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VOTING OPERATORS IN THE SPACE OF CHOICE FUNCTIONS

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Abstract

Assuming the individual and collective opinions given as choice functions the new formalization of voting problem is considered. The notions of local functional operator and the closedness of domains in choice-functional space relative to local operators are introduced. The problem of voting is reduced to the analysis of three kinds of operators' classes and their mutual relations.

The functional analogues of well-known in the theory of Arrow paradox results are established.

Key words: Voting problem, Choice function, Functional profile, Local operator, Operators' classes, Operator's closedness.

1. Introduction

An axiomatic approach to the problems of voting first suggested in Arrow (1963) was further developed in two directions. According to the first of them it is assumed that both individual opinions and collective decision are formalised as orderings of variants (or as binary relations of a more general type). The rule which performs the transformation of individual binary

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relations into a collective one may be called a voting operator of type I.

Investigations of operators of type I have been summarised in a general form in Ferejohn and Fishburn (1979), and Aizerman and Aleskerov (1983b).

The second direction deals with the axiomatic analysis of the voting problems with individual opinions formalised as before in the form of orderings (binary relations), while a collective decision is obtained as a choice function (see Sen (1973), Richelson (1978a), Grether and Plott (1982) and others).

The operators which perform such kind of transformation may be accordingly called as operators of type[1] II.

The above two trends have directly led to the analysis of operators of a different type which are specified on the choice functions of voters and determines a collective choice function; whereas the "input" and "output" choice functions are not necessarily classically rational. Such operators will be called here the operators of type[2] III.

This paper[3] is concerned with the operators of this type III.

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1) According to Sen (1973) operator of type II is called "functional" (contrary to the "relational" rule for operators of type I). The term "functional operator" is used here in a different sense - its definition is given below.

2) The authors do not know publications in which such operators were analyzed. The only two papers (Blair (1975) and Parks (1976)) were concerned with the examples of operators of type III for checking of whether these operators had met the characteristic conditions similar to those of Arrow.

3) The summary of this paper has been published in Aizerman and Aleskerov (1983a).

The notion of "ordering of variants" is not used hereafter altogether.

The paper consists of seven sections.

Section 2 introduces basic notions and the statement of the problem, defines a local functional operator and their list representation[4]. All the results described have been obtained using the list representation.

Section 3 introduces characteristic properties of local functional operators and studies relations between the classes of operators isolated according to such properties.

Section 4 considers various "mechanisms" of generating the operators and the classes which correspond to them in the operator set.

When the operators of type III are regarded the problem arises of replacing the standard constraints (like transitivity, acyclicity, etc. used in the analysis of operators of types I and II) with other constraints which have to be imposed just on the choice functions. To solve the problem the notion of closedness of domains in choice-functional space relative to the operators in question is introduced.

Section 5 analyses the conditions which have to be met by the operators used to "restore" in a specified domain the collective choice function using the individual choice functions from the same domain.

Section 6 establishes relations between the classes of operators relative to which the domains in the choice functions

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4) A similar representation was used in Aizerman and Aleskerov (1983b) to describe operators of type I.

space are closed with the operator classes analysed in sections 3 to 5. Particular importance is given to analysis of operators belonging to a so- called Central Region delineated by the conditions natural for the voting operators.

Finally, section 7 discusses a possibility of obtaining a collective choice function from the specified domain using a fixed operator and compares the results obtained in section 6 with those from Aizerman and Aleskerov (1983 b) pertinent to the operators of type I.

2. Local functional operators and their list-form representation

Choice of variants is assumed to be performed here as follows (see also Aizerman and Malishevsky (1981)): a finite set A which consists of $m > 2$ variants $x_1, \ldots, x_m$ is given; every non-empty subset $X \in \mathcal{A}$, where $\mathcal{A} = 2^A \setminus \{\phi\}$, may be presented for choice, and the choice consists in picking out the subset $Y \subseteq X$ (the case $Y = \phi$, i.e. a refusal of choice, is allowed too). The set of pairs $\{X, Y\}$, $\forall X \in \mathcal{A}$ is said to be a choice function $Y = C(\cdot)$. The space(set) of such functions is denoted as $\mathcal{C} : C(\cdot) \in \mathcal{C}$.

Suppose that a group consists of $n \geqslant 2$ voters; n is finite, and $N = \{1, \ldots, n\}$ is a set of voters' indises. Each voter $i \in N$ independently of others defines his own function, i.e. a subset $Y_i = C_i(\cdot)$ for all $X \in \mathcal{A}$. The totality of choice functions $C_i(\cdot)$, $i \in N$ is said to be a functional profile (or, for brevity, a profile) and denoted as $\{C_i(\cdot)\}$. The profile $\{C_i(\cdot)\}$ has to be transformed into a choice function $C^*(\cdot)$, which defines a collective choice $Y^* = C^*(X)$, for all $X \in \mathcal{A}$.

The mapping F, which transforms each profile $\{C_i(\cdot)\}$ into a collective choice function $C^*(\cdot)$ is referred to as functional voting operator (of type III). An operator F is defined on n-tuples of functions from $\mathcal{C}$ and it values are functions from $\mathcal{C}$ too.

Let $V(x, X; \{C_i(\cdot)\})$ be a set of voters' indices for which

$$V(x, X; \{C_i(\cdot)\}) = \{ i \in N \mid x \in C_i(X) \}$$

holds, i.e. a set $V(x, X; \{C_i(\cdot)\})$ consists of such voters $i \in N$ who choose a variant x from X by their choice functions $C_i(\cdot)$.

<u>Definition 1.</u> An operator F is referred to as a local functional one (for brevity, L-operator) if for any two profiles $\{C_i(\cdot)\}$ and $\{\tilde{C}_i(\cdot)\}$, and arbitrary x, X $(x \in X \in \mathcal{A})$, which satisfy the condition $V(x, X; \{C_i(\cdot)\}) = V(x, X; \{\tilde{C}_i(\cdot)\})$, $x \in C^*(X)$ iff $x \in \tilde{C}^*(X)$ holds, where $C^*(\cdot) = F(\{C_i(\cdot)\})$, $\tilde{C}^*(\cdot) = F(\{\tilde{C}_i(\cdot)\})$.

It can be easily seen that local operators imply that the inclusion of the variant x in the collective choice $C^*(X)$ is independent of other variants y from $X \setminus \{x\}$.

The set of all local operators is hereafter denoted as $\mathcal{L}$.

Let us introduce a list-form representation of local functional operators.

<u>Definition 2.</u> An arbitrary subset ω of N $(\omega \subseteq N)$ is said to be a group. Let us put into correspondence to each pair (x, X), where $x \in X \in \mathcal{A}$, a set of groups $\Omega(x, X) = \{\omega_1^{(x,X)}, \dots, \omega_s^{(x,X)}\}$. The set $\Omega(x, X)$ is referred to as a list for the pair (x, X). We will say that operator F has a list-form representation if there exists a totality of lists $\{\Omega(x, X)\}_{x \in X \in \mathcal{A}}$ such that for all x, X, where $x \in X \in \mathcal{A}$, and all profiles $\{C_i(\cdot)\}$, where $C_i(\cdot) \in \mathcal{C}$, $\forall i \in N$, the following expression holds

$$x \in C^*(X) \Leftrightarrow V(x, X; \{C_i(\cdot)\}) \in \Omega(x, X)$$

The class of all operators for which there exists a list-form representation is denoted as Φ . Definitions 1 and 2 immediately imply that the classes Φ and $\mathcal{L}$ coincide, i.e. every L-operator has a list-form representation and, vice versa, every operator which has a list-form representation is local.

In sect.3 to 5 various constraints are imposed on lists and such constraints are interpreted in terms of L-operators' properties.

Remark. To analyse the operators of type I Murakami (1968) used the boolean functions' technique.

The similar technique can be used in our study of operators of type III.

Let us introduce the following logical variables $\ell_i(x, X)$ ($i = 1, \ldots, n$):

$$\ell_i(x, X) = \begin{cases} 1, \text{ if } & x \in C_i(X), \\ 0, \text{ if } & x \notin C_i(X) \end{cases}$$

Then locality of a voting operator is equivalent to the existence of a boolean function $f_{(x,X)}(\ell_1, \ldots, \ell_n)$, which satisfies the condition

$$f_{(x,X)}(\ell_1, \ldots, \ell_n) = \begin{cases} 1, \text{ if } & x \in C^*(X), \\ 0, \text{ if } & x \notin C^*(X) \end{cases}$$

The list-form representation of L-operators is directly related to the above boolean function. Namely, each group $\omega^{(x,X)}$ in the list $\Omega(x,X)$ corresponds to some totality $\ell_1, \ldots, \ell_n$ of variables for which $f_{(x,X)}(\ell_1, \ldots, \ell_n) = 1$, and $\omega^{(x,X)}$

consists just of indices i of those variables ℓ_i in this totality which are equal to 1.

If the function $f(x,X)$ is represented in a perfect disjunctive normal form (PDNF) , then the list $\Omega(x,X)$ consists of groups $\omega^{(x,X)}$ which correspond to conjunctive terms of such PDNF. Each such group $\omega^{(x,X)}$ consists of indices i of those variables which are included in a given term positively, i.e. as ℓ_i , not $\bar{\ell}_i$.

On the other hand, the representation of a function $f(x,X)$ in a perfect conjunctive normal form (PCNF) creates a possibility to construct another list $\mathcal{E}$ (a set of groups of indices). Such a list consists of groups $\varepsilon^{(x,X)}$ which correspond to disjunctive terms of that PCNF and each group as previosly includes indices j of such variables ℓ_j which belong to the conformable term in a positive way. Using this "dual" list $\mathcal{E}(x,X)$ a "dual" list-form representation for given L-operator can be obtained

$$[x \notin C^*(X)] \Longleftrightarrow [\exists\, \varepsilon \in \mathcal{E}(x,X) : V(x\, X;\{C_i(\cdot)\}) = N \setminus \varepsilon].$$

3. Characteristic classes in the set of local operators

Introduce some characteristic conditions[5] similar to those considered in Arrow (1963), and following papers investigated operators of type I. These conditions and the corresponding classes in $\mathcal{L}$ are numbered as 1°,2°,...,5°. In the formulations below $C^*(\cdot) = F(\{C_i(\cdot)\})$

1. Sovereignty (non-imposedness). This condition is divided into two conditions:

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5) Notice that the analogue of Arrow's main "independence of irrelevant alternatives" condition – the locality one – is not picked out separately. This condition is included, by definition 1, in the formulation of "operator's locality", i.e. in the definition of the set $\mathcal{L}$.

1°_+. Positive sovereignty – for every $X \in \mathcal{A}$ and $x \in X$ there exists a profile $\{C_i(\cdot)\}$ such that $x \in C^*(X)$;

1°_-. Negative sovereignty – for every $X \in \mathcal{A}$ and $x \in X$ there exists a profile $\{C_i(\cdot)\}$ such that $x \notin C^*(X)$

The case when both conditions 1°_+ and 1°_- are satisfied is referred to as sovereignty condition 1°.

The conditions 1° guarantee for every $X \in \mathcal{A}$ that a collective choice $Y^* = C^*(X)$ is dependent on the profile and is not pre-defined by the "voting system" (by L-operator) independently of the voters' "opinions".

2°. Monotonicity. Let $\{C_i(\cdot)\}$ and $\{\tilde{C}_i(\cdot)\}$ be two profiles and for some $X \in \mathcal{A}$ holds $x \in C^*(X)$. Let an inclusion $V(x, X; \{C_i(\cdot)\}) \subseteq V(x, X; \{\tilde{C}_i(\cdot)\})$ hold. Then $x \in \tilde{C}^*(X)$, where $\tilde{C}^*(\cdot) \in F(\{\tilde{C}_i(\cdot)\})$.

The monotonicity condition guarantees that if all voters who have elected a given variant x in the choice from X maintain their "opinions", and some voters who have formerly rejected this variant in choice change possibly their "judgements" and now elect it, then this variant is included in the new collective choice[6], if this variant x have been included previously.

The monotonicity condition is a reinforcement of locality condition[7].

3°. Neutrality to variants. Let $\{C_i'(\cdot)\}$ and $\{C_i''(\cdot)\}$ be profiles. If for every X' X'', x', x'' such that $x' \in X'$, $x'' \in X''$ holds $x' \in C_i'(X')$ iff $x'' \in C_i''(X'')$ for all i, then $x' \in C^{*\prime}(X')$ iff $x'' \in C^{*\prime\prime}(X'')$, where $C^{*\prime}(\cdot) = F(\{C_i'(\cdot)\})$ and $C^{*\prime\prime}(\cdot) = F(\{C_i''(\cdot)\})$.

6) Voters' "opinions" with respect to othervariants y from set $X \setminus \{x\}$ may change arbitrarily.

7) If an operator F which is not assumed to be local satisfies the monotonicity condition, then F obeys the locality condition wittingly.

The condition 3° is essentially a combination of the two following conditions.

3°-a. Independence of content (on X). Let $\{C_i'(\cdot)\}$ and $\{C_i''(\cdot)\}$ be profiles. If for every X', X'' and some x such that $x \in X', x \in X''$ hold $x \in C_i'(X')$ iff $x \in C_i''(X'')$ for all i, then $x \in C^{*\prime}(X')$ iff $x \in C^{*\prime\prime}(X'')$

3°-b. Independence of variant (on x). Let $\{C_i'(\cdot)\}$ and $\{C_i''(\cdot)\}$ be profiles. If for every x', x'' and some X such that $x' \in X$, $x'' \in X$ holds $x' \in C_i'(X)$ iff $x' \in C_i''(X)$ for all i, then $x' \in C^{*\prime}(X)$ iff $x'' \in C^{*\prime\prime}(X)$

Each of the two conditions 3°-a and 3°-b is a reinforcement of the locality condition.

4°. Neutrality to voters. Let η be a one-to-one mapping (a bijection) of N on N, where $N = \{1, \dots, n\}$ is a set of voters' indices. Then $F(\{C_i(\cdot)\}) = F(\{C_{\eta(i)}(\cdot)\})$.

The conditions 3° and 4° guarantee the voting system to predict "the same attitude" to different variants (and to their sets X) and to different voters accordingly.

5°. Unanimity condition (Pareto principle). If $x \in C_i(X)$ for all $i \in N$, then $x \in C^*(X)$ ("positive unanimity", denoted as - 5°_+); if $x \notin C_i(X)$ for all $i \in N$, then $x \notin C^*(X)$ ("negative unanimity", denoted as - 5°_-).

