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SOPHISTICATED VOTING WITH SEPARABLE PREFERENCES

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I. INTRODUCTION

Suppose there is a legislature with a set of members, M, each of whose preferences, R_i, is a weak order defined on a set of alternatives X. Assume further that X has more than two elements and that M contains m ($>$ 1) legislators. The legislature has some method for aggregating member preferences in such a way that a single alternative will be chosen for any given set of preferences. We may denote this aggregation procedure as a function F that takes m-tuples of preference relations into elements of X. F may be thought of as a voting procedure of some sort that takes the preferences that individuals give it as data and produces a unique social choice, $F(R_1, \ldots, R_m) \in X$.

Gibbard [1973] and Satterthwaite [1974] investigated conditions under which no voter has an incentive to misrepresent his true preferences under the procedure F. They proved that if $|X| > 2$, every F that is nondictatorial has the property that some individual voter will on occasion have an incentive to announce a preference ordering that is not his own.

In this paper, I will impose a relatively weak preference restriction and a class of voting procedures with the property that correct revelation of preferences is a Nash equilibrium. The theorem proved here, while quite simple mathematically, is of importance because the class of procedures proposed is similar to those employed in legislatures and because the preference restriction seems to be acceptable as a description of legislator tastes.

In order to establish this result, we shall make use of a theorem of Farquharson [1969]. Farquharson studied a class of voting procedures that worked as follows. A given set of alternatives X is divided into two subsets X_1, X_2 with the property that $X_1 \neq X$ and $X_2 \neq X$ and $X_1 \cup X_2 = X$. The legislature must decide which of the two alternative subsets it prefers according to a nondictatorial voting procedure that satisfies a positive association axiom. Then, the subset that wins on the first division is divided into two subsets in the same fashion and the legislature must decide between these two subsets. The procedure continues until a unique alternative is chosen. Such a procedure is called binary.

Each legislator is therefore faced with the problem of choosing a strategy which tells him how to vote at each division. Farquharson distinguished the following types of strategies.

Definition: A strategy is called sincere if and only if at each division a legislator always chooses the subset which contains his most preferred alternative. In case of a tie, he picks the subset with the highest second place element, etc..

Definition: A strategy is called straightforward if and only if, given the procedure, it dominates (in a game theoretic sense) all other strategies.

Definition: A strategy is called admissible if and only if it is undominated. One can see that if all legislators have strong preference orders (xR_iy and $yR_ix \implies x = y$) then there is only one sincere voting strategy. Further, if there is a straightforward strategy, it is unique.

Farquharson introduced the notion of a sophisticated voting strategy in the following manner. The set of admissible strategies is called the set of primarily admissible strategies. Under the assumption that no one else will play an inadmissible strategy, each legislator may apply the definition of admissibility again and obtain the set of secondarily admissible strategies. The set of k-arily admissible strategies is obtained by finding the set of admissible strategies under the assumption that the others will each employ (k - 1)-arily admissible strategies.

Definition: A strategy is sophisticated if and only if it is k-arily admissible for all k.

Farquharson proved that if everyone adopts a sophisticated strategy under a binary procedure, and if preferences are strong, then there is a determinate outcome in the sense that if any subset of voters switch to alternative sophisticated strategies the outcome is unchanged (note that sophisticated strategies are not generally unique). Further, if everyone chooses a sophisticated strategy, the legislature is at a Nash equilibrium. We shall demonstrate that for a class of procedure and a particular preference restriction, the unique sophisticated voting strategy for each legislator is his sincere voting strategy. Thus, *a fortiori* sincere voting in this case is an equilibrium configuration.

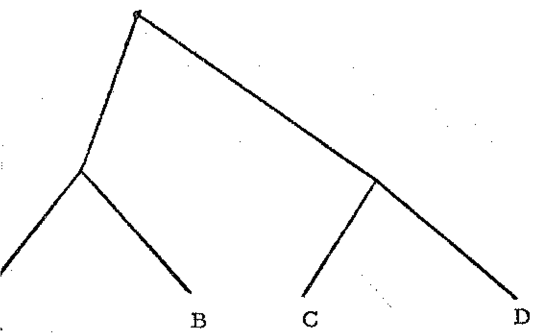
II. THE MODEL

We shall assume that, as in actual legislative bodies, the set of alternatives X is constructed through the passage or failure of a set of n bills. We may therefore write X as a Cartesian product as follows: $X = \{0, 1\}^n$ where 0 denotes the failure of a bill and 1, its passage. Elements of X are simply ordered n-tuples that tell which bills passed and which did not. The class of procedures under examination here is described informally as follows. The legislature takes up the bills in a given order and decides on passage or failure one bill at a time. An example will illustrate the sort of restriction that is being described. Suppose there are two bills and, therefore, four alternatives constructed as follows:

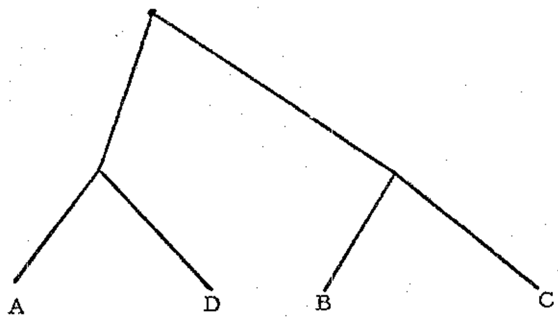
A	PP
B	PF
C	FP
D	FF

Consider the following three agendas:

