]$ .

Corollary 4.1 clearly implies that asymptotic analysis of test statistics must consider explicitly the transient learning. The derivation of properties of test statistics as if the economy were at its rational expectations equilibrium from the initial trading round on cannot be justified by the knowledge that the economy will move eventually toward rational expectations.

I should emphasize that this negative result does not depend on differences in beliefs *per se*. A careful analysis of the proofs confirms that if investors all start with the same beliefs, a similar result would follow. In other words, standard asymptotics fail to hold in the mere presence of learning. This contrasts with the findings in the previous section: convergence to rational expectations (at least in my economy) would always obtain under homogeneous beliefs.

## 5 Properties of statistics when convergence to rational expectations does not follow

In the general case ( $v$  is drawn according to a measure  $\mu$ ), the averages in (8) and (9) do not converge to zero, and, hence, the corresponding statistics diverge. It is possible, however, to establish the speed of divergence.

### Theorem 5

$$\frac{1}{\sqrt{T}} \left( \sqrt{T} \left( \frac{1}{T\sqrt{V_{1T}}} \sum_{t=1}^T (p_t - x_{t+1}) \right) \right) \rightarrow \frac{\ln(\frac{\varepsilon}{2})(v - \theta^{*2})}{(\ln(\frac{\varepsilon}{2}))^2(v - \theta^{*2})^2 + 2\theta^{*4}},$$

$$\frac{1}{\sqrt{T}} \left( \sqrt{T} \left( \frac{1}{T\sqrt{V_{1T}}} \sum_{t=1}^T (p_t - x_{t+1})p_t \right) \right)$$

$$\rightarrow \frac{(\ln(\frac{\varepsilon}{2}))^2(v - \theta^{*2})^2 - \theta^{*2} \ln(\frac{\varepsilon}{2})(v - \theta^{*2})}{\{(\ln(\frac{\varepsilon}{2}))^2(v - \theta^{*2})^2 + 2\theta^{*4}\} \{(\ln(\frac{\varepsilon}{2}))^2(v - \theta^{*2})^2 + \theta^{*2}(\theta^{*2} + 2 \ln(\frac{\varepsilon}{2})(v - \theta^{*2}))\}},$$

both in  $M_{\theta^*, \infty}$ .

Theorem 5 not only provides an estimate of the speed of convergence (of the order of the square root of the sample size), but indicates what variables the statistics converge to (in probability) when suitably scaled. The first statistic, corresponding to the unbiasedness test, when multiplied by  $1/\sqrt{T}$ , converges to a random variable that depends on  $v - \theta^{*2}$ .  $v - \theta^{*2}$  varies across economies and can be both positive and negative. Consequently, we expect to see both positive and negative values for the unbiasedness statistic in cross-section, and they should increase (or decrease) in proportion with the square root of the sample size. The second statistic, used in the predictability test, converges to a variable that is more likely to be positive (because of the quadratic term  $(\ln(\frac{\varepsilon}{2}))^2(v - \theta^{*2})^2$ ). Consequently, we expect to see a higher proportion of positive values for this statistic in cross-section. They as well should be increasing with the square root of the sample size.

## 6 Concluding Remarks

The analysis of this paper seems to be particularly relevant in view of some longstanding puzzles about asset prices, in particular, forward rates. Errors from forward rates as predictors of future spot rates have persistently been found to be biased and predictable. This is most obvious in the case of interest rates (see, e.g., Fama [1986]). On the average, one-month forward rates implicit in two-month U.S. Treasury bill prices have been

above next month's one-month spot rate, and the prediction error, multiplied by the forward rate, is positive on average. As a matter of fact, explaining this finding is what term structure theory is about. While various equilibrium rational-expectations models attribute it to the presence of a (time-varying) risk premium (e.g., Cox, Ingersoll and Ross [1985]), the empirical success of such models has been mixed at best. Some even wonder whether they will ever fit the data (e.g., Den Haan [1991]). In contrast, this paper indicates that speculation may explain the empirical regularities.