Introduce now a list of boolean functions' properties I-V which are equivalent to the corresponding L-operators' properties specified by the conditions 1°-5°: $\forall x, X : x \in X \in \mathcal{A}$

I⁺. $f_{(x,X)}(\ell) \not\equiv 0$; I⁻. $f_{(x,X)}(\ell) \not\equiv 1$;

II. a function $f_{(x,X)}(\ell)$ is monotonic;

III. $f_{(x,X)}(\ell) = f_{(x',X')}(\ell)\ \forall x', X' : x' \in X' \in \mathcal{A}$, i.e. $f_{(x,X)}(\ell) \equiv f(\ell)$ (f is independent of x and X);

III-a. $f_{(x,X)}(\ell) \equiv f_{(x)}(\ell)$ (f is independent of X);

III-b. $f_{(x,X)}(\ell) \equiv f_{(X)}(\ell)$ (f is independent of x);

IV. A function f is symmetric on its arguments;

V $^+$. $f_{(x,X)}(1,\dots,1)=1$; V$^-$. $f_{(x,X)}(0,\dots,0)=0$

It may be easily shown by some examples, that there exist L-operators which satisfy or do not satisfy each of these characteristic conditions. Hence the classes of L-operators (the classes in the set $\mathcal{L}$) are isolated with these conditions.

The classes in $\mathcal{L}$ which are isolated with these conditions 1°–5° are denoted as $\Lambda^{1°},\dots,\Lambda^{5°}$.

The class in $\mathcal{L}$ in which the conditions 1° and 2° are observed is further referred to as Basic Region; the class in which the conditions 1°,2° and 3° are observed is referred to as Central Region, and the class in which the conditions 1°,2°,3° and 4° are observed is referred to as Symmetrically – Central Region. These Regions are denoted as Λ^{BR}, Λ^{CR} and Λ^{SCR}, accordingly.

Apart from these Regions the following four special operators are picked out in $\mathcal{L}$ – two "trivial" and two "extreme" operators.

"Trivial" operators: 1) operator generating for all $X\in\mathcal{A}$ $C^*(X)=X$ independently of the profile $\{C_i(\cdot)\}$; 2) operator generating for all $X\in\mathcal{A}$ $C^*(X)=\emptyset$ independently of the profile $\{C_i(\cdot)\}$.

Extreme operators: 1) the operator "unanimity" which generates a function $C^*(\cdot)$ in the following way: $\forall X\in\mathcal{A}$

$$C^*(X)=\bigcap_{i\in N} C_i(X),$$

2) the operator "at least one vote ay", which generates

$$C^*(X)=\bigcup_{i\in N} C_i(X), \ \forall X\in\mathcal{A}$$

The four operators are hereafter denoted as $1, 0, \cup$ and $\vee$.

Consider now a list-form representation of L-operators and the subsets of the set Φ (which is equal to $\mathcal{L}$), which are picked out with the following conditions: $\forall X \in \mathcal{A}$ and $\forall x \in X$:

I^+. $\Omega(x,X) \neq \emptyset$;

I^-. $\Omega(x,X) \neq 2^N$;

II. Let $\omega(x,X) \in \Omega(x,X)$. Then for every $\tilde{\omega}$ such that $\omega(x,X) \subseteq \tilde{\omega} \subseteq N$ holds $\tilde{\omega} \in \Omega(x,X)$;

III. $\Omega(x,X) = \Omega(x',X')$ $\forall x', X' : x \in X' \in \mathcal{A}$;

III-a. $\Omega(x,X) = \Omega(x,X')$ $\forall X' \in \mathcal{A}$;

III-b. $\Omega(x,X) = \Omega(x',X)$ $\forall x' \in X$;

IV. Let $\Omega(x,X) = \{\omega_1^{(x,X)}, \dots, \omega_{S(x,X)}^{(x,X)}\}$ and η be a one-to-one mapping of N on N. Denote $\eta(\omega_i^{(x,X)}) = \{\eta(j) \mid j \in \omega_i^{(x,X)}\}$ and $\eta(\Omega(x,X)) = \{\eta(\omega_1^{(x,X)}), \dots, \eta(\omega_{S(x,X)}^{(x,X)})\}$. Then[8)] $\eta(\Omega(x,X)) = \Omega(x,X)$;

V^+. $N \in \Omega(x,X)$;

V^-. $\emptyset \notin \Omega(x,X)$.

Theorem 1. Classes in the set $\mathcal{L}$ picked out with the following characteristic condition 1°_+, 1°_-, 2°, 3°, 3°-a, 3°-b, 4°, 5°_+, 5°_- coincide with the classes of the set Φ, which are isolated with

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8) A list-form representation of L-operators which satisfy the condition 4° of neutrality to voters may be as well as by condition IV characterised by the following equivalent condition

IV'. $\forall X, x : x \in X \in \mathcal{A}$, $\omega \in \Omega(x,X)$, $\mathrm{card}(\omega') = \mathrm{card}(\omega) \Rightarrow \omega' \in \Omega(x,X)$

conditions I^+, I^-, II, III, III-a, III-b, IV, V^+ and V^-, accordingly.

The proof of Theorem 1 is immediately seen to follow from comparison of definitions 1 and 2 ~~and~~ from conditions 1°-5° and I-V considered above.

If a class of L-operators for which the lists satisfy some of conditions I-V is denoted as $\Lambda^{\overline{I}}, \ldots, \Lambda^{\overline{V}}$ then the formulation of theorem 1 may be written down as $\Lambda^{1^\circ_+} = \Lambda^{\overline{I}^+}, \ldots, \Lambda^{5^\circ_-} = \Lambda^{\overline{V}^-}$

It is easily seen that for two classes isolated with two conditions i and j the equality $\Lambda^{i \cap j} = \Lambda^{i} \cap \Lambda^{j}$ holds.

If the conditions I-V of L-operator F given in the list-form are re-formulated as the corresponding properties of the boolean function $f_{(x, X)}$ one can be convinced of the coincidence of those with the conditions $I^+, \ldots, V^-$ of the boolean functions, which have been established when such re-formulation of conditions $1^\circ_+, \ldots, 5^\circ_-$ have been performed. This remark may be regarded as the undirect proof of Theorem 1.

Theorem 1 explains the "arrangement" of L-operators which meet the conditions 1°-5° and in particular, the conditions 3° and 4°.

When for example the condition 3°-a (independence of the context) is observed none of the lists $\Omega(x, X)$ depends on the set X: $\Omega(x, X) \equiv \Omega(x)$ for all X, i.e. L-operator is in the case defined with totality of m lists ($m = card(A)$, where A is a set of all variants) of the type $\Omega(x_1), \ldots, \Omega(x_m)$. The class $\Lambda^{3^\circ b}$ of L-operators satisfying the condition 3°-b of independence of variants, consists of operators defined by $2^m - 1$ lists $\Omega(X_1), \ldots, \Omega(X_{2^m-1})$ for all non-empty subsets X of A. Finally, every L-operator for which the neutrality to variants condition holds, may be represented by the unique list $\Omega = \{\omega_1, \ldots, \omega_s\}$ for all x and X. In particular, when

$\Omega = 2^N$ the list defines the L-operator 1. On the other hand the case $\Omega = \phi$ represents the L-operator 0.

Consider now mutual relations between L-operators' classes which meet the conditions 1°–5°, i.e. establish the non-empty intersections of the classes. These relations are conveniently represented using the Euler-Venn's diagrams.

Theorem 2. Let a profile $\{C_i(\cdot)\}$ consist of arbitrary choice functions and $C^*(\cdot)$ may be an arbitrary one too[9]. Then the classes $\Lambda^{1^\circ_+}, \ldots, \Lambda^{5^\circ_-}$ are mutually related in the set $\mathcal{L}$ in such way as it is shown[10] on Fig.1 and 2.

Proof. There are 128 conjunctions of the conditions $1^\circ_+, \ldots, 5^\circ_-$ and their negations. The implications considered below hold. Some of these implications are obvious, and the other ones are easily shown to follow from the list-form representation of L-operators. It is obvious that $5^\circ_+ \Rightarrow 1^\circ_+$, $5^\circ_- \Rightarrow 1^\circ_-$, $\bar{1}^\circ_+ \cap 3^\circ \Rightarrow 2^\circ$, $\bar{1}^\circ_- \cap 3^\circ \Rightarrow 2^\circ$, $\bar{1}^\circ_- \cap 3^\circ \Rightarrow 5^\circ_+$, $\bar{1}^\circ_+ \cap 3^\circ \Rightarrow 5^\circ_-$, $1^\circ_+ \cap 2^\circ \Rightarrow 5^\circ_+$, $1^\circ_- \cap 2^\circ \Rightarrow 5^\circ_-$. Furthermore, $\bar{1}^\circ_+ \cap \bar{1}^\circ_- \cap 3^\circ = \phi$. Indeed, $\bar{1}^\circ_- \cap 3^\circ$ implies $\Omega(x,X) = \phi$, $\forall x, X : x \in X \in \mathcal{A}$. The intersection $\bar{1}^\circ_+ \cap 3^\circ$ implies $\Omega(x,X) = 2^N, \forall x, X$. Finally, $\bar{1}^\circ_+ \cap 3^\circ \Rightarrow 4^\circ$ and $\bar{1}^\circ_- \cap 3^\circ \Rightarrow 4^\circ$. Indeed, the conjunction $\bar{1}^\circ_+ \cap 3^\circ$ defines a list-form representation for which $\Omega(x,X) = \phi$, $\forall x, X$, and the conjunction $\bar{1}^\circ_- \cap 3^\circ$ defines the list $\Omega(x,X) = 2^N$, $\forall x, X$. Obviously, no such totalities of groups are changed when any one-to-one mapping of N on N is performed. Only 38 conjunctions from possible 128 ones maintain the above properties. The other 90 conjunctions isolate empty classes, which do not include any L-operators.

Those 38 conjunctions of the conditions $1^\circ_+, \ldots, 5^\circ_-$ and their

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9) The cases when functions $C_i(\cdot)$ as well as $C^*(\cdot)$ are somehow imposed are studied in section 5.

10) Authors ask to excuse them for using figures in the statement of Theorem 2 and other theorems. Unfortunately, any other statement would have been cumbersome and non-illustrative.

negations pick out non-empty classes of $\llcorner$-operators:

1: $\overline{1^\circ_+} \cap \overline{1^\circ_-} \cap \overline{2^\circ} \cap \overline{3^\circ} \cap \overline{4^\circ} \cap \overline{5^\circ_+} \cap \overline{5^\circ_-}$

2: $1^\circ_+ \cap \overline{1^\circ_-} \cap \overline{2^\circ} \cap \overline{3^\circ} \cap \overline{4^\circ} \cap \overline{5^\circ_+} \cap 5^\circ_-$

3: $\overline{1^\circ_+} \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap \overline{4^\circ} \cap \overline{5^\circ_+} \cap \overline{5^\circ_-}$

4: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap 3^\circ \cap \overline{4^\circ} \cap 5^\circ_+ \cap \overline{5^\circ_-}$

5: $\overline{1^\circ_+} \cap \overline{1^\circ_-} \cap 2^\circ \cap \overline{3^\circ} \cap \overline{4^\circ} \cap \overline{5^\circ_+} \cap \overline{5^\circ_-}$

6: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap 3^\circ \cap \overline{4^\circ} \cap \overline{5^\circ_+} \cap \overline{5^\circ_-}$

7: $\overline{1^\circ_+} \cap \overline{1^\circ_-} \cap \overline{2^\circ} \cap \overline{3^\circ} \cap 4^\circ \cap \overline{5^\circ_+} \cap \overline{5^\circ_-}$

8: $1^\circ_+ \cap \overline{1^\circ_-} \cap \overline{2^\circ} \cap \overline{3^\circ} \cap 4^\circ \cap \overline{5^\circ_+} \cap \overline{5^\circ_-}$

9: $\overline{1^\circ_+} \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap 4^\circ \cap \overline{5^\circ_+} \cap \overline{5^\circ_-}$

10: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap 4^\circ \cap \overline{5^\circ_+} \cap \overline{5^\circ_-}$

11: $\overline{1^\circ_+} \cap \overline{1^\circ_-} \cap 2^\circ \cap \overline{3^\circ} \cap 4^\circ \cap \overline{5^\circ_+} \cap \overline{5^\circ_-}$

12: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap 3^\circ \cap 4^\circ \cap \overline{5^\circ_+} \cap \overline{5^\circ_-}$

13: $1^\circ_+ \cap \overline{1^\circ_-} \cap \overline{2^\circ} \cap \overline{3^\circ} \cap \overline{4^\circ} \cap 5^\circ_+ \cap \overline{5^\circ_-}$

14: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap \overline{4^\circ} \cap 5^\circ_+ \cap \overline{5^\circ_-}$

15: $1^\circ_+ \cap \overline{1^\circ_-} \cap 2^\circ \cap \overline{3^\circ} \cap \overline{4^\circ} \cap 5^\circ_+ \cap \overline{5^\circ_-}$

16: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap 3^\circ \cap \overline{4^\circ} \cap 5^\circ_+ \cap \overline{5^\circ_-}$

17: $1^\circ_+ \cap \overline{1^\circ_-} \cap \overline{2^\circ} \cap \overline{3^\circ} \cap 4^\circ \cap 5^\circ_+ \cap \overline{5^\circ_-}$

18: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap 4^\circ \cap 5^\circ_+ \cap \overline{5^\circ_-}$

19: $1^\circ_+ \cap \overline{1^\circ_-} \cap 2^\circ \cap \overline{3^\circ} \cap 4^\circ \cap 5^\circ_+ \cap \overline{5^\circ_-}$

20: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap 3^\circ \cap 4^\circ \cap 5^\circ_+ \cap \overline{5^\circ_-}$

21: $1^\circ_+ \cap \overline{1^\circ_-} \cap 2^\circ \cap 3^\circ \cap 4^\circ \cap 5^\circ_+ \cap \overline{5^\circ_-}$

22: $\overline{1^\circ_+} \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap \overline{4^\circ} \cap 5^\circ_+ \cap 5^\circ_-$

23: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap \overline{4^\circ} \cap \overline{5^\circ_+} \cap 5^\circ_-$

24: $\overline{1^\circ_+} \cap 1^\circ_- \cap 2^\circ \cap \overline{3^\circ} \cap \overline{4^\circ} \cap \overline{5^\circ_+} \cap 5^\circ_-$

25: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap 3^\circ \cap \overline{4^\circ} \cap \overline{5^\circ_+} \cap 5^\circ_-$

26: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap 4^\circ \cap \overline{5^\circ_+} \cap 5^\circ_-$

27: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap 4^\circ \cap \overline{5^\circ_+} \cap 5^\circ_-$

28: $\overline{1^\circ_+} \cap 1^\circ_- \cap 2^\circ \cap \overline{3^\circ} \cap 4^\circ \cap \overline{5^\circ_+} \cap 5^\circ_-$

29: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap 3^\circ \cap 4^\circ \cap \overline{5^\circ_+} \cap 5^\circ_-$

30: $\overline{1^\circ_+} \cap 1^\circ_- \cap 2^\circ \cap 3^\circ \cap 4^\circ \cap \overline{5^\circ_+} \cap 5^\circ_-$

31: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap \overline{4^\circ} \cap 5^\circ_+ \cap 5^\circ_-$

32: $1^\circ_+ \cap 1^\circ_- \cap 2^\circ \cap \overline{3^\circ} \cap \overline{4^\circ} \cap 5^\circ_+ \cap 5^\circ_-$

33: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap 3^\circ \cap \overline{4^\circ} \cap 5^\circ_+ \cap 5^\circ_-$

34: $1^\circ_+ \cap 1^\circ_- \cap 2^\circ \cap 3^\circ \cap \overline{4^\circ} \cap 5^\circ_+ \cap 5^\circ_-$

35: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap \overline{3^\circ} \cap 4^\circ \cap 5^\circ_+ \cap 5^\circ_-$

36: $1^\circ_+ \cap 1^\circ_- \cap 2^\circ \cap \overline{3^\circ} \cap 4^\circ \cap 5^\circ_+ \cap 5^\circ_-$

37: $1^\circ_+ \cap 1^\circ_- \cap \overline{2^\circ} \cap 3^\circ \cap 4^\circ \cap 5^\circ_+ \cap 5^\circ_-$

38: $1^\circ_+ \cap 1^\circ_- \cap 2^\circ \cap 3^\circ \cap 4^\circ \cap 5^\circ_+ \cap 5^\circ_-$

Examples of $\llcorner$-operators for 26 conjunctions with their numbers given are shown in Table 1. Each of this conjunctions corresponds to its column which consists of four rows indicating lists for the following "variant-set" pairs: (x, A), $(x, \{x, y\})$, $(x, \{x, z\})$ and $(x, \{x\})$, when $A = \{x, y, z\}$ and $\mathcal{N} = 1, 2, 3$. For other pairs of type (t, X), where $t \neq x$, $X \in \mathcal{A}$, the lists may be exactly like those for the corresponding pairs of the type (x, X). Such lists are easily checked to satisfy the corresponding conjunction and, hence, the class defined by this conjunction is not empty.