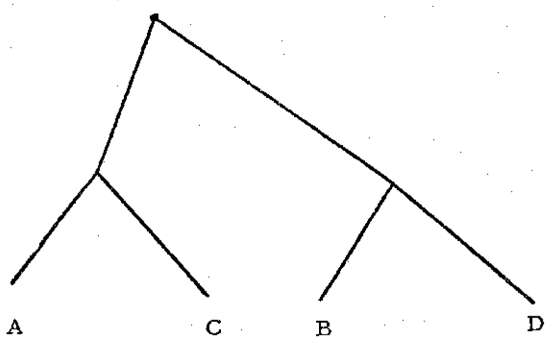
Agenda one



Agenda two



Agenda three



Agendas one and three satisfy the above definition while agenda two does not. Agendas that satisfy this definition are called bill-by-bill (BBB).

Each of the voters is assumed to have a strong preference order on X that satisfies the following condition.

Definition: A preference ordering R on $X = \{0, 1\}^m$ is separable if and only if for any subset of bills Q and $x, y, \overline{x}, \overline{y} \in X$ such that

$$x_i = y_i \text{ and } \overline{x}_i = \overline{y}_i \quad \forall_i \in Q \quad \text{and}$$

$$\overline{x}_i = x_i \text{ and } \overline{y}_i = y_i \quad \forall_i \notin Q \quad \text{then}$$

$$\overline{x} R \overline{y} \text{ if and only if } x R y.$$

That is, a legislator's preference for the passage or failure of a particular bill does not depend on the disposition of the other bills. The following example shows that if a legislator has separable preferences, and if the agenda is BBB, then he does not necessarily have a straightforward strategy.

Example 1: Assume there are two issues and four alternatives labeled as above and that agenda three is in use. Assume the legislator has the following preferences:

A
B
C
D

Then depending on how the other legislators vote, A may beat C or C may beat A and B may or may not beat D. If the first legislator believes that A will beat C then he votes for {A, C} on the first division. If he thinks that B beats D and C beats A, then he should vote for {B, D} on the first division. Evidently he does not have a straightforward strategy.

Nevertheless, we may demonstrate the following theorem.

Theorem: If all legislators have strong separable preferences, and if the agenda is bill-by-bill, then the unique sophisticated voting strategy for each legislator is to vote sincerely.

Proof: Consider an arbitrary bill-by-bill agenda. At the final vote, once all but one of the bills has been decided, each legislator will vote sincerely in every primarily admissible strategy. Thus, each will vote for passage or against passage of this bill no matter what happened on the other bills. Thus, the disposition of the final bill is certain to each legislator under the assumption that everyone plays a primarily admissible strategy and has separable preferences.

On the next to last vote, the disposition of the first n-2 bills is known, and if everyone plays primarily admissible strategies so is the disposition of the last bill. Thus, as before, each legislator will vote sincerely on this bill in all secondarily admissible strategies. Since his preferences are separable, he will either vote for passage regardless of the disposition of the other bills or for failure.

Repeating this argument, it is seen that the only ultimately admissible strategy is the sincere voting strategy.

Q. E. D.

Note that this result depends on the assumption that everyone has separable preferences. If not, the situation in example one could arise: C might beat A while B beats D at the last vote. In this situation, a sophisticated legislator with separable preferences would vote insincerely on the first bill.

Further, this theorem depends crucially on the assumption that the agenda is bill-by-bill. Consider the following example using agenda two given above.

Example 2: Assume that three legislators have the following preferences and that the passage of bills is decided by simple majority rule:

1	2	3
A	D	D
B	B	B
C	C	C
D	A	A

On the second division, D beats A and B beats C so that if Mr. 1 is sophisticated he will vote insincerely on the first division.

III. DISCUSSION

The simple little theorem in this paper demonstrates that, in a certain sense, we have been able to find a class of voting procedures for deciding among sets of more than three elements that seem to be "unmanipulable" and nondictatorial. How does this square with the Gibbard-Satterthwaite theorem? Of course, our notion of manipulability is somewhat different than theirs. In the Gibbard-Satterthwaite sense, binary bill-by-bill procedures are manipulable since one can imagine a set of announced preferences in all but one of the legislators that leaves the last legislator an incentive to misreveal his preferences. Exactly such a case was given in example two. For the Gibbard-Satterthwaite definition of nonmanipulability to be satisfied, a procedure would have to present each member with a straightforward strategy no matter what his preferences are. We have implicitly offered a weaker definition. A procedure is unmanipulable with respect to a preference restriction if and only if for all admissible preferences the strategy m-tuple in which all members vote sincerely is a Nash equilibrium.

The importance of the preference restriction in the above discussion should be emphasized. Using the Gibbard-Satterthwaite theorem, we can easily show that if legislators can hold any m-tuple of weak orders then the sincere-voting strategy m-tuple cannot be an equilibrium for all preference configurations. Assume the contrary. Using the Gibbard-Satterthwaite theorem, if the procedure is nondictatorial there is an (m-1)-tuple of strategy choices by all but one of the voters that gives the last voter an incentive to misreveal his preferences. Simply choose a preference configuration so that the (m-1)-tuple consists of sincere revelation by the other voters. Thus, some sort of restriction on preferences is needed to guarantee that sincere voting is an equilibrium strategy. It is pleasing that the required restriction seems relatively weak as a descriptive statement about preferences.

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