Among other things, speculation not only explains why the one-month forward rates are biased and the prediction error multiplied by the forward rate is positive on average, but why the same phenomenon is true for two-month, three-month, four-month and five-month forward rates as well (the  $n$ -month forward rate predicts the one-month spot rate  $n$  months in the future, or the  $n - 1$ -month forward rate one month ahead). Speculation also explains why the corresponding statistics (scaled averages) often increase with the square root of the sample size.

Figure 1 illustrates this. It plots, as a function of sample size, (i) the unbiasedness statistic, (ii) the predictability statistic, (iii) the averages in (8) and (9), for the three-month forward interest rate (the results are similar for other forward rates<sup>1</sup>). A square root function is also fitted to (i) and (ii), in order to calibrate trends. All time series are constructed using the Fama Treasury bill files on the CRSP tapes. The sample consists of monthly values over the period 1959–86. I took the  $n$ -month forward rate *in excess of* the  $n - 1$ -month forward rate as a predictor of the *change* in the  $n - 1$ -month forward rate, in order to avoid well-known nonstationarity problems (the 0-month forward rate is the spot rate). In the notation of the model in the previous sections,  $p_t$  is the  $n$ -month forward rate in excess of the  $n - 1$ -month forward rate, and  $x_{t+1}$  is the change in the  $n - 1$ -month forward rate.

Prediction errors in forward foreign exchange rates exhibit the same behavior (see, e.g., Fama [1984]). Their averages are positive or negative, but, when multiplied by the forward rate, their averages are always positive. The corresponding statistics often grow proportional to the square root of the sample size. Figure 2 illustrates this. It plots the same variables as Figure 1 for one-month forward Japanese yen rates (the figures are similar for the deutsche mark, British pound, Swiss Frank, Canadian Dollar and French franc). Again, the forward rate in excess of the spot rate is taken as a predictor of the change in the spot rate, in order to avoid problems with nonstationarity. The data are sampled in intervals of four weeks over the period 1973–90. The data were acquired from DRI.

As with forward interest rates, the behavior of forward foreign exchange rates has been linked to equilibrium models within the rational expectations framework, but attempts to fit such models to the data have hitherto met little success (for an overview, see Hodrick

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<sup>1</sup>The complete set of figures can be obtained from the author.

[1987]). While they should not be ignored, data-related problems do not explain the findings either (Bossaerts and Hillion [1991]). Speculation, however, provides a viable alternative (or complement), well worth further investigation.

An asset pricing theory based on speculation has the added advantage that it readily explains trading volume, something which is difficult within a rational expectations framework (see Harris and Raviv [1991]). It also provides an alternative justification for trade in options (Bossaerts and Hillion [1992]). In a model of rational expectations and frictionless markets, trading in derivative securities such as options can only be linked to market completion.

A closer look at Figures 1 and 2 indicates that part of the puzzle is left unexplained. In particular, while the statistics of the predictability test increase in proportion to the square root of the sample size, this is not the case for the unbiasedness test. The square root function fits miserably. The unscaled averages, however, still reveal evidence of slowly changing beliefs: they change substantially as the sample size increases, i.e., they do not quickly converge to some population average and stay there forever.

In my economy, when prices do not converge to their rational expectations equivalent, the statistics corresponding to the unbiasedness and predictability tests increase in absolute value, proportional to the square root of the sample size. The former, however, will equally likely be positive or negative. The latter will be more often positive. Data from the forward interest rate market and the forward foreign exchange market confirm this. At present, however, I am unable to grasp the intuition behind this result.