Examples of $\lfloor$-operators for the conjunctions with the numbers of 6,12,16,20,21,25,29,30,33,34,37 and 38 (which are not shown in Table 1) may be obtained by the examples for the conjunctions with the numbers of 4,10,14,18,19,23,27,28,31,32,35, and 36 accordingly, assuming that 3° is observed and, the list Ω is assumed to be equal to $\Omega(x, A)$.

$\lfloor$-operator with $\Omega(x,X)=\phi, \forall x,X: x\in X\in\mathcal{A}$, i.e. the operator O is defined by conjunction 30. Conjunction 21 defines an $\lfloor$-operator with $\Omega(x,X)=2^N \ \forall x,X: x\in X\in\mathcal{A}$, i.e. operator 1.

The fact that 38 conjunctions are non-empty and the other 90 conjunctions are empty is illustrated on Figs. 1 and 2.

Figs 1 and 2 imply that the four conditions 1°_+, 1°_-, 2° and 4° are jointly independent, i.e. there exist $\lfloor$-operators, which satisfy each of 16 possible conjunctions of these conditions and their negations; class $\Lambda^{5^\circ_+}$ lies within class $\Lambda^{1^\circ_+}$ and the intersection of classes $\Lambda^{5^\circ_+}$ and Λ^{2° coincides with the intersection of classes $\Lambda^{1^\circ_+}$ and Λ^{2°; class $\Lambda^{5^\circ_-}$ is located within class $\Lambda^{1^\circ_-}$ and the class $\Lambda^{1^\circ_- \cap 2^\circ}$ coincides with $\Lambda^{5^\circ_- \cap 2^\circ}$; class Λ^{3° is put strictly within the intersection $\Lambda^{1^\circ_+ \cap 1^\circ_-}$ except for two trivial operators 1 and O: These operators are located in the classes $\Lambda^{1^\circ_+ \cap 2^\circ}$ and $\Lambda^{1^\circ_- \cap 2^\circ}$ accordingly.

In Fig.1 shows relation between the conditions 1°,2°3° and 5°. The Basic Region Λ^{BR}, in which the conditions 1° and 2° are simultaneously satisfied, shaded. On Fig.2 with condition 5° instead of condition 4° shown, the Central Region Λ^{CR} also shades. In this Central Region the conditions 1°,2° and 3° (or, equivalently, 1°_+, 2°, 3° and 5°_-, or 1°_-, 2°, 3° and 5°_+, or 2°,3°,5°_+ and 5°_-) hold.

4. Mechanisms which generate legal operators

Consider now mechanisms generating L-operators. Being intuitively apparent the notion can be formulated as some deterministic rule according to which variant $x \in X$ is either included or not into the choice $C^*(X)$ depending on the inclusion of x into choice $C_i(X)$ of some voters. One can say that the enumeration of such voters just define the rule.

The formulae defining operators U and V (see sect. 3) may be considered as examples of such mechanisms.

Using Ω - as well as $\mathcal{E}$-representation introduce now some mechanisms which generate L-operators from Basic Region $\Lambda^{BR} = \Lambda^{1^\circ \cap 2^\circ}$, but first, we define the notion of a basic list for given list $\Omega(x,X)$

<u>Definition 3.</u> The totality of groups $\{\omega^{(x,X)}\}$ from the given list $\Omega(x,X)$ is referred to as a basic list for $\Omega(x,X)$ and denoted as $\Omega^B(x,X)$ $(\Omega^B(x,X) \subseteq \Omega(x,X))$ if

a) every superset of each of these groups $\omega^{(x,X)}$ belongs to $\Omega(x,X)$, i.e. $\forall \tilde{\omega} \;\; \tilde{\omega} \supseteq \omega^{(x,X)} \Rightarrow \tilde{\omega} \in \Omega(x,X)$,

b) no proper subset of any of these groups $\omega^{(x,X)}$ belongs to $\Omega(x,X)$, i.e. $\forall \tilde{\omega} : \tilde{\omega} \subset \omega^{(x,X)} \Rightarrow \tilde{\omega} \notin \Omega(x,X)$

For the L-operators, which satisfy the condition 2° (in the case the list $\Omega(x,X)$ satisfies condition II) it is obvious that the basic list $\Omega^B(x,X)$ is the totality of groups $\omega^{(x,X)}$ from $\Omega(x,X)$ minimal in set-inclusion (i.e. the list $\Omega^B(x,X)$ is defined only by condition b), and, therefore, given the basic list $\Omega^B(x,X)$ the whole list $\Omega(x,X)$ for such operators is defined in a unique way.

<u>Definition 4.</u> L-operator F is referred to as a "union of intersection" operator (marking UI), if $\Omega^B(x,X) = \{\omega_1^{(x,X)}, \ldots, \omega_{S(x,X)}^{(x,X)}\}$, where $\omega_i^{(x,X)} \subseteq N$ $(\forall i = 1, \ldots, S(x,X);\ S(x,X) \neq 0)$ are arbitrary nonempty subsets of N .

Using the boolean representation $f_{(x,X)}(\ell_1,\dots \ell_n)$ of L-operator F, operator "union of intersection" may be written down as

$$f_{(x,X)}(\ell_1,\dots,\ell_n) = \bigvee_{j=1}^{S(x,X)} \bigwedge_{i\in\omega_j^{(x,X)}} \ell_i ;$$

i.e. the variant x is included in the choice $C^*(X)$, if all voters from at least one group $\omega_j^{(x,X)}$ include x in the choice $C_i(X)$

Using $\mathcal{E}$-representation of operators F the operator UI may be represented in another form, namely

$$f_{(x,X)}(\ell_1,\dots,\ell_n) = \bigwedge_{j=1}^{t(x,X)} \bigvee_{i\in\mathcal{E}_j^{(x,X)}} \ell_i$$

Such an operator is referred to as an "intersection of unions" operator and marked as IU.

The classes of operators UI and IU are hereafter denoted as $\Lambda^{UI(x,X)}$ and $\Lambda^{IU(x,X)}$ accordingly, where the presence of the pair (x,X) in parenthesis emphasize the fact that the totalities $\{\omega_i^{(x,X)}\}_1^{S(x,X)}$ are defined for every pair (x,X) separately.

From the definitions regarded above it is obvious that

(1) $$\Lambda^{BR} = \Lambda^{UI(x,X)} = \Lambda^{IU(x,X)}$$

Now some particular cases of operators "union of intersection" and "intersection of unions" will be analyzed.

<u>Definition 5.</u> L-operator F is referred to as a "partial intersection operator" (marking - PI) if $\Omega^B(x,X) = \{\omega_1^{(x,X)}\}$ In the case, apparently

$$f_{(x,X)}(\ell_1,\dots,\ell_n) = \bigwedge_{i\in\omega_1^{(x,X)}} \ell_i$$

<u>Definition 6.</u> $\mathbb{L}$-operator F is referred to as a "partial union" operator (marking – PU) if $\Omega^B(x,X)=\{\omega_1^{(x,X)},\dots,\omega_{t(x,X)}^{(x,X)}\}$, card $(\omega_j^{(x,X)})=1$ $\forall j = 1,\dots, t(x,X)$.

Basic $\mathcal{E}$-list for operator PU is represented as $\mathcal{E}^B(x,X)= \{\mathcal{E}_1^{(x,X)}\}$. Hence

$$f_{(x,X)}(\ell_1,\dots,\ell_n)=\bigvee_{i\in\mathcal{E}_1^{(x,X)}}\ell_i$$

<u>Definition 7.</u> $\mathbb{L}$-operator F is referred to as a "decisive voter" (dictator) operator (marking – $\mathcal{D}V$), if $\Omega^B(x,X)=\{\omega_1^{(x,X)}\}$, card ($\omega_1^{(x,X)}$) = 1, i.e. $\omega_1^{(x,X)}=\{i^*\}$

In the case

$$f_{(x,X)}(\ell_1,\dots,\ell_n)=\ell_{i*}$$

The classes of operators PI , PU and $\mathcal{D}V$ are denoted as $\Lambda^{PI(x,X)}$, $\Lambda^{PU(x,X)}$ and $\Lambda^{\mathcal{D}V(x,X)}$, accordingly.

<u>Definition 8.</u> $\mathbb{L}$-operator F is referred to as a " $k(x,X)$-plurality" operator (marking kP), if $\Omega^B(x,X)=\{\omega_1^{(x,X)},\dots, \omega_{S(x,X)}^{(x,X)}\}$, card $(\omega_j^{(x,X)})=k(x,X)$ $\forall j=1,\dots,S(x,X)$ and $S(x,X)=\binom{k(x,X)}{n}$, where $\binom{k(x,X)}{n}$ is the number of combinations of n things $k(x,X)$ at a time.

For " $k(x,X)$-plurality" operators holds

$$f_{(x,X)}(\ell_1,\dots,\ell_n)=\bigvee_{j=1}^{S(x,X)}\bigwedge_{i\in\omega_j^{(x,X)}}\ell_i$$

The class of all kP-operators is denoted as $\Lambda^{kP(x,X)}$. When $k(x,X)\geqslant\left[\frac{n}{2}\right]+1$ an operator " $k(x,X)$ —plurality" may be understood as the "plurality rule" (see, e.g., Richelson (1978b)) defined for each fixed x

and[11] X .

Theorem 3. The intersection of the class of all PI-operators with the class of all PU-operators coincides with the class of all $\mathcal{D}V$-operators. The intersection of the class of all kP-operators with the class of operators: a) "decisive voter", b) "partial intersection" and c) "partial union" in the case a) is empty, in the case b) consists of the unique operator U, in the case c) consists of the unique operator V, i.e.

$$\Lambda^{PI(x,X)} \cap \Lambda^{PU(x,X)} = \Lambda^{\mathcal{D}V(x,X)}, \quad \Lambda^{kP(x,X)} \cap \Lambda^{\mathcal{D}V(x,X)} = \phi,$$

$$\Lambda^{kP(x,X)} \cap \Lambda^{PI(x,X)} = U; \quad \Lambda^{kP(x,X)} \cap \Lambda^{PU(x,X)} = V$$

Proof. Let us proove the first statement of Theorem 3. For PI-operator the list $\Omega^B(x,X)$ is defined with unique group $\omega_1^{(x,X)}$ and for PU-operator every group $\omega^{(x,X)}$ from $\Omega^B(x,X)$ has the cardinality 1. It immediately proves the statement.

a) For kP-operator when $k = 1$ the list $\Omega^B(x,X)$ has to consist of all one-element subsets of N, and for operator $\mathcal{D}V$ the list $\Omega^B(x,X)$ consists of the unique index i^*.

b) An operator PI is defined by the list $\Omega^B(x,X) = \{\omega_1^{(x,X)}\}$. If this operator is a kP-operator too, then $\omega_1^{(x,X)} = N, \forall x, X$: $x \in X \in A$, i.e $\Omega^B(x,X) = \Omega = \{N\}$. That is exactly the list-form representation of an operator U.

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11) When $k < \frac{n}{2}$ the application of the term "plurality" to kP-operator is in some sense conditional. In particular, when $k = 1$ variant x by operator kP has to be included in the choice from X if this variant is included in the choice from X by at least one voter; in other words the operator "$1(x,X)$-plurality" is the operator V. Apparently the operator "$n(x,X)$-plurality" is the operator U.

c) An operator PU is defined by list $\Omega^B(x,X) = \{\omega_1^{(x,X)}, \ldots, \omega_{t(x,X)}^{(x,X)}\}$, when card $(\omega_i^{(x,X)}) = 1$, $\forall i = 1,\ldots, t(x,X)$. If this operator is a kP-operator too, then apparently $\Omega^B(x,X) = \{\{1\}, \ldots, \{n\}\}$, $\forall x, X : x \in X \in \mathcal{A}$. That is exactly the list-form representation of an operator V.

Fig. 3 illustrates the mutual relations between operators' classes considered, which follow from definitions 4–8 and Theorem 3.

Consider now some important subclasses of the operators' classes regarded above. Operator UI is referred to as an independent one, if the boolean function $f(x,X)$ is independent of parameter X, i.e. $f(x,X) \equiv f(x)$ or, in terms of Ω-representation $\Omega(x,X) \equiv \Omega(x)$). The classes of independent operators, which are subclasses of those stated in definitions 4–8, are denoted herein as $\Lambda^{UI(x)}$, $\Lambda^{IU(x)}$, $\Lambda^{PU(x)}$, $\Lambda^{PI(x)}$, $\Lambda^{\oplus V(x)}$ and $\Lambda^{kP(x)}$.

Operator UI is referred to as neutral one if the function $f(x,X)$ is independent of X as well as of x, i.e. $f(x,X) \equiv f$ (in terms of Ω-representation, accordingly, $\Omega(x,X) \equiv \Omega$)

The classes of neutral operators, which correspond to those stated in definitions 4–8, are hereafter denoted as Λ^{UI}, Λ^{IU}, Λ^{PI}, etc.

Theorem 3, the equality (1) and the corresponding definitions are easily seen to imply

<u>Corollary 1.</u> $\Lambda^{BR} \cap \Lambda^{4^\circ} = \Lambda^{kP(x,X)}$

<u>Corollary 2.</u> $\Lambda^{BR} \cap \Lambda^{3^\circ\text{-}a} = \Lambda^{UI(x)} = \Lambda^{IU(x)}$, $\Lambda^{BR} \cap \Lambda^{3^\circ\text{-}a} \cap \Lambda^{4^\circ} = \Lambda^{kP(x)}$

<u>Corollary 3.</u> $\Lambda^{BR} \cap \Lambda^{3^\circ} = \Lambda^{CR} = \Lambda^{IU} = \Lambda^{UI}$, $\Lambda^{CR} \cap \Lambda^{4^\circ} = \Lambda^{SCR} = \Lambda^{kP}$

Using neutral operators allows defining the value $C^*(X)$ of the collective choice function by the values $C_i(X)$ of functions

$C_i(\cdot)$ only with the set-theory operations. Formally, when the "neutral union of intersection" operator is used, then for each value $C^*(X)$ holds

$$C^*(X) = \bigcup_{j=1}^{s} \bigcap_{i \in \omega_j} C_i(X)$$

For neutral operator IU : $C^*(X) = \bigcap_{j=1}^{t} \bigcup_{i \in \mathcal{E}_j} C_i(X)$;

neutral operator PI : $C^*(X) = \bigcap_{i \in \omega_1} C_i(X)$;

neutral operator PU : $C^*(X) = \bigcup_{i \in \mathcal{E}_1} C_i(X)$;

neutral operator $\not\!\!\exists V$: $C^*(X) = C_{i^*}(X)$

neutral ${}_kP$-operator $C^*(X) = \bigcup_{j=1}^{s} \bigcap_{i \in \omega_j} C_i(X)$; where card $(\omega_j) = k$, $\forall j = 1, \ldots, s$, $s = \binom{k}{n}$

Classes of the above operators in the "independent subset" of $\mathcal{I}$ satisfy the relations which have been established by Theorem 3, when corresponding markings are substituted. Furthermore, these classes are mutually related as well as the classes [illegible] and $\Lambda^{IU(x,X)}$ and their subclasses.(see Fig. 3). Obviously, the classes of neutral operators are related in the same way. In particular, the classes Λ^{PI}, Λ^{PU} and $\Lambda^{\not\exists V}$ meet the condition $\Lambda^{PI} \cap \Lambda^{PU} = \Lambda^{\not\exists V}$ and when card$(N) = n$ the class $\Lambda^{\not\exists V}$ consists of exactly n operators (when $i^* = 1, \ldots, n$, accordingly).

Remark. An operator "union of intersections" may be redefined to envolve the cases when for some X either $\Omega(x,X) = \emptyset$, or non-empty totality $\Omega(x,X) = \{\omega_i^{(x,X)}\}_1^{s(x,X)}$ consist of at least one empty set of indices $\omega_i^{(x,X)} = \emptyset$, i.e. $\emptyset \in \Omega(x,X)$. In the first case let $f_{(x,X)}(\ell_1, \ldots, \ell_n) \equiv 0$, and in the second one $f_{(x,X)}(\ell_1, \ldots, \ell_n) \equiv 1$

The operator "intersection of unions" in the analogous cases is re-defined accordingly in the following way: when $\mathcal{E}(x,X) = \emptyset$ let $f_{(x,X)} \equiv 1$, and when $\emptyset \in \mathcal{E}(x,X)$ - let $f_{(x,X)} \equiv 0$ .Notice that the case $\Omega(x,X) = \phi$ for operator UI corresponds to $\mathcal{E}(x,X) \ni \phi$ for operator IU, and, vice-versa, the case $\Omega(x,X) \ni \phi$ corresponds to $\mathcal{E}(x,X) = \phi$.

Such re-defined operators are naturally referred to as trivial ones. When the operators "neutral union of intersections" and "neutral intersection of unions" are regarded, the re-definition fixes the operators 0 and 1. In particular, for the operator "neutral union of intersections" the case $\Omega = \phi$ corresponds to the operator 0, and the case $\Omega \ni \phi$ corresponds to the operator 1. Apparently, the classes of operators "union of intersection" and "intersection of unions" ("independent union of intersection" and "independent intersection of unions", "neutral union of intersection" and "neutral intersection of unions") re-defined in such a way coincides with each other too.

The operators' classes which consist of the corresponding trivial operators is referred to as a "replenished class" of operators and denoted as $\sim$. For instance the replenished class of operators "neutral union of intersections" is easily seen to be a union of the trivial operators 0 and 1, and the class "neutral union of intersection", i.e. $\Lambda^{\widetilde{UI}} = \Lambda^{UI} \cup \{0\} \cup \{1\}$

The equality (1) may be generalised when the re-definition of the classes in question is performed in the following way

$$(2) \quad \Lambda^{2^\circ} = \Lambda^{\widetilde{UI}(x,X)} = \Lambda^{\widetilde{IU}(x,X)}$$

Thus the equality (2) establishes the form of operators which satisfy the monotonicity condition, and therefore from (2) some statements about the form of operators from intersections $\Lambda^{2^\circ} \cap \Lambda^{3^\circ a}$, $\Lambda^{2^\circ} \cap \Lambda^{3^\circ}$, $\Lambda^{2^\circ} \cap \Lambda^{4^\circ}$ ....., $\Lambda^{2^\circ} \cap \Lambda^{3^\circ} \cap \Lambda^{4^\circ}$ follow as simple corollaries.

5. Closedness of characteristic domains in the choice-functional space

In previous sections we assumed that a profile $\{C_i(\cdot)\}$ may consist of and L-operator F may generate any functions from $\mathcal{C}$, i.e. the domain of operator F is n-tuple direct product $\mathcal{C} \times \mathcal{C} \times \ldots \times \mathcal{C}$ and the range of operator F is $\mathcal{C}$. It is however interesting to consider an operator F with its arguments - functions $C_i(\cdot)$ - belong not to the entire space $\mathcal{C}$, but to some restricted domain in this space. Such domain may be named the setting domain of operator F [12] and denoted as Q_S ($Q_S \subset \mathcal{C}$.

Moreover, it is interesting to restrict the range of the operator F to some predetermined domain $Q_r \subset \mathcal{C}$. In this case we will assume that $Q_S = Q_r = Q$, i.e. choice functions which are acceptable as individual choice functions are acceptable as collective ones as well and vice-versa. This situation is naturally interpreted as closedness of the domain Q to operator F.

Definition 9. The domain of choice functions $Q \subset \mathcal{C}$ is said to be closed relative to operator $F \in \mathcal{L}$ if $F(\{C_i(\cdot)\}) \in Q$ holds for any profile $\{C_i(\cdot)\}$ comprising the functions $C_i(\cdot) \in Q$, the domain $Q \in \mathcal{C}$ is treated as "closed relative to the class of operators $\mathfrak{F} \subseteq \mathcal{L}$" if it is closed relative to every operator $F \in \mathfrak{F}$. Such class $\mathfrak{F}$ will be called a "class of operator closedness" for $Q \subseteq \mathcal{C}$. The largest, in set-inclusion class Λ of operator closedness for $Q \subset \mathcal{C}$ (i.e. the class Λ composed of all operators F relative to which the given domain Q

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12) In this case the domain of operator F is n-tuple direct product $Q_S \times Q_S \times \ldots \times Q_S$ against n-tuple direct product $\mathcal{C} \times \ldots \times \mathcal{C}$ in a general case.

is closed) will be called a complete class of operator closedness for $\mathcal{Q}$ and denoted as $\Lambda_{\mathcal{Q}}$.

The following lemma holds

Lemma 1. Let $\Lambda_{\mathcal{Q}'}$ and $\Lambda_{\mathcal{Q}''}$ be complete classes of operator closedness for $\mathcal{Q}'$ and $\mathcal{Q}''$. Then for complete class of operator closedness for intersection $\mathcal{Q}' \cap \mathcal{Q}''$ the inclusion $\Lambda_{\mathcal{Q}'} \cap \Lambda_{\mathcal{Q}''} \subseteq \Lambda_{\mathcal{Q}' \cap \mathcal{Q}''}$ holds.

Proof. Suppose $F \in \Lambda_{\mathcal{Q}'} \cap \Lambda_{\mathcal{Q}''}$. Consider an arbitrary profile $\{C_i(\cdot)\}$ such that $C_i(\cdot) \in \mathcal{Q}' \cap \mathcal{Q}''$ $\forall i$. For $C_i(\cdot) \in \mathcal{Q}'$ and $C_i(\cdot) \in \mathcal{Q}''$ it follows from closedness of $\mathcal{Q}'$ and $\mathcal{Q}''$ relative to F that $F(\{C_i(\cdot)\}) \in \mathcal{Q}'$ and $F(\{C_i(\cdot)\}) \in \mathcal{Q}''$ and just $F(\{C_i(\cdot)\}) \in \mathcal{Q}' \cap \mathcal{Q}''$. This correlation means exactly the closedness of domain $\mathcal{Q}' \cap \mathcal{Q}''$ relative to F, i.e. $F \in \Lambda_{\mathcal{Q}' \cap \mathcal{Q}''}$ ■

For the domains in $\mathcal{C}$ considered below more rigid result than established in Lemma 1 may be stated, namely, the inclusion is transformed into equality $\Lambda_{\mathcal{Q}'} \cap \Lambda_{\mathcal{Q}''} = \Lambda_{\mathcal{Q}' \cap \mathcal{Q}''}$. This result will be established after the proof of Theorem 4.

Let us now introduce some characteristic properties of choice functions, which isolate some domains in space $\mathcal{C}$.

Definition 10. Following[13] to Aizerman and Malishevsky (1981) a function $C_i(\cdot) \in \mathcal{C}$ is said to satisfy the condition of

- H (Heritage): $\forall X_1, X_2 : X_1 \subseteq X_2 \Rightarrow C(X_1) \supseteq C(X_2) \cap X_1$;
- C (Concordance): $\forall X_1, X_2 \Rightarrow C(X_1) \cap C(X_2) \subseteq C(X_1 \cup X_2)$;

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13) The papers in which the conditions H, C, O and K or their analogues are introduced are cited in Aizerman and Malishevski (1981).

– O (Independence of rejecting the outcast variants): $\forall X_1, X_2$: $X_1 \subseteq X_2 \setminus C(X_2) \Rightarrow C(X_2 \setminus X_1) = C(X_2)$;

– K (Constancy): $\forall X_1, X_2 : X_1 \subseteq X_2\ , X_1 \cap C(X_2) \neq \phi \Rightarrow$
$\Rightarrow C(X_1) = C(X_2) \cap X_1$.

Sen (1973)(see also Aizerman and Malishevski (1981)) show that in the space of all non-empty choice functions the condition K and intersections $H \cap C$ and $H \cap C \cap O$ isolate functions, which are generated by pair-dominant choice on an arbitrary relation of the weak order, acyclic relation and strict partial order, resp. In space $\mathcal{C}$ of all choice functions the intersection $H \cap C$ isolates functions, which are generated by the pair-dominant choice on an arbitrary binary relation. Besides, the condition K and intersection $H \cap C \cap O$ fix the functions which are generated by maximizing choice with any one scalar criterion and n-tuple of criteria, resp.

The domain in $\mathcal{C}$ which consist of functions meeting the conditions H,C,O and K will be denoted by the same letters.

We will now analyze conditions ensuring the closedness of fundamental domains H,C and O and their intersections. The complete classes of operator closedness for these domains are denoted as Λ_H, Λ_C, Λ_O, $\Lambda_{H \cap C}$, etc.

Let us introduce the second group of conditions in list terms.

VI. $\forall X\ \ X', x : x \in X' \subset X \in \mathcal{A},\ \omega \in \Omega(x, X) \Rightarrow (\forall \tilde{\omega} : \omega \subseteq \tilde{\omega} \subseteq N \Rightarrow \tilde{\omega} \in \Omega(x, X'))$;

VII. $X', X'', x, \omega', \omega'', \tilde{\omega} : X' \nsubseteq X'', X'' \nsubseteq X';\ x \in X' \in \mathcal{A},\ x \in X'' \in \mathcal{A}$
$\omega' \in \Omega(x, X'),\ \omega'' \in \Omega(x, X''),\ \omega' \cap \omega'' \subseteq \tilde{\omega} \subseteq N \Rightarrow \tilde{\omega} \in \Omega(x, X' \cup X'')$;

VIII. $\forall X,\ \forall x,y\in X,\ \forall \omega,\omega',\tilde{\omega}: \omega\in\Omega(x,X),\ \omega'\notin\Omega(y,X),$
$\omega\setminus\omega'\subseteq\tilde{\omega}\subseteq\omega\cup\omega' \Rightarrow \tilde{\omega}\in\Omega(x,X\setminus\{y\})$; $\forall z\in X,\ z\neq y,$
$\forall\tilde{\omega}'',\omega'': \omega''\notin\Omega(z,X),\ \omega''\setminus\omega'\subseteq\tilde{\omega}''\subseteq\omega''\cup\omega' \Rightarrow$
$\Rightarrow \tilde{\omega}''\notin\Omega(z,X\setminus\{y\})$.

Theorem 4. The conditions VI, VII and VIII are necessary and sufficient for closedness of domains H (condition VI), C (condition VII), O (condition VIII), resp., relative to the set $\mathcal{L}$ of all local operators, i.e. $\Lambda^{\mathrm{VI}}=\Lambda_H$, $\Lambda^{\mathrm{VII}}=\Lambda_C$ and $\Lambda^{\mathrm{VIII}}=\Lambda_O$.

The proof is given for the classes Λ_H, Λ_C and Λ_O separately.

1) Class Λ_H. Necessity. Assume VI to be fulfilled. Show that $C^*(\cdot)\in H$. Let on the contrary there exist a profile $\{C_i(\cdot)\}, X, X'$ x such that $x\in X'\subset X\in\mathcal{A}$, $x\in C^*(X)$ and $x\notin C^*(X')$. Since $x\in C^*(X)$, then $V(x,X;\{C_i(\cdot)\})\in\Omega(x,X)$.

Since $C_i(\cdot)\in H,\ \forall i\in N$: $i\in V(x,X;\{C_i(\cdot)\})$ and $\forall X'\subset X\Rightarrow x\in C_i(X')$, then $V(x,X;\{C_i(\cdot)\})\subseteq V(x,X';\{C_i(\cdot)\})$. Since $x\in C^*(X')$, i.e. $V(x,X';\{C_i(\cdot)\})\notin\Omega(x,X')$ and the condition VI fails.