## Appendix

*Proof of Theorem 1:*

From Chebychev's inequality:

$$\begin{aligned} M_{\theta^*, \infty} & \{ |p_t - \theta^{*2} - \ln(\frac{e}{2})(v - \theta^{*2})| > \epsilon \} \\ & \leq \frac{M_{\theta^*, \infty} |p_t - \theta^{*2} - \ln(\frac{e}{2})(v - \theta^{*2})|}{\epsilon} \end{aligned}$$

From (5)

$$p_t = \frac{1}{t} \sum_{j=1}^t \left( \frac{j+2}{j+t} v + \frac{t}{j+t} w_t \right).$$

Hence,

$$\begin{aligned} & M_{\theta^*, \infty} |p_t - \theta^{*2} - \ln(\frac{e}{2})(v - \theta^{*2})| \\ & \leq \mu |v - \theta^{*2}| \left| \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} - \ln(\frac{e}{2}) \right| \\ & + P_{\theta^*, \infty} |w_t - \theta^{*2}| \left| \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right| \\ & + \theta^{*2} \left| \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right| \end{aligned}$$

Now:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} & = \lim_{t \rightarrow \infty} \int_0^1 \frac{[\xi t] + 3}{[\xi t] + t + 1} d\xi \\ & = \int_0^1 \frac{\xi}{\xi + 1} d\xi \\ & = \ln(\frac{e}{2}), \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow \infty} \left| \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right| & \leq \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{j=1}^t \frac{t}{t} \\ & = \lim_{t \rightarrow \infty} \frac{t^2}{t^2} \\ & = 1 \end{aligned}$$

Also

$$P_{\theta^*, \infty} |w_t - \theta^{*2}| \rightarrow 0$$

and

$$\frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \rightarrow 0$$

as  $t \rightarrow \infty$ . Consequently:

$$\lim_{t \rightarrow \infty} M_{\theta^*, \infty} |p_t - \theta^{*2} - \ln\left(\frac{e}{2}\right)(v - \theta^{*2})| = 0.$$

□

*Proof of Corollary 1.1*

From (6), the rational expectations price equals:  $p^{re} = \theta^{*2}$ .

The result then follows immediately from Theorem 1.

□

*Proof of Theorem 2*

First, consider  $\frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) - \ln\left(\frac{e}{2}\right)(v - \theta^{*2})$ :

$$\begin{aligned}
& \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) - \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \\
&= \frac{1}{T} \sum_{t=1}^T \left( (p_t - \theta^{*2}) - \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \right) - \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) \\
&= (v - \theta^{*2}) \frac{1}{T} \sum_{t=1}^T \left( \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} - \ln\left(\frac{e}{2}\right) \right) \\
&\quad + \frac{1}{T} \sum_{t=1}^T (w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) + \theta^{*2} \frac{1}{T} \sum_{t=1}^T \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \\
&\quad - \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}).
\end{aligned}$$

Using Chebychev's inequality:

$$\begin{aligned}
& M_{\theta^{*}, \infty} \left\{ \left| \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) - \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \right| > \epsilon \right\} \\
&\leq \frac{M_{\theta^{*}, \infty} \left| \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) - \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \right|}{\epsilon}.
\end{aligned}$$

The numerator of the right-hand side can be rewritten as:

$$\begin{aligned}
& M_{\theta^{*}, \infty} \left| \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) - \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \right| \\
&\leq \mu |v - \theta^{*2}| \frac{1}{T} \sum_{t=1}^T \left| \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} - \ln\left(\frac{e}{2}\right) \right| \\
&\quad + \frac{1}{T} \sum_{t=1}^T P_{\theta^{*}, \infty} |w_t - \theta^{*2}| \left| \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right| + \theta^{*2} \frac{1}{T} \sum_{t=1}^T \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \\
&\quad + P_{\theta^{*}, \infty} \left| \frac{1}{T} \sum_{t=1}^T x_{t+1} - \theta^{*2} \right|.
\end{aligned}$$

The last term converges to zero as  $T \rightarrow \infty$ . The terms in the sum (over  $t$ ) in the first term converge to zero, hence, they are Cesàro summable, and their average converges to zero as well. The average (over  $j$ ) in the second term is finite, for all  $t$ . Since  $P_{\theta^*, \infty} |w_t - \theta^{*2}| \rightarrow 0$ , the terms in this sum (over  $t$ ) are Cesàro summable, and the expression converges to zero as well. The third term is also a Cesàro sum with limit 0.