Sufficiency. Let $C^*(\cdot)\in H$. Show that condition VI holds. Let on the contrary there exist $X, X', x, \omega, \tilde{\omega}$ such that $x\in X'\subset X\in\mathcal{A}$, $\omega\in\Omega(x,X)$, $\tilde{\omega}\supseteq\omega$ and $\tilde{\omega}\notin\Omega(x,X')$. Consider the profile $\{C_i(\cdot)\}$ such that $\forall i\in(N\setminus\omega)\cap(N\setminus\tilde{\omega})\Rightarrow x\notin C_i(X)$, $x\notin C_i(X')$; $\forall i\in(N\setminus\omega)\cap\tilde{\omega}\Rightarrow x\notin C_i(X)$, $x\in C_i(X')$; $\forall i\in\omega\cap\tilde{\omega}\Rightarrow$ $\Rightarrow x\in C_i(X)$, $x\in C_i(X')$. It is obvious that each of these demands does not contradict the condition H. Then from $V(x,X;\{C_i(\cdot)\})=\omega$, $V(x,X';\{C_i(\cdot)\})=\tilde{\omega}$ it follows that $x\in C^*(X)$ and $x\notin C^*(X')$, which fails the supposition $C^*(\cdot)\in H$.

The statement of theorem 4 about coincidence of classes $\Lambda^{\overline{VI}}$ and Λ_H is proofed.

2) <u>Class Λ_C</u>. Necessity. Assume VII to be fulfilled. Show that $C^*(\cdot)\in C$. Let on the contrary there exist a profile $\{C_i(\cdot)\}$, X', X'', x such that $x\in C^*(X')\cap C^*(X'')$ and $x\notin C^*(X'\cup X'')$. Since $x\in C^*(X')$, then $V(x,X';\{C_i(\cdot)\})\in \Omega(x,X')$. Similarly, $V(x,X'';\{C_i(\cdot)\})\in\Omega(x,X'')$. Let us consider $i\in V(x,X';\{C_i(\cdot)\})\cap V(x,X'';\{C_i(\cdot)\})$. Since $C_i(\cdot)\in C$, $x\in C_i(X'\cup X'')$, then $V(x,X';\{C_i(\cdot)\})\cap V(x,X'';\{C_i(\cdot)\})\subseteq V(x,X'\cup X'';\{C_i(\cdot)\})$. For $x\notin C^*(X'\cup X'')$ it immediately follows $V(x,X'\cup X'';\{C_i(\cdot)\})\notin\Omega(x,X'\cup X'')$ and just failure of condition VII.

Sufficiency. Suppose that $C^*(\cdot)\in C$. Show that condition VII holds. Let on the contrary there exist $X', X'', x, \omega', \omega'', \tilde\omega$ such that $X'\nsubseteq X''$, $X''\nsubseteq X'$, $x\in X'\cup X''$, $\omega'\in\Omega(x,X')$, $\omega''\in\Omega(x,X')$, $\omega'\cap\omega''\subseteq\tilde\omega\subseteq N$ and $\tilde\omega\notin\Omega(x,X'\cup X'')$. Only 8 intersections of sets ω',ω'' and $\tilde\omega$ and their complements are acceptable, one of which is empty due to $\omega'\cap\omega''\subseteq\tilde\omega$. Let us construct a profile $\{C_i(\cdot)\}$ satisfying the following condition (where $X'=\{x,y\}$, $X''=\{x,z\}$. $C_i(X)=X$, if $Card(X)=1$):

a) $\forall i\in\omega'\cap\omega''\cap\tilde\omega \Rightarrow C_i(X')=\{x\};\ C_i(X'')=\{x\},\ C_i(X'\cup X'')=\{x\};$

b) $\forall i\in\omega'\cap(N\setminus\omega'')\cap\tilde\omega\Rightarrow C_i(X')=\{x\};\ C_i(X'')=\{z\},\ C_i(X'\cup X'')=\{x,y\};$

c) $\forall i\in(N\setminus\omega')\cap\omega''\cap\tilde\omega\Rightarrow C_i(X')=\{y\};\ C_i(X'')=\{x\},\ C_i(X'\cup X'')=\{x,z\};$

d) $\forall i\in\omega'\cap(N\setminus\omega'')\cap(N\setminus\tilde\omega)\Rightarrow C_i(X')=\{x\};\ C_i(X'')=\{z\},\ C_i(X'\cup X'')=\{z\};$

e) $\forall i\in(N\setminus\omega')\cap\omega''\cap(N\setminus\tilde\omega)\Rightarrow C_i(X')=\{x\};\ C_i(X'')=\{x\},\ C_i(X'\cup X'')=\{y\};$

f) $\forall i\in(N\setminus\omega')\cap(N\setminus\omega'')\cap\tilde\omega\Rightarrow C_i(X')=\{y\};\ C_i(X'')=\{z\},\ C_i(X'\cup X'')=X'\cup X'';$

g) $\forall i\in(N\setminus\omega')\cap(N\setminus\omega'')\cap(N\setminus\tilde\omega)\Rightarrow C_i(X')=C_i(X')=C_i(X'\cup X'')=\emptyset.$

None of these conditions a)–g) contradicts the condition C Hence, the profile $\{C_i(\cdot)\}$, $C_i(\cdot) \in C$, $\forall i$, for which $V(x, X'; \{C_i(\cdot)\}) = \omega'$, $V(x, X''; \{C_i(\cdot)\}) = \omega''$ and $V(x, X' \cup X''; \{C_i(\cdot)\}) = \tilde{\omega}$ are fulfilled may be constructed

It follows $x \in C^*(X')$, $x \in C^*(X'')$ and $x \notin C^*(X' \cup X'')$ which contradicts $C^*(\cdot) \in \mathcal{C}$

The statement of theorem 4 about coincidence of classes Λ^{VII} and Λ_C is prooved.

3) <u>Class Λ_O</u>. Two lemmas will introduce the proof of the case.

<u>Lemma 2</u>. Let $x, y \in X$, $\omega \in \Omega(x, X)$, $\omega' \notin \Omega(y, X)$, $\omega \setminus \omega' \subseteq \tilde{\omega} \subseteq \omega \cup \omega'$. Then there exists the profile $\{C_i(\cdot)\}$, $C_i(\cdot) \in O$, $\forall i$, and $V(x, X; \{C_i(\cdot)\}) = \omega$, $V(y, X; \{C_i(\cdot)\}) = \omega'$, $V(x, X \setminus \{y\}; \{C_i(\cdot)\}) = \tilde{\omega}$.

<u>Proof of lemma 2</u>. Only 8 intersections of sets $\omega, \omega', \tilde{\omega}$ and their complements are acceptable three of which from $\omega \subseteq \tilde{\omega}$ and $\tilde{\omega} \subseteq \omega \cup \omega'$ are empty. Let us consider the profile $\{C_i(\cdot)\}$ shown in Table 2 where $X = \{x, y, z\}$, $X = \{x, y\}$. In this profile $C_i(\cdot) \in O, \forall i$, and the correlation required is fulfilled.

<u>Lemma 3</u>. Let $x, y, z \in X$, $\omega \in \Omega(x, X)$, $\omega' \notin \Omega(y, X)$, $\omega'' \notin \Omega(z, X)$, $\omega \setminus \omega' \subseteq \tilde{\omega} \subseteq \omega \cup \omega'$, $\omega'' \setminus \omega' \subseteq \tilde{\omega}'' \subseteq \omega'' \cup \omega'$.

Then there exists a profile $\{C_i(\cdot)\}$, $C_i(\cdot) \in O$, $\forall i$, and $V(x, X; \{C_i(\cdot)\}) = \omega$, $V(y, X; \{C_i(\cdot)\}) = \omega'$, $V(z, X; \{C_i(\cdot)\}) = \omega''$, $V(x, X \setminus \{y\}; \{C_i(\cdot)\}) = \tilde{\omega}$, $V(z, X \setminus \{y\}; \{C_i(\cdot)\}) = \tilde{\omega}''$

<u>Proof of lemma 3</u>. Let us consider without loss of generality $X = \{x, y, z\}$. Only 32 intersections of sets $\omega, \omega', \omega'', \tilde{\omega}, \tilde{\omega}''$ and their complements ninetem of which from correlations $\omega \subseteq \tilde{\omega}$, $\tilde{\omega} \subseteq \omega \cup \omega'$, $\omega'' \subseteq \tilde{\omega}''$, $\tilde{\omega}'' \subseteq \omega'' \cup \omega'$ are empty are acceptable. Consider the profile $\{C_i(\cdot)\}$ shown in Table 3. In this

profile $C_i(\cdot)\in O$, $\forall i$, and all relations required are satisfied.

Proof of theorem 4 about class Λ_o .

Necessity. Su pose VIII. Let us show that $C_i(\cdot)\in O$.

Assume on the contrary that for some X either A) $x\in C^*(X)$, $y\notin C^*(X)$, $x\notin C^*(X\setminus\{y\})$, or B) $z\notin C^*(X)$, $y\notin C^*(X)$, $z\in C^*(X\setminus\{y\})$ are satisfied.

A) Since $x\in C^*(X), y\notin C^*(X)$, then $\omega = V(x, X; \{C_i(\cdot)\})\in\Omega(x,X)$, $\omega' = V(y,X;\{C_i(\cdot)\})\notin\Omega(y,X)$ and $\tilde\omega = V(x,X\setminus\{y\};\{C_i(\cdot)\})\notin\Omega(x,X\setminus\{y\})$. Show that $\omega\setminus\omega'\subseteq\tilde\omega\subseteq\omega\cup\omega'$ is satisfied. Indeed, $\forall i\in\omega\setminus\omega'\Rightarrow i\in\tilde\omega$ since $i\in\omega\setminus\omega'$, then $x\in C_i(X), y\notin C_i(X)$. For $C_i(\cdot)\in O$ it follow $x\in C_i(X\setminus\{y\})$, and therefore, $i\in\tilde\omega$. Furthermore, $\forall i\in\tilde\omega\Rightarrow i\in\omega\cup\omega'$. The following four case are possible: $a_1)$ $x,y\notin C_i(X)$. Then $x\notin C_i(X\setminus\{y\})$ and, therefore, $i\notin\tilde\omega$; $a_2)$ $x\in C_i(X), y\notin C_i(X)$. In this case $i\in\omega$ and, moreover, $i\in\omega\cup\omega'$; $a_3)$ $x\notin C_i(X), y\in C_i(X)$. Then $i\in\omega'$, and hence $i\in\omega\cup\omega'$; $a_4)$ $x\in C_i(X), y\in C_i(X)$. Then $i\in\omega\cap\omega'$ and, all the more, $i\in\omega\cup\omega'$. We have shown that the inclusion $\omega\setminus\omega'\subseteq\tilde\omega\subseteq\omega\cup\omega'$ holds. Since $\tilde\omega\notin\Omega(x,X\setminus\{y\})$ the first part of the condition VIII fails.

The second possibility is $x\in C^*(X)$, $y\notin C^*(X)$, $z\notin C^*(X)$, $z\in C^*(X\setminus\{y\})$. Then $V(y,X;\{C_i(\cdot)\}) = \omega'\notin\Omega(y,X)$, $V(z,X;\{C_i(\cdot)\}) = \omega''\notin\Omega(z,X)$; $V(z,X\setminus\{y\};\{C_i(\cdot)\}) = \tilde\omega''\in\Omega(z,X\setminus\{y\})$. As previously demonstrated it may be shown that the inclusion $\omega''\setminus\omega'\subseteq\tilde\omega\subseteq\omega''\cup\omega'$ holds and since on assumption $\tilde\omega''\in\Omega(z,X\setminus\{y\})$ then the second part of the condition VIII is not met.

Sufficiency. Let $C^*(\cdot)\in O$. Show that condition VIII holds. Assume on the contrary that condition VIII is not met. In the case however according to lemma 2 and 3 for all $X, x, y, z, \omega, \omega', \omega'', \tilde{\omega}, \tilde{\omega}''$ there exists a profile $\{C_i(\cdot)\}$ for which the relations $V(x, X; \{C_i(\cdot)\}) = \omega$. $V(y, X; \{C_i(\cdot)\}) = \omega'$. $V(z, X; \{C_i(\cdot)\}) = \omega''$. $V(x, X \setminus \{y\}; \{C_i(\cdot)\}) = \tilde{\omega}$. $V(z, X \setminus \{y\}; \{C_i(\cdot)\}) = \tilde{\omega}''$ holds. The failure of condition VIII implies $x \in C^*(X)$, $z, y \notin C^*(X)$ and either $x \notin C^*(X \setminus \{y\})$ or $z \in C^*(X \setminus \{y\})$. i.e. the failure of the assumption $C^*(\cdot) \in O$.

Theorem 4 is completely proved. ■

Three remarks to the Theorem 4 will be given below.

Remark 1. When we have proved Theorem 4 in the part which deals with the class Λ_0 we have actually considered the condition of closedness not of the domain O , but some other domain O' . This domain O' is picked out with the following condition $\forall y, X : y \in X \setminus C(X) \Rightarrow C(X \setminus \{y\}) = C(X)$. An arbitrary choice function obeying condition O satisfies condition O' obviously.

For $X = \{x, y, z\}$ condition VIII therefore must be added to the condition characterized the case when the subset $\{x, y\}$ is rejected from X . This additional condition may be written down as

$$\forall \tilde{\omega} : \omega \setminus (\omega' \cup \omega'') \subseteq \tilde{\omega} \subseteq \omega \cup \omega' \cup \omega'' \Rightarrow \tilde{\omega} \in \Omega(x, X \setminus \{y, z\})$$

The condition VIII characterizes hence the closedness of the domain O' , but it will be shown that the condition of closedness of the domain O may be written analogously and it consists of the number of conditions characterizing the closedness of the domains O', O'' , etc. These domains are identified by the conditions of rejecting of every τ-element subsets ($\tau = 1, 2, \ldots, \tau \le m$) of the set X.

In this case it turns out if the condition VIII is added to the closedness conditions of domains O', O'', etc., the properties of the operators class Λ_O is not changed, i.e. the main properties of the class Λ_O are defined by the very condition VIII and therefore when the closedness of the domain O is considered it may be restricted to study only the class Λ^{VIII}.

Remark 2. Among the main assumptions of the paper there is the condition $card(A) = m \geqslant 3$. The closedness of the domains H and O however as it have been implied by the proof of theorem 4 may be established just in the case when $card(A)=2$

Remark 3. In this paper we use the notion of the complete class of operator closedness Λ_Q for the domain Q. Generalizing this notion we could use the notion of a class $\Lambda(Q' \Rightarrow Q'')$ consisting of all operators which "construct" by the profile $\{C_i(\cdot)\}, C_i(\cdot) \in Q', \forall i$, a function $C^*(\cdot) \in Q''$

It is found that when the domains H, C and O as Q'' and the domains $H \cap C$, $H \cap O$, $C \cap O$ and $H \cap C \cap O$ as Q' when $Q' \subseteq Q''$ are considered the corresponding classes $\Lambda(Q' \Rightarrow Q'')$ are defined with the same conditions VI–VIII.

Hence the following generalization of Theorem 4 are permitted

Theorem 4'. For the function $C^*(\cdot)$ to be in the class

1) H whereas $Q' = H$ or $Q' = H \cap O$ or $Q' = H \cap C$ or $Q' = H \cap C \cap O$,
2) C whereas $Q' = C$ or $Q' = C \cap O$ or $Q' = H \cap C \cap O$;
3) O whereas $Q' = O$ or $Q' = H \cap O$ or $Q' = C \cap O$ or $Q' = H \cap C \cap O$

it is necessary and sufficient that the list-representation of L-operator F $(C^*(\cdot) = F(\{C_i(\cdot)\}), C_i(\cdot) \in Q')$ satisfies the conditions 1 VI; 2) VII; 3) VIII, resp.

In the formulation of Theorem 4' the condition which defines the class $\Lambda(H \cap C \to C)$ is not established. This condition is the modification of the condition VII and may be written down in the following way:

$$\text{VII}' \ \forall X', X'', x, \omega', \omega'', \tilde{\omega}: \ X' \nsubseteq X'', \ X'' \nsubseteq X'', \ x \in X' \in \mathcal{A}, \\ x \in X'' \in \mathcal{A}, \ \omega' \in \Omega(x, X'), \ \omega'' \in \Omega(x, X''), \\ \omega' \cap \omega'' = \tilde{\omega} \Rightarrow \tilde{\omega} \in \Omega(x, X' \cup X'')$$

Apparently when the monotonicity condition is satisfied the condition VII' coincides with the condition VII.

The results established in the Theorem 4' are easily implied by the proof of Theorem 4.

Theorem 4' immediately implies also the possibility of using for the complete classes of operator closedness the correlation $\Lambda_{Q' \cap Q''} = \Lambda_{Q'} \cap \Lambda_{Q''}$ for the domains H, C and O of the space $\mathcal{C}$.

Theorem 4 establishes the list-form representation of L-operators the complete classes Λ_H, Λ_C and Λ_O of operator closedness consist of. These representations give a possibility for studying the "disposition" of classes Λ_H, Λ_C and Λ_O in the set $\mathcal{L}$ and their relation with classes $\Lambda^{1^\circ_+}, \dots, \Lambda^{5^\circ_-}$.

<u>Theorem 5</u>. Classes $\Lambda^{1^\circ_+}, \dots, \Lambda^{4^\circ}$ and classes Λ_H, Λ_C and Λ_O are mutually related as shown[14] by the Euler-Venn's diagrams on Fig. 4 (for Λ_H), Fig. 5 (for Λ_C), and Fig. 6 (for Λ_O).

Proof. 1) <u>The class Λ_H</u>. (Fig. 4). From 38 non-empty possible conjunctions of the conditions 1°–5° and their negations (see sect. 3) in this case only 30 conjunctions with the numbers 1–5, 7–11, 13–15, 17–19, 21–24, 26–28, 30–32, 34–36, 38 are acceptable.

It follows from the relation $1^\circ_+ \cap 3^\circ \Rightarrow 2^\circ$ which is implied by condition VI. The latter implies that the Central Region is

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14) Classes 5°_+ and 5°_- are not shown for clearness on Fig. 4–6, but in the proof of theorem 5 these conditions are studied as well as the other conditions 1°–4°.

included in Λ_H.

2) <u>Class Λ_C</u> (Fig. 5). In this case, because the relation $1^\circ_+ \cap 3^\circ \Rightarrow 2^\circ$ (see VII) is satisfied the same thirty conjunctions are acceptable.

Condition VII however implies that Λ_C includes not the entire Central Region. Indeed, in Central Region (i.e. when conditions 1°, 2° and 3° hold) condition VII may be written in the following form: $\forall \omega', \omega'', \tilde{\omega} : \omega', \omega'' \in \Omega, \omega' \cap \omega'' \subseteq \tilde{\omega} \subseteq N \Rightarrow \tilde{\omega} \in \Omega$ hence a narrower class than Λ^{CR} is isolated.

3) <u>Class Λ_O</u> (Fig. 6). In this case we will first establish some correlations between the classes of L-operators.

a) $\bar{1}^\circ_+ \cap 1^\circ_- \cap \bar{3}^\circ = \phi$

Let us show that there exists $x \in A$ such that $\Omega(x,A) \neq \phi$ Assume on the contrary that $\forall x \in A \quad \Omega(x,A) = \phi$. Then since $C^*(\cdot) \in O$ the correlation $\Omega(t,X) = \phi, \forall t \in X$ and $\forall X \in \mathcal{A}$ holds, i.e. condition 3° holds.

Let us show that $\forall y \in A \quad \Omega(y,A) \neq \phi$. Assume on the contrary that there exists $y \in A$ such that $\Omega(y,A) = \phi$. Since for some $x \in A \quad \Omega(x,A) \neq \phi$, then there exist $\omega \subseteq N$ and $\omega \in \Omega(x,A)$ and since $\Omega(y,A) = \phi$, then $N \notin \Omega(y,A)$ It follows that $\forall \tilde{\omega} : \omega \setminus N = \phi \subseteq \tilde{\omega} \subseteq \omega \cup N = N \Rightarrow \tilde{\omega} \in \Omega(x, A \setminus \{y\})$ i.e. $\Omega(x, A \setminus \{y\}) = 2^N$ which contradicts 1°_-.

Thus $\forall t \in A \quad \Omega(t,A) \neq \phi$. Let us show, that $\Omega(t,X) \neq \phi \quad \forall t \in X$ and $X \subset A$. Since 1°_- holds, then $\Omega(t,X) \neq 2^N, \forall t \in X$ and $X \subseteq A$. So for as $\Omega(t,A) \neq \phi$, $\forall t \in A$, and since condition 1°_- holds for all $x, y \in A$ there exist $\omega', \omega'', \omega'''$ and ω'''' such that $\omega' \notin \Omega(x,A)$, $\omega'' \in \Omega(x,A)$, $\omega''' \notin \Omega(y,A)$, $\omega'''' \in \Omega(y,A)$. Then $\forall \tilde{\omega}' : \omega'' \setminus \omega''' \subseteq \tilde{\omega}' \subseteq \omega'' \cup \omega''' \quad \tilde{\omega}' \in \Omega(x, A \setminus \{y\})$ and $\forall \tilde{\omega}'' : \omega'''' \setminus \omega' \subseteq \tilde{\omega}'' \subseteq \omega'''' \cup \omega' \quad \tilde{\omega}'' \in \Omega(y, A \setminus \{x\})$. It may be shown in the same way that $\Omega(t,X) \neq \phi \quad \forall X \in \mathcal{A}$ and $\forall t \in X$

By the property a) the conjunctions 3,9,14,22,26 and 28 are excluded.

b) $1^\circ \cap 3^\circ \Rightarrow 2^\circ$. This property immediately follows from condition VIII.

By the property b) the conjunctions 6,12,16,20,25,29,33,37 are excluded.

c) $1^\circ_+ \cap 1^\circ_- \cap 3^\circ_+ = \phi$

c1) Let $card(X) > 1$ and there exists $x \in X$ such that $N \notin \Omega(x, X)$. From 1°_+ $\forall y \in X\ \exists \omega \in \Omega(y, X)$ holds. Then from condition VIII $\forall \tilde{\omega} : \omega \setminus N = \phi \subseteq \tilde{\omega} \subseteq \omega \cup N = N \Rightarrow \tilde{\omega} \in \Omega(y, X \setminus \{x\})$, i.e. $\Omega(y, X \setminus \{x\}) = 2^N$ which contradicts 1°_-.

c2) Let $N \notin \Omega(x, \{x\})$. Consider $X = \{x, y\}$. If 1°_- holds there exists $\omega' \notin \Omega(y, X)$ and from c1) $N \in \Omega(x, X)$. Then $\forall \tilde{\omega} : N \setminus \omega' \subseteq \tilde{\omega} \subseteq N \cup \omega' = N \Rightarrow \tilde{\omega} \in \Omega(x, \{x\})$ and therefore $N \in \Omega(x, \{x\})$.

This property excludes the conjunctions 4,10,23,27.

d) $1^\circ_+ \cap 1^\circ_- \Rightarrow 3^\circ$.

d1) $\forall \omega \in \Omega(x, X)$ and $\forall X' \subset X$, $\omega \in \Omega(x, X')$. From 1°_- $\forall y \in X\ \exists \omega' : \omega' \notin \Omega(y, X)$ holds. Then $\forall \tilde{\omega} : \omega \setminus \omega' \subseteq \tilde{\omega} \subseteq \omega \cup \omega'$ $\tilde{\omega} \in \Omega(x, X \setminus \{y\})$. But ω satisfies this correlation and therefore $\omega \in \Omega(x, X \setminus \{y\})$.

d2) $\forall \omega \in \Omega(x, X)$ and $\forall X' \supset X$ $\omega \in \Omega(x, X')$. Assume on the contrary that $\omega \notin \Omega(x, X')$. Let us consider a profile $\{C_i(\cdot)\}$ such that $\forall i \in \omega\ C_i(X') = \{x\}$ $\forall i \notin \omega\ C_i(X') = \phi$.

Then obviously $C^*(X') = \phi$ and since $C^*(\cdot) \in O \Rightarrow C^*(X) = \phi$. But $C_i(\cdot) \in O$, then $C_i(X) = \{x\}$, $\forall i \in \omega$, and $C_i(X) = \phi$ $\forall i \notin \omega$, and $C^*(X) = \{x\}$ which contradicts $C^*(\cdot) \in O$.

This implies that $\forall \omega \in \Omega(x, X)$ and $\forall X' \subseteq A$, $\omega \in \Omega(x, X')$ (for this it is sufficient to consider the set $X'' = X \cup X'$, $X'' \supset X$ and $X'' \supset X'$).

It is shown that the condition 3°-a holds, i.e. the lists $\Omega(x, X)$ are represented as $\Omega(x)$. Let us show that condition 3°-b holds.

d3) Let $\omega' \in \Omega(x)$. Show that $\forall \tilde{\omega} : \omega' \subseteq \tilde{\omega}, \tilde{\omega} \in \Omega(x)$. Let $\tilde{\omega} \notin \Omega(x)$. Let us consider $\omega'' \notin \Omega(y)$. Such set ω'' exists since condition 1° holds. Then $\forall \tilde{\omega}''$: $\tilde{\omega} \setminus \omega'' \subseteq \tilde{\omega}'' \subseteq \tilde{\omega} \cup \omega''$ $\Rightarrow \tilde{\omega}'' \notin \Omega(x)$ — . Thus $\forall \tilde{\omega}'''$: $\omega'' \setminus (\omega \cup \omega') = \phi \subseteq \tilde{\omega}'' \subseteq \omega \cup \omega'' \Rightarrow \tilde{\omega}''' \notin \Omega(y)$. In this case $\forall \tilde{\tilde{\omega}} : \tilde{\omega} \setminus (\omega'' \cup \tilde{\omega}) = \phi \subseteq \tilde{\tilde{\omega}} \subseteq \tilde{\omega} \cup \omega'' \Rightarrow \tilde{\tilde{\omega}} \notin \Omega(x)$

Since $\tilde{\omega} \supset \omega'$ it follows that $\omega' \notin \Omega(x)$.

d4) $\forall \omega \in \Omega(x) \Rightarrow \omega \in \Omega(y)$. Suppose on the contrary that $\omega \notin \Omega(y)$. Then $\forall \tilde{\omega} : \omega \setminus \omega = \phi \subseteq \tilde{\omega} \subseteq \omega \Rightarrow \tilde{\omega} \in \Omega(x)$ and from d3) $\Omega(x) = 2^N$ follows. This implies $\forall x \in A \ \Omega(x) \equiv \Omega$, i.e. the condition 3° holds.

The property d) excludes conjunctions 14,18,31,35 and 36. The fact that the intersections of the class and the operator classes determined by conjunctions 1,2,5,7,8,11,13,15,17,19, 21,30,34 and 38 are non-empty may be illustrated by specific operators.

Figures 4-6 particularly imply that the domain H,C and O may be closed relative to L-operators satisfying neither all conditions 1°-5° nor any of them.

Making use of the conditions VI-VIII, and Figures 4-6 we obtain mutual relations between the complete classes of operator closedness Λ_H, Λ_C and Λ_O and the Basic, Central and Symmetrically—Central Regions. Theorems 6 and 7 given below establish these relations. These theorems are in fact the corollaries of theorem 5 and the correlation $\Lambda_{Q' \cap Q''} = \Lambda_{Q'} \cap \Lambda_{Q''}$ and therefore the proofs of these theorems are shortened.

Denote as $\bar{\Lambda}_Q$ the implementation of the class Λ_Q in $\mathcal{L}$ (Q is a domain in $\mathcal{C}$).

Theorem 6. In space $\mathcal{Z}$ all eight classes $\Lambda_H \cap \Lambda_C \cap \Lambda_O$, $\bar{\Lambda}_H \cap \Lambda_C \cap \Lambda_O, \ldots, \bar{\Lambda}_H \cap \bar{\Lambda}_C \cap \bar{\Lambda}_O$ are non-empty.

Proof. Examples of the operators when $A = \{x, y, z\}$ and $N = \{1, 2, 3\}$ are shown in Table 4, in which the lists only for the pairs $\Omega(x, A)$, $\Omega(x, \{x, y\})$, $\Omega(x, \{x, z\})$ are demonstrated. For other pairs (t, X) let us put the corresponding lists for the operators from classes $\bar{\Lambda}_H \cap \bar{\Lambda}_O \cap \bar{\Lambda}_C$, $\Lambda_H \cap \bar{\Lambda}_O \cap \bar{\Lambda}_C$, $\bar{\Lambda}_H \cap \bar{\Lambda}_O \cap \Lambda_C$, $\Lambda_H \cap \bar{\Lambda}_O \cap \Lambda_C$ being equal N, and for the operators from classes $\bar{\Lambda}_H \cap \Lambda_O \cap \bar{\Lambda}_C$, $\bar{\Lambda}_H \cap \Lambda_O \cap \Lambda_C$ and $\Lambda_H \cap \Lambda_O \cap \bar{\Lambda}_C$ being equal to 2^N. It may be immediately checked that such operators satisfy the corresponding conditions VI–VIII or their intercrosses.

The statement of theorem 6 is illustrated on Fig.7. The classes Λ_H, Λ_C and Λ_O are shown as circles. The trivial operators 0 and 1 are disposed within the class $\Lambda_H \cap \Lambda_C \cap \Lambda_O$

Consider now how the basic, Central and Symmetrically-Central Regions are located relative to the classes Λ_H, Λ_O and Λ_C.