Next, consider  $\frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})p_t - \left(\ln\left(\frac{e}{2}\right)\right)^2 (v - \theta^{*2})^2 - \theta^{*2} \ln\left(\frac{e}{2}\right)(v - \theta^{*2})$ .

$$\begin{aligned}
& \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})p_t - \left(\ln\left(\frac{e}{2}\right)\right)^2 (v - \theta^{*2})^2 - \theta^{*2} \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \\
&= \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2})p_t - \left(\ln\left(\frac{e}{2}\right)\right)^2 (v - \theta^{*2})^2 - \theta^{*2} \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \\
&\quad - \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})p_t \\
&= \frac{1}{T} \sum_{t=1}^T \left( (p_t - \theta^{*2})^2 - \left(\ln\left(\frac{e}{2}\right)\right)^2 (v - \theta^{*2})^2 \right) \\
&\quad + \theta^{*2} \frac{1}{T} \sum_{t=1}^T \left( (p_t - \theta^{*2}) - \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \right) \\
&\quad - \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(p_t - \theta^{*2}) - \theta^{*2} \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}).
\end{aligned}$$

Again appealing to Chebychev's inequality, convergence to zero of the following must be verified:

$$\begin{aligned}
& M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T \left( (p_t - \theta^{*2})^2 - \left(\ln\left(\frac{e}{2}\right)\right)^2 (v - \theta^{*2})^2 \right) \right. \\
&\quad \left. + \theta^{*2} \frac{1}{T} \sum_{t=1}^T \left( (p_t - \theta^{*2}) - \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \right) \right. \\
&\quad \left. - \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(p_t - \theta^{*2}) - \theta^{*2} \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) \right| \\
&\leq M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T \left( (p_t - \theta^{*2})^2 - \left(\ln\left(\frac{e}{2}\right)\right)^2 (v - \theta^{*2})^2 \right) \right| \\
&\quad + \theta^{*2} M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T \left( (p_t - \theta^{*2}) - \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \right) \right|
\end{aligned}$$

$$\begin{aligned}
& + M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(p_t - \theta^{*2}) \right| \\
& + \theta^{*2} M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) \right|.
\end{aligned}$$

Convergence of the last term is easily-established. Convergence of the second term follows from the first part of this proof. Consider the third term:

$$\begin{aligned}
& M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(p_t - \theta^{*2}) \right| \\
& \leq M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(v - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} \right) \right| \\
& \quad + M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \right| \\
& \quad + \theta^{*2} \frac{1}{T} \sum_{t=1}^T P_{\theta^*, \infty} |x_{t+1} - \theta^{*2}| \left| \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right|.
\end{aligned}$$

Convergence to zero of the last term follows from Cesàro summability. As to the second term, write:

$$\begin{aligned}
& M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \right| \\
& \leq \frac{1}{T} \sum_{t=1}^T P_{\theta^*, \infty} |x_{t+1} - \theta^{*2}| P_{\theta^*, \infty} |w_t - \theta^{*2}| \left| \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right|,
\end{aligned}$$

and convergence to zero follows again from Cesàro summability. Next, the first term can be rewritten as follows:

$$\begin{aligned}
& M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(v - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} \right) \right| \\
& \leq \mu |v - \theta^{*2}| P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} \right) \right|
\end{aligned}$$

Define:  $s_t = \sum_{\tau=1}^t (x_\tau - \theta^{*2})$  and  $W_T(\eta) = \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2\theta^{*2}}} s_{[\eta T]}$ . Then:

$$\begin{aligned}
& P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} \right) \right| \\
&= P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T \sqrt{T} \sqrt{2\theta^{*2}} \left( \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2\theta^{*2}}} s_{t+1} - \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2\theta^{*2}}} s_t \right) \left( \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} \right) \right| \\
&\leq \sqrt{2\theta^{*2}} P_{\theta^*, \infty} \left| \int_{1/T}^{T+1/T} \frac{\sqrt{T}}{[\eta T]} \int_0^{[\eta T]/T} \frac{[\eta T] + 3}{[\xi T] + [\eta T] + 1} d\xi dW_T(\eta) \right|.
\end{aligned}$$

As  $T \rightarrow \infty$ ,

$$\int_0^{[\eta T]/T} \frac{[\xi T] + 3}{[\xi T] + [\eta T] + 1} d\xi \rightarrow \int_0^\eta \frac{\xi}{\xi + \eta} d\xi = \eta \ln\left(\frac{e}{2}\right),$$

but

$$\frac{\sqrt{T}}{[\eta T]} \rightarrow 0.$$

Consequently, this term converges to zero as well.