Theorem 7. The intersections of the Basic Region $(\Lambda^{1^\circ \cap 2^\circ})$ with the classes $\bar{\Lambda}_H \cap \bar{\Lambda}_O \cap \bar{\Lambda}_C$, $\Lambda_H \cap \bar{\Lambda}_O \cap \bar{\Lambda}_C$, $\bar{\Lambda}_H \cap \bar{\Lambda}_O \cap \Lambda_C$, $\Lambda_H \cap \bar{\Lambda}_O \cap \Lambda_C$, $\Lambda_H \cap \Lambda_O \cap \bar{\Lambda}_C$, $\Lambda_H \cap \Lambda_O \cap \Lambda_C$ are non-empty. The intersection of the Basic Region with the class Λ_O is located strictly inside the intersection of the class Λ_H and the basic Region, i.e. $\Lambda^{BR} \cap \Lambda_O \subset \Lambda^{BR} \cap \Lambda_H$.

The Central Region lies strictly inside the class Λ_H and includes strictly the intersection of the class Λ_O and the Basic Region, i.e. $\Lambda^{BR} \cap \Lambda_O \subset \Lambda^{CR} \subset \Lambda_H$.

Symmetrically-Central Region lies strictly inside the class Λ_H and has only one V operator in its intersection with Λ_O and only one U in its intersection with Λ_C, i.e. $\Lambda^{SCR} \subset \Lambda_H$, $\Lambda^{SCR} \cap \Lambda_O = \{V\}$, $\Lambda^{SCR} \cap \Lambda_C = \{U\}$.

Proof. The first statement of theorem 7 about the non-emptiness of the intersection of the Basic Region with classes $\overline{\Lambda}_H \cap \overline{\Lambda}_o \cap \overline{\Lambda}_c$, $\Lambda_H \cap \overline{\Lambda}_o \cap \overline{\Lambda}_c$, $\overline{\Lambda}_H \cap \overline{\Lambda}_o \cap \Lambda_c$ and $\Lambda_H \cap \overline{\Lambda}_o \cap \Lambda_c$ is easily seen from the examples used in the proof of theorem 6.

The inclusion $\Lambda^{BR} \cap \Lambda_o \subset \Lambda^{CR}$ follows immediately from Fig. 6, and the inclusion $\Lambda^{CR} \subset \Lambda_H$ - from Fig. 4 (see the proof of theorem 5).

Consider the examples of operators showing that the intersections of the Basic Region and classes $\Lambda_H \cap \Lambda_o \cap \overline{\Lambda}_c$ and $\Lambda_H \cap \Lambda_o \cap \Lambda_c$ are non-empty. Making use of the inclusion $\Lambda^{BR} \cap \Lambda_o \subset \Lambda^{CR}$ let us construct for the class $\Lambda_H \cap \Lambda_o \cap \overline{\Lambda}_c$ an operator, for which the basic list is represented as $\Omega^B = \{\{1\},\{2\}\}$, and for the class $\Lambda_H \cap \Lambda_o \cap \Lambda_c$ - the operator with $\Omega^B = \{\{1\}\}$. It is easily checked that these operators satisfy the conditions required.

The statement $\Lambda^{SCR} \subset \Lambda_H$ follows immediately from the inclusion $\Lambda^{SCR} \subset \Lambda^{CR} \subset \Lambda_H$. •

To prove the last two statements of theorem 7 consider now the list-form conditions VII and VIII in the Symmetrically-Central Region, i.e. when the conditions 1°-4° hold. For this consider first, the form of these conditions in Central Region. Obviously

$$\Lambda^{VII} \cap \Lambda^{CR} : \forall \omega', \omega'' \in \Omega, \forall \tilde{\omega} : \omega' \cap \omega'' \subseteq \tilde{\omega} \subseteq N \Rightarrow \tilde{\omega} \in \Omega,$$

$$\Lambda^{VIII} \cap \Lambda^{CR} : \forall \omega, \omega', \omega'', \tilde{\omega}, \tilde{\omega}'' : \omega \in \Omega,\ \omega' \notin \Omega,$$
$$\omega'' \notin \Omega,\ \omega \setminus \omega' \subseteq \tilde{\omega} \subseteq N,\ \omega'' \setminus \omega' \subseteq \tilde{\omega}'' \subseteq N \Rightarrow \tilde{\omega} \in \Omega,$$
$$\tilde{\omega}'' \notin \Omega$$

Show that the operators from the class $\Lambda^{VIII} \cap \Lambda^{CR}$ are determined by the basic list of the type $\Omega^B = \{\omega_1, \dots, \omega_S\}$ where $card(\omega_i) = 1$, $i = 1, \dots, S$. Indeed, let on the contrary $\omega = \{i,j\} \in \Omega$ and $\{i\} \notin \Omega$, $\{j\} \notin \Omega$. Then by the condition VIII $\tilde{\omega} = \{i,j\} \setminus \{i\} = \{j\} \in \Omega$. This contradiction proves the statement. The fulfillment of condition 4° yields the unique operator from the class $\Lambda^{VIII} \cap \Lambda^{CR}$, for which the basic list is $\Omega^B = \{N\}$, and from the class $\Lambda^{VIII} \cap \Lambda^{CR}$ – the unique operator with $\Omega^B = \{\{1\}, \ldots, \{n\}\}$. These basic lists exactly correspond to the list-form representations of operators $\mathcal{U}$ and $\mathcal{V}$. ∎

Theorem 7 is illustrated on Fig. 8a. A single-dashed line shows in this Figure the Central Region, and a double-dashed line – the Symmetrically-Central Region.

The fragment of Fig. 8-a – the Central Region and its intersections with the intersections of complete classes of operator closedness and their implementations are shown on Fig. 8-b. The single-dashed line in this Figure shows the intersections of the Central Region with the classes $\Lambda_H \cap \Lambda_0$ and $\Lambda_H \cap \Lambda_C$, and the double-dashed line shows the intersection $\Lambda^{CR} \cap \Lambda_H \cap \Lambda_C \cap \Lambda_0$

The last two statements of theorem 7 give rise to the following as corollaries:

<u>The first functional principle of mutually-exclusive neutralities</u>. Except for the operators"unanimity" the Basic Region does not contain the operators which satisfy condition 3° of neutrality to variants and 4° of neutrality to voters and relative to which the domain C in the space $\mathcal{C}$ is closed.

<u>The second functional principle of mutually exclusive neutralities</u>. Except for the operator "at least one vote ay the Basic Region does not contain the operators which satisfy condition 3° of neutrality to variants and 4° of neutrality to voters and relative to which the domain O in the space $\mathcal{C}$ is closed.

These principles imply the existence in the Symmetrically-Central Region of only two operators $\mathcal{U}$ and $\mathcal{V}$ relative to which the domains C and O are respectively closed.

Making use of lemma 1 and theorems 5–7 we obtain

Corollary 1. Except for the operator "unanimity" there are no L-operaotrs safisfying the conditions 1°–4° and relative to which the class $H \cap C$ is closed

Corollary 2. There are no L-operators satisfying conditions 1°–4° and relative to which the class $H \cap C \cap O$ is closed.

Thus, not only the common plurality rule but each L-operator satisfying what seems to be equally natural conditions 1°–4° transforms the classically-rational individual functions to a collective choice function which does not in a general case satisfy these conditions.

6. Mutual relations between the complete classes of operator closedness Λ_H, Λ_O and Λ_C and the Basic Region

In this section we study the mutual relations between the classes Λ_H, Λ_O and Λ_C and the classes $\Lambda^{UI(x,X)}$, $\Lambda^{UI(x)}$ and Λ^{UI} and their subclasses regarded in section 4.

Consider conditions VI–VIII for the case when conditions 3°–a, of independence of the context, and 3° of neutrality to variants are fulfilled additionally. Otherwise speaking we consider the list-form representation of L-operators from the intersections $\Lambda_H \cap \Lambda^{3°\text{-}a}$, $\Lambda_H \cap \Lambda^{3°}$, $\Lambda_O \cap \Lambda^{3°\text{-}a}$, etc. Modifications of conditions VI–VIII will be denoted by the same numbers with the upper index pointing out to an additional condition.

Consider only three modifications of these conditions, namely $\text{VI}^{3°\text{-}a}$–$\text{VIII}^{3°\text{-}a}$.

$\text{VI}^{3°\text{-}a}$ $\forall X \in \mathcal{A} : \omega \in \Omega(x), \omega \subseteq \tilde{\omega} \subseteq N \Rightarrow \tilde{\omega} \in \Omega(x)$;

$\text{VII}^{3°\text{-}a}$ $\forall X \in \mathcal{A} : \omega' \in \Omega(x), \omega'' \in \Omega(x), \omega' \cap \omega' \subseteq \tilde{\omega} \subseteq N \Rightarrow \tilde{\omega} \in \Omega(x)$;

$\text{VIII}^{3°\text{-}a}$ $\forall X \in \mathcal{A} : \omega \in \Omega(x), \omega' \notin \Omega(y), \omega \setminus \omega' \subseteq \tilde{\omega} \subseteq \omega \cup \omega' \Rightarrow \tilde{\omega} \in \Omega(x)$,

$\omega'' \notin \Omega(z), \omega'' \setminus \omega' \subseteq \tilde{\omega}'' \subseteq \omega'' \cup \omega' \Rightarrow \tilde{\omega}'' \notin \Omega(z)$

The other three modifications for the case when condition 3° is satisfied are obvious since in this case the list-form representation of L-operator is determined by the unique list for all $X \in \mathcal{A}$ and $\forall$ all $x \in X$. The conditions $\mathrm{VI}^{3°-a}$ and $\mathrm{VII}^{3°-a}$ satisfy apparently the monotonicity condition and $\mathrm{VI}^{3°-a}$ coincides with the conjunction of the conditions II and III-a.

These facts give rise to the following corollaries.

<u>Corollary 1</u>. The intersection Λ_H with the class $\Lambda^{3°-a}$ coincides with the intersection $\Lambda^{2°}$ with $\Lambda^{3°-a}$ and coincides with the replenished class of operators "independent union of intersections", i.e. $\Lambda_H \cap \Lambda^{3°-a} = \Lambda^{2°} \cap \Lambda^{3°-a} = \Lambda^{\widetilde{uI}(x)}$.

Show that the class $\Lambda^{2°}$ is "wider" than the intersection $\Lambda_H \cap \Lambda^{3°-a}$.

<u>Example</u>. Let $A = \{x,y,z\}$ and $N = \{1,2,3\}$ and the totality of lists for variant x is $\Omega^B(x, A) = \{\{1\}\}$, $\Omega^B(x, \{x,y\}) = \{N\}$, $\Omega^B(x, \{x,z\}) = \{\{z\}\}$, $\Omega(x, \{x\}) = \emptyset$

The condition 2° is obviously satisfied, but none of the conditions VI–VIII are satisfied.

<u>Corollary 2</u>. The intersection Λ_c with the class $\Lambda^{3°-a}$ lies strictly inside the intersection $\Lambda^{2°}$ with $\Lambda^{3°-a}$ and coincides with the replenished class of operators "independent partial intersection", i.e. $\Lambda^{\widetilde{pI}(x)} = \Lambda_c \cap \Lambda^{3°-a} \subset \Lambda^{2°} \cap \Lambda^{3°-a}$.

The results dealing with the L-operators from the intersections Λ_H and Λ_c with the classes $\Lambda^{1°}$ or $\Lambda^{3°}$ follow from corollaries 1 and 2 and will be stated below.

We study now a complete class of operator closedness Λ_o.

The condition VIII immediately implies that the class Λ_o comprises the operators satisfying the condition 3°-a and unlike Λ_H and Λ_c do not in general satisfy the monotonicity condition 2°. But Fig. 6 implies that the intersection of Λ_o with $\Lambda^{1°}$ consists of the operators satisfying the neutrality condition 3°. The operators, which comprise this intersection are isolated by

Lemma 4. The intersection of Λ_o with the class Λ^{1° coincides with the class of operators "neutral partial union", i.e. $\Lambda_o \cap \Lambda^{1^\circ} = \Lambda^{PU}$.

Proof. Show that $\forall F \in \Lambda_o \cap \Lambda^{1^\circ} \Rightarrow F \in \Lambda^{PU}$. Fig. 6 implies that $\Lambda_o \cap \Lambda^{1^\circ} \subset \Lambda^{CR}$. Proving theorem 7 we have shown that the basic list for the operators from $\Lambda_o \cap \Lambda^{CR}$ is represented as $\Omega^B = \{\omega_1, \dots, \omega_S\}$ where $\forall i = 1, \dots, S \quad card(\omega_i) = 1$. This representation coincides exactly with those for the operators "neutral partial union".

The proof of the converse statement $\Lambda^{PU} \subseteq \Lambda_o \cap \Lambda^{1^\circ}$ is obvious. ▯

The results established above are allowed studying the explicit form of operators which comprise the intersections of the complete classes of operators closedness with the Basic, Central and Symmetrically-Central Regions.

Consider two categories of operator classes: a) determined by the intersections of complete classes of operators closedness and their complements, and b) determined by the totalities of characteristic conditions.

Theorem 8. The first category classes given in the intercrosses of the columns of Table 5 and the second category classes given in the rows of Table 5 coincide with the corresponding classes of L-operators indicated in the intercrosses of rows and columns of Table 5.

Proof. Let us prove step by step the correlations the numbers of which are indicated in the intercrosses of the rows and columns of Table 5.

1) Lemma 4 implies that $\Lambda_o \cap \Lambda^{1^\circ} = \Lambda^{PU}$. Hence $\Lambda^{2^\circ} \cap \Lambda_o \cap \Lambda^{1^\circ} = \Lambda^{PU} \cap \Lambda^{2^\circ}$ and since $\Lambda^{PU} \cap \Lambda^{2^\circ} = \Lambda^{PU}$, then $\Lambda^{BR} \cap \Lambda_o = \Lambda^{PU}$.

2) Follows from 1) and the inclusion $\Lambda^{PU} \subset \Lambda^{CR} \subset \Lambda_H$

3) $\Lambda^{BR} \cap \Lambda_o \cap \Lambda_C = \Lambda^{\vartheta V}$. The condition VII $^{3°}$ determines the operators with $\Omega^B = \{\omega_1\}$ and since $\Lambda^{BR} \cap \Lambda_o = \Lambda^{PU}$, then $card(\omega_1) = 1$.

4) Follows from 2) and 3)

5) Follows from the coincidence of condition $\overline{VI}^{3°-a}$ with the definition of the operator "independent union of intersections"

6),9) The proof is analogous to 5)

7),8) Follows from 1) and 2)

10),11) Follows from 3) and 4) so far as $\Lambda^{\vartheta V} \cap \Lambda^{3°-a} = \Lambda^{\vartheta V}$

12) Follows from the coincidence of condition $\overline{VI}^{3°}$ with the definition of the operator "neutral union of intersections".