Finally, to show convergence to zero of the first term, write:

$$\begin{aligned}
& M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T \left( (p_t - \theta^{*2})^2 - \left( \ln\left(\frac{e}{2}\right) \right)^2 (v - \theta^{*2})^2 \right) \right| \\
&= M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T \left( (v - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} \right) + (w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \right. \right. \\
&\quad \left. \left. + \theta^{*2} \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) \right)^2 - \left( \ln\left(\frac{e}{2}\right) \right)^2 (v - \theta^{*2})^2 \right| \\
&\leq \mu |v - \theta^{*2}|^2 \frac{1}{T} \sum_{t=1}^T \left| \left( \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} \right)^2 - \left( \ln\left(\frac{e}{2}\right) \right)^2 \right| \\
&\quad + \frac{1}{T} \sum_{t=1}^T P_{\theta^*, \infty} |w_t - \theta^{*2}|^2 \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right)^2 \\
&\quad + \theta^{*4} \frac{1}{T} \sum_{t=1}^T \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right)^2
\end{aligned}$$

$$\begin{aligned}
& + 2\mu|v - \theta^{*2}| \frac{1}{T} \sum_{t=1}^T P_{\theta^{*}, \infty} |w_t - \theta^{*2}| \left( \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} \right) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \\
& + 2\theta^{*2}\mu|v - \theta^{*2}| \frac{1}{T} \sum_{t=1}^T \left( \frac{1}{t} \sum_{j=1}^t \frac{j+2}{j+t} \right) \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) \\
& + 2\theta^{*2} \frac{1}{T} \sum_{t=1}^T P_{\theta^{*}, \infty} |w_t - \theta^{*2}| \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \left( \frac{1}{t} \sum_{j=0}^t \frac{2}{j+t} \right).
\end{aligned}$$

Cesàro summability implies that the 3rd, 5th and 6th terms converge to zero. The arguments of the first part of this proof immediately lead to convergence to zero of the 1st, 2nd and 4th terms.

□

*Proof of Corollary 2.1*

Immediate from Theorem 2.

□

*Proof of Theorem 3*

Borrowing results from the proof of Theorem 2, write:

$$\begin{aligned} & \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) \right) \\ &= \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \right) \\ & \quad + \sqrt{T} \theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T \frac{1}{t} \sum_{j=t}^t \frac{2}{j+t} \right) - \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) \right). \end{aligned}$$

Define  $s_t$  and  $W_T(\eta)$  as in the proof of Theorem 2. Then:

$$\begin{aligned} & \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \right) \\ &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{\sqrt{T}}{t^2} \sqrt{2} \theta^{*2} \left( \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2} \theta^{*2}} s_t \right) \left( \sum_{j=1}^t \frac{t}{j+t} \right) \\ &= \sqrt{2} \theta^{*2} \int_{1/T}^{T+1/T} \frac{T^2}{[\eta T]^2} \int_0^{[\eta T]/T} \frac{[\eta T]}{[\xi T] + [\eta T] + 1} d\xi W_T(\eta) d\eta. \end{aligned}$$

Using a version of the continuous mapping theorem that allows the functional to depend on the sample size (Billingsley [1968], Theorem 5.5, p. 34), and noting that, as  $T \rightarrow \infty$ ,

$$\int_0^{[\eta T]/T} \frac{[\eta T]}{[\xi T] + [\eta T] + 1} d\xi \rightarrow \int_0^\eta \frac{\eta}{\xi + \eta} d\xi = \eta \ln 2,$$

one obtains:

$$\sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \right) \rightsquigarrow \sqrt{2} \theta^{*2} \ln 2 \int_0^1 \frac{1}{\eta} W(\eta) d\eta,$$



where  $W(\eta)$  is the value of a standard Brownian motion at  $\eta(\in[0, 1])$ .

Next consider:

$$\begin{aligned} \sqrt{T}\theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) &= \theta^{*2} \int_0^1 \frac{T}{[\eta T] + 1} \int_0^{([\eta T] + 1)/T} \frac{2\sqrt{T}}{[\xi T] + [\eta T] + 2} d\xi d\eta \\ &\rightarrow 0 \end{aligned}$$

as  $T \rightarrow \infty$ . Finally,

$$\begin{aligned} \sqrt{T} \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) &= \sqrt{2}\theta^{*2} \left( \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2\theta^{*2}}} s_T \right) + \frac{1}{\sqrt{T}} (x_{T+1} - x_1) \\ &\rightsquigarrow \sqrt{2}\theta^{*2} W(1), \end{aligned}$$

by Donsker's Theorem (Billingsley [1965], Theorem 16.1, p. 137).

As for the second part of the theorem, consider:

$$\begin{aligned} &\sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) p_t \right) \\ &= \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2})^2 \right) + \sqrt{T}\theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2}) \right) \\ &\quad - \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(p_t - \theta^{*2}) \right) - \sqrt{T}\theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) \right). \end{aligned}$$

The fourth term is easiest:

$$\begin{aligned} &\sqrt{T}\theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) \right) \\ &= \sqrt{2}\theta^{*4} \left( \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2\theta^{*2}}} s_T \right) + \theta^{*2} \frac{1}{\sqrt{T}} (x_{T+1} - x_1) \\ &\rightsquigarrow \sqrt{2}\theta^{*4} W(1), \end{aligned}$$

by Donsker's Theorem. The second term was shown in the first part of this proof to converge to:

$$\sqrt{T}\theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2}) \right) \rightsquigarrow \sqrt{2}\theta^{*4} \ln 2 \int_0^1 \frac{1}{\eta} W(\eta) d\eta.$$

Using arguments from the proof of Theorem 2,

$$\begin{aligned} & \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(p_t - \theta^{*2}) \right) \\ &= \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \right) \\ & \quad + \sqrt{T}\theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) \right). \end{aligned}$$

Using Chebychev's inequality, the last term converges to zero in probability. To see this, consider:

$$\begin{aligned} & M_{\theta^*, \infty} \left| \sqrt{T}\theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) \right) \right| \\ &= \theta^{*2} P_{\theta^*, 1} |x_1 - \theta^{*2}| \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) \right). \end{aligned}$$

The first part of this proof showed that  $\sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) \right) \rightarrow 0$  as  $T \rightarrow \infty$ .

Using arguments from the proof of Theorem 2,

$$\begin{aligned} & \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (x_{t+1} - \theta^{*2})(w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \right) \\ &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \sqrt{T} \sqrt{2}\theta^{*2} \left( \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2}\theta^{*2}} s_{t+1} - \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2}\theta^{*2}} s_t \right) \\ & \quad \frac{\sqrt{T}}{t} \sqrt{2}\theta^{*2} \left( \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2}\theta^{*2}} s_t \right) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \\ &= 2\theta^{*4} \int_{1/T}^{T+1/T} \frac{T^{3/2}}{[\eta T]^2} \int_0^{[\eta T]/T} \frac{[\eta T]}{[\xi T] + [\eta T] + 1} d\xi W_T(\eta) dW_T(\eta). \end{aligned}$$

As  $T \rightarrow \infty$ ,

$$\int_0^{[\eta T]/T} \frac{[\eta T]}{[\xi T] + [\eta T] + 1} d\xi \rightarrow \int_0^\eta \frac{\eta}{\xi + \eta} d\xi = \eta \ln 2,$$

but

$$\frac{T^{3/2}}{[\eta T]^2} \rightarrow 0.$$

Consequently, the above expression converges to zero.

Finally,

$$\begin{aligned} & \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2})^2 \right) \\ &= \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T \left( (w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) + \theta^{*2} \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) \right)^2 \right) \\ &= \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (w_t - \theta^{*2})^2 \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right)^2 \right) \\ &\quad + \sqrt{T} \theta^{*4} \left( \frac{1}{T} \sum_{t=1}^T \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right)^2 \right) \\ &\quad + 2\sqrt{T} \theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T (w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) \right). \end{aligned}$$

The first term converges to zero:

$$\begin{aligned} & \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (w_t - \theta^{*2})^2 \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right)^2 \right) \\ &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{T}{t^2} 2\theta^{*4} \left( \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2}\theta^{*2}} s_t \right)^2 \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right)^2 \\ &= 2\theta^{*4} \int_{1/T}^{T+1/T} \frac{T^{3/2}}{[\eta T]^2} \left( \frac{T}{[\eta T]} \int_0^{[\eta T]/T} \frac{[\eta T]}{[\xi T] + [\eta T] + 1} d\xi \right)^2 (W_T(\eta))^2 d\eta, \end{aligned}$$

and, as  $T \rightarrow \infty$

$$\frac{T}{[\eta T]} \int_0^{[\eta T]/T} \frac{[\eta T]}{[\xi T] + [\eta T] + 1} d\xi \rightarrow \int_0^\eta \frac{1}{\xi + \eta} d\xi = \ln 2,$$

but

$$\frac{T^{3/2}}{[\eta T]^2} \rightarrow 0.$$

The second term converges to zero as well:

$$\begin{aligned} & \sqrt{T}\theta^{*4} \left( \frac{1}{T} \sum_{t=1}^T \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right)^2 \right) \\ &= \theta^{*4} \int_0^1 \left( \frac{T}{[\eta T] + 1} \int_0^{([\eta T] + 1)/T} \frac{2T^{1/4}}{[\xi T] + [\eta T] + 2} d\xi \right)^2 d\eta, \end{aligned}$$

and, as  $T \rightarrow \infty$ ,

$$\frac{T}{[\eta T] + 1} \int_0^{([\eta T] + 1)/T} \frac{2T^{1/4}}{[\xi T] + [\eta T]} d\xi \rightarrow 0.$$

Finally,

$$\begin{aligned} & 2\sqrt{T}\theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T (w_t - \theta^{*2}) \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) \right) \\ &= 2\sqrt{T}\theta^{*2} \left( \frac{1}{T} \sum_{t=1}^T \frac{\sqrt{T}}{t} \sqrt{2}\theta^{*2} \left( \frac{1}{\sqrt{T}} \frac{1}{\sqrt{2}\theta^{*2}} s_t \right) \right. \\ & \quad \left. \left( \frac{1}{t} \sum_{j=1}^t \frac{t}{j+t} \right) \left( \frac{1}{t} \sum_{j=1}^t \frac{2}{j+t} \right) \right) \\ &= 2\sqrt{2}\theta^{*4} \int_{1/T}^{T+1/T} \frac{T^3}{[\eta T]^3} \left( \int_0^{[\eta T]/T} \frac{[\eta T]}{[\xi T] + [\eta T] + 1} d\xi \right) \left( \int_0^{[\eta T]/T} \frac{2}{[\xi T] + \eta T + 1} d\xi \right) \\ & \quad W_T(\eta) d(\eta). \end{aligned}$$

Since  $\int_0^{[\eta T]/T} \frac{2}{[\xi T] + [\eta T] + 1} d\xi \rightarrow 0$  as  $T \rightarrow \infty$ , the above expression converges to zero.

□

*Proof of Theorem 4*

To prove the first claim, verify whether

$$P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})^2 - 2\theta^{*4} \right|$$

converges to zero. Convergence in  $P_{\theta^*, \infty}$  then follows from Chebychev's inequality. Rearrange this expression:

$$\begin{aligned} & P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})^2 - 2\theta^{*4} \right| \\ & \leq P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2})^2 \right| + P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (\theta^{*2} - x_{t+1})^2 \right. \\ & \quad \left. - 2\theta^{*4} \right| + 2P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (p_t - \theta^*)(\theta^{*2} - x_{t+1}) \right|. \end{aligned}$$

Borrowing a result from the proof of Theorem 2, the first term converges to zero. Convergence to zero of the second term follows immediately from the maintained distributional assumptions. As shown in the proof of Theorem 2, the third term converges to zero by Cesàro summability.

Consider the second claim. Convergence in  $P_{\theta^*, \infty}$  will follow, again by Chebychev's inequality, if it is shown that:

$$P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})^2 - 2\theta^{*4} \right| \rightarrow 0 \text{ as } T \rightarrow \infty,$$

and

$$P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T p_t^2 - \theta^{*4} \right| \rightarrow 0 \text{ as } T \rightarrow \infty.$$

The former was shown before. For the latter, write:

$$\begin{aligned} & P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T p_t^2 - \theta^{*4} \right| \\ & \leq P_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2})^2 \right| + 2\theta^{*2} \left| \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2}) \right|. \end{aligned}$$

From the proof of Theorem 2, both terms converge to zero.

□

*Proof of Corollary 4.1*

Follows immediately from dividing the results in Theorem 3 by the those of Theorem 2.

□

*Proof of Theorem 5*

Borrowing results from Theorem 2:

$$\begin{aligned} & \frac{1}{\sqrt{T}} \left( \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) \right) \right) \\ &= \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) \\ & \rightarrow \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \end{aligned}$$

in  $M_{\theta^*, \infty}$ . The denominator of the first statistic can be rewritten as follows:

$$\begin{aligned} & \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})^2 \\ &= \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2})^2 + \frac{1}{T} \sum_{t=1}^T (\theta^{*2} - x_{t+1})^2 \\ & \quad + \frac{2}{T} \sum_{t=1}^T (p_t - \theta^{*2})(\theta^{*2} - x_{t+1}). \end{aligned}$$

The proof of Theorem 2 indicates that the first term converges to  $(\ln(\frac{e}{2}))^2 (v - \theta^{*2})^2$  in  $M_{\theta^*, \infty}$ , while the third term converges to zero. Because of the distributional assumptions, the second term converges to  $2\theta^{*4}$ . Combining the numerator and denominator produces the first claim of Theorem 5.

Analogously,

$$\begin{aligned} & \frac{1}{\sqrt{T}} \left( \sqrt{T} \left( \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) p_t \right) \right) \\ &= \frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1}) p_t \\ & \rightarrow (\ln(\frac{e}{2}))^2 (v - \theta^{*2})^2 - \theta^{*2} \ln\left(\frac{e}{2}\right)(v - \theta^{*2}) \end{aligned}$$

in  $M_{\theta^*, \infty}$ . Decompose the denominator into  $\frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})^2$  and  $\frac{1}{T} \sum_{t=1}^T p_t^2$ . By the above argument,

$$\frac{1}{T} \sum_{t=1}^T (p_t - x_{t+1})^2 \rightarrow (\ln(\frac{e}{2}))^2 (v - \theta^{*2})^2 + 2\theta^{*4}$$

in  $M_{\theta^*, \infty}$ . Use Chebychev's inequality to show that  $\frac{1}{T} \sum_{t=1}^T p_t^2$  converges in  $M_{\theta^*, \infty}$  to  $(\ln(\frac{e}{2}))^2 (v - \theta^{*2})^2 + \theta^{*2}(\theta^{*2} + 2 \ln(\frac{e}{2})(v - \theta^{*2}))$ :

$$\begin{aligned} & M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T p_t^2 - (\ln(\frac{e}{2}))^2 (v - \theta^{*2})^2 - \theta^{*2}(\theta^{*2} + 2 \ln(\frac{e}{2})(v - \theta^{*2})) \right| \\ & \leq M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2})^2 - (\ln(\frac{e}{2}))^2 (v - \theta^{*2})^2 \right| \\ & \quad + 2\theta^{*2} M_{\theta^*, \infty} \left| \frac{1}{T} \sum_{t=1}^T (p_t - \theta^{*2}) - \ln(\frac{e}{2})(v - \theta^{*2}) \right|. \end{aligned}$$

From the proof of Theorem 2, both terms converge to zero.

□



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