13),16) Proof is analogous to 12)

14),15) Follows from 7) and 8)

17),18) Follows from 10) and 11)

19) Consider how condition $\overline{VI}^{3°}$ transforms when the condition $\overline{IV}^1$ (equivalent to 4°) holds. In this case apparently there exists number K such that $\forall \omega_i \in \Omega^B \Rightarrow card(\omega_i) = K$ and there is no $\tilde{\omega} \notin \Omega^B$ with $card(\tilde{\omega}) > K$. Since the condition 4° holds all subsets of N with cardinality K belong to Ω^B and we obtain the definition of KP-operator.

20),21) See the proof of theorem 6.

22),23) Follows from 20),21) owing to the correlations $\Lambda^{SCR} \subset \Lambda_H$ and $U \in \Lambda_H, V \in \Lambda_H$

24),25) Follows from 17),18) and the obvious fact that $\Lambda^{\vartheta V} \cap \Lambda^{4°} = \emptyset$. ∎

Table 5 shows no operators from the intersections $\Lambda^{BR} \cap \Lambda_H$, $\Lambda^{BR} \cap \Lambda_C$ and $\Lambda^{BR} \cap \Lambda_H \cap \Lambda_C$. Section 4 has established the coincidence of the Basic Region with the class of operators "union of intersections". Hence the intersections in question consist of the operators of the type UI, but satisfying additionally the corresponding condition VI or VII.

The correlations established in the last rows of Table 5 are illustrated in Fig. 9. This Figure is obtained by combining Fig. 6-b with Fig. 3 (when Fig. 3 is considered for the case of neutral operators).

Let us generalize now the results of theorem 8 shown in the last two rows of Table 5.

Consider again two categories of operator classes: a) determined by the intersections of complete classes of operator closedness and their complements, and b) determined by the totalities of characteristic conditions

<u>Theorem 9</u>. The first category classes given in the intercrosses of the columns of Table 6 and the second category classes given in the rows of Table 6 coincide with the corresponding classes of Λ-operators indicated in the intercrosses of rows and columns of Table 6.

Proof. Let us prove successively the correlations the numbers of which are indicated in the intercrosses of the rows and columns of Table 5.

1) The condition $\underline{\overline{\mathrm{VI}}}^{3^\circ}$ is easily seen to coincide with 2°. In sect. 4 we have demonstrated that $\Lambda^{2^\circ} = \Lambda^{\widetilde{UI}(x,X)}$. Hence

$$\Lambda_H \cap \Lambda^{2^\circ} \cap \Lambda^{3^\circ} = \Lambda^{2^\circ} \cap \Lambda^{3^\circ} \quad \Lambda^{\widetilde{UI}(x,X)} \cap \Lambda^{3^\circ} = \Lambda^{\widetilde{UI}}$$

2),4) Follows from the fact that condition $\underline{\overline{\mathrm{VIII}}}^{3^\circ}$ coincides with the definition of the operator "neutral partial intersection" with additional possibilities $\Omega = \phi$ or $\Omega^B = \{\phi\}$.

3),5) Follows from the fact that condition $\underline{\mathrm{VIII}}^{3^\circ}$ coincides with the definition of the operator "neutral partial union" with additional possibilities $\Omega = \phi$ or $\Omega^B = \{\phi\}$.

6),7). The joint fulfilment of conditions $\overline{VII}\ 3^\circ$ and $\overline{VIII}\ 3^\circ$ determines operators with $\Omega^B = \{\omega_1\}$ where $Card(\omega_1) = 1$ or operators with $\Omega = \phi$.

8). Follows from 1) and the following obvious correlation

$$\Lambda^{\widetilde{\omega I}} \cap \Lambda^{4^\circ} = \Lambda^{\widetilde{kP}}$$

9)–14). Follows from 2)–7), resp., taking into account that operators 0 and 1 satisfy condition 4°. #

Theorems 8 and 9 give rise to the following as corollaries

Corollary 1. The domain $H \cap C$ is closed relative to $\mathbb{L}$-operator from the Central Region if and only if this operator is an operator "neutral partial intersections".

Corollary 2. The domain $H \cap O$ which consists of all choice functions satisfying the Plott's condition of "independence of the path" is closed relative to $\mathbb{L}$-operator from the Central Region if and only if this operator is an operator "neutral partial union".

Corollary 3. The domain $H \cap C \cap O$ is closed relative to $\mathbb{L}$-operator from the Central Region if and only if this operator is an operator "decisive voter".

Remark. It is interesting to consider the problem about the closedness of class K , which is narrower than the classes studied above. It may be shown that this class is closed relative to $\mathbb{L}$-operator from Central Region if and only if this operator is an operator "decisive voter".

7. Discussion

Discussing the problems arisen and solved in this paper we will draw the readers' attention to the main ideas and results established in the new formulation of voting problem, i.e. when the profile comprising of individual choice function has to be transformed to the collective choice function. The introduction of locality condition (see sect.2) which defines the operators performing such transformation allows to use the following general scheme for analise these operators - to stady three categories of operators classes which are: a) isolated with characteristic conditions (sect. 3), b) defined as some mechanisms or deterministic rules (sect. 4) and, finally, c) picked out with closedness condition of domains in space $\mathcal{C}$ (sect. 5). The main results are obtained comparing these categories of operators classes (sect. 5 and 6). We want to emphasize two following results: a) two principles of mutually exclusive neutralities (sect. 5) which establish that natural demands to voting systems lead to inconsistency of neutralities - to voters and to variants; b) the establishing of exact structure of Central Region comprising natural voting systems in terms of its mutual relations with two last categories of operator classes (see Theorem 8 and Fig.9).

Moreover this comparison mentioned above shows also the emptiness of intersection of some classes of operators (see sect. 6) or in other words, the inconsistency of some natural conditions to which it seems voting operators have to be satisfied. This situation is analogous in some sense to that one established by K.J.Arrow and his followers when they regarded voting operators which transform the totality of binary relations to collective binary relation.

We discuss also in this section two problems pertinent to the additional analysis of local operators. The former is concerned with generating collective decisions in a specified class of choice functions for the fixed voting operator of type III. In the papers on the Arrow's paradox in its conventional interpretation (see Sen (1970) and the references thereof) we can outline a trend which studies a possibility of obtaining a resultant binary relation in a class of linear quasiorders using a "neutral simple majority" rule (i.e. the rule of "neutral k-majority" for $k = [\frac{n}{2}] + 1$).

In a general case when a profile consists of arbitrary linear quasi-orders this is not possible. Hence, the above papers of this trend are striving among other things, to answer the following question: what mutual constraints should be observed by the initial "preference relations" of the voters for the resultant relation to belong a priori to a class of linear quasi-orders?

It is interesting to obtain similar results for a problem of collective choice in its "functional" interpretation used herein. For the case when the operator $\bigvee$ "at least one vote ay" is used as an operator of the group choice one of the results permitting such an interpretation has been obtained in Aleskerov, Zavalishin and Litvakov (1979) in the form of mutual constraints for n functions $C_i(\cdot)$ from the class $H \cap C \cap O$, which comprise the profile $\{C_i(\cdot)\}$. The observation of these constraints by operator $\bigvee$ results in the function $C^*(\cdot)$ in the class $H \cap C \cap O$, although this is not possible in a general case, as has already been mentioned.

Another problem that we discuss in this section deals with the comparison of the above results with those of Aizerman and Aleskerov (1983b) obtained when considering the voting operators of type I (i.e. considering the Arrow's problem in its conventional statement).

Let us consider Table 7. Its left part gives some results referring to L-operators from the Central Region. The first column shows domains[15] Q in the space $\mathcal{C}$, and the second column - L-operators from the Central Region which ensure that domain Q is closed (the intersection $\Lambda_Q \cap \Lambda^{CR}$) The right part gives a similar table for the voting operators of type I (see Aizerman and Aleskerov (1983b). Its first column describes regions $\mathcal{R}$ in the space of all binary structures, the second column - the operators which constitute the intercross of a complete class $\Lambda_{\mathcal{R}}$ of the operator closedness for $\mathcal{R}$ and the Central Region (for the voting operators of type I the Central Region is determined by the totality of conditions similar to 1°_{+}, 1°_{-}, 2°, and 3° - see Aizerman and Aleskerov (1983b).

The raws of these two tables coincide as follows: the first column of the left table shows a domain $Q \subset \mathcal{C}$ which is realized by a conventional choice from binary structures of the domain $\mathcal{R}$ which is indicated in the first column of the right table.

For rows 4 and 5 of the right table there is no correspondence since classes $H \cap O$ and H cannot be realised by the pair-dominant mechanisms of choice.

From Table 7 one can immediately see the "shift" of statements concerning L-operators relative to the statements on

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15) The domains $\hat{Q}$ given in the first two rows of this column consist of functions belonging to the intersection of the domain Q with the space $\hat{\mathcal{C}}$ of non-empty choice functions, i.e. $\hat{Q} = Q \cap \hat{\mathcal{C}}$

the voting operators of type I. Thus, for instance, for a domain of binary relations of the partial order the operator closedness (using the terminology of this paper) is ensured by the operators of "partial intersection" (see Aizerman and Aleskerov (1983b)). On the other hand, a domain in $\mathcal{C}$ generated by a pair-dominant rule of choice in this class of binary relations is closed only to L-operators "decisive voter". Meanwhile a considerably wider domain of choice functions, the domain C, generated by the "non-conventional" rules of choice on "non-conventional" structures (see Aizerman and Malishevski (1981)) which have completely been omitted from the discussion of the Arrow's paradox is closed to the arbitrary L-operators "partial intersection".

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Table 1

lists \ numbers	1	2	3	4	5	7	8	9	10	11	13	14	15
$\Omega(x,A)$	ϕ	$\{1\}$	ϕ	$\{\phi\},\{1\}$	$\{1\},\{1,2\},\{1,3\},N$	$\{1\},\{2\},\{3\}$	$\{1\},\{2\},\{3\}$	$\{1\},\{2\},\{3\}$	$\{\phi\},\{1\},\{2\},\{3\}$	N	$\{1\},N$	$\{\phi\},\{1\},N$	$\{1,2\}$
$\Omega(x,\{x,y\})$	2^N	2^N	ϕ	$\{\phi\}$	ϕ	ϕ	2^N	ϕ	$\{\phi\}$	ϕ	2^N	N	2^N
$\Omega(x,\{x,z\})$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	2^N	2^N	2^N	ϕ	$\{\phi\}$	2^N	2^N	N	N
$\Omega(x,\{x\})$	ϕ	$\{1\}$	$\{\phi\}$	$\{\phi\}$	ϕ	ϕ	2^N	$\{\phi\}$	$\{\phi\}$	2^N	2^N	N	N
lists \ numbers	**17**	**18**	**19**	**22**	**23**	**24**	**26**	**27**	**28**	**31**	**32**	**35**	**36**
$\Omega(x,A)$	$\{1\},\{2\},\{3\},N$	$\{1\},\{2\},\{3\},N$	2^N	$\{1\}$	$\{1\}$	$\{1,2\},N$	$\{1\},\{2\},\{3\}$	$\{1\},\{2\},\{3\}$	ϕ	$\{1\},N$	$\{1,2\},N$	$\{1\},\{2\},\{3\},N$	$\{1,2\},\{1,3\},\{2,3\},N$
$\Omega(x,\{x,y\})$	2^N	N	$\{\phi\},N$	ϕ	$\{1\},\{2\}$	ϕ	ϕ	N	$\{1\},\{2\},\{3\},N$	N	N	N	N
$\Omega(x,\{x,z\})$	N	N	N	ϕ	$\{1\}$	ϕ	ϕ	N	ϕ	N	N	N	N
$\Omega(x,\{x\})$	N	N	N	ϕ	$\{1\}$	ϕ	ϕ	N	ϕ	N	N	N	N

Table 2

X / $C_i(\cdot)$	$\{x,y,z\}$	$\{x,z\}$	$\{x,y\}$	$\{x\}$
$i \in (N \setminus \omega) \cap (N \setminus \omega') \cap (N \setminus \tilde{\omega})$	ϕ	ϕ	ϕ	ϕ
$i \in (N \setminus \omega) \cap \omega' \cap (N \setminus \tilde{\omega})$	y	ϕ	y	ϕ
$i \in (N \setminus \omega) \cap \omega' \cap \tilde{\omega}$	y	x	y	x
$i \in \omega \cap (N \setminus \omega') \cap \tilde{\omega}$	x	x	x	x
$i \in \omega \cap \omega' \cap \tilde{\omega}$	x, y	x	x, y	x

Table 3

$C_i(\cdot)$ \ X	$\{x,y,z\}$	$\{x,z\}$
$i\in(N\setminus\omega)\cap(N\setminus\omega')\cap(N\setminus\omega'')\cap(N\setminus\tilde{\omega})\cap(N\setminus\tilde{\omega}'')$	ϕ	ϕ
$i\in(N\setminus\omega)\cap\omega'\cap(N\setminus\omega'')\cap(N\setminus\tilde{\omega})\cap(N\setminus\tilde{\omega}'')$	y	ϕ
$i\in\omega\cap(N\setminus\omega')\cap(N\setminus\omega'')\cap\tilde{\omega}\cap(N\setminus\tilde{\omega}'')$	x	x
$i\in(N\setminus\omega)\cap\omega'\cap(N\setminus\omega'')\cap\tilde{\omega}\cap(N\setminus\tilde{\omega}'')$	y	x
$i\in(N\setminus\omega)\cap\omega'\cap(N\setminus\omega'')\cap(N\setminus\tilde{\omega})\cap\tilde{\omega}''$	y	z
$i\in(N\setminus\omega)\cap(N\setminus\omega')\cap\omega''\cap(N\setminus\tilde{\omega})\cap\tilde{\omega}''$	z	z
$i\in\omega\cap\omega'\cap(N\setminus\omega'')\cap\tilde{\omega}\cap(N\setminus\tilde{\omega}'')$	x,y	x
$i\in(N\setminus\omega)\cap\omega'\cap\omega''\cap(N\setminus\tilde{\omega})\cap\tilde{\omega}''$	y,z	z
$i\in(N\setminus\omega)\cap\omega'\cap(N\setminus\omega'')\cap\tilde{\omega}\cap\tilde{\omega}''$	y	x,z
$i\in(N\setminus\omega)\cap\omega'\cap\omega''\cap\tilde{\omega}\cap\tilde{\omega}''$	y,z	x,z
$i\in\omega\cap(N\setminus\omega')\cap\omega''\cap\tilde{\omega}\cap\tilde{\omega}''$	x,z	x,z
$i\in\omega\cap\omega'\cap(N\setminus\omega'')\cap\tilde{\omega}\cap\tilde{\omega}''$	x,y	x,z
$i\in\omega\cap\omega'\cap\omega''\cap\tilde{\omega}\cap\tilde{\omega}''$	x,y,z	x,z

Table 4

lists \ classes	$\bar{\Lambda}_H \cap \bar{\Lambda}_O \cap \bar{\Lambda}_C$	$\Lambda_H \cap \bar{\Lambda}_O \cap \bar{\Lambda}_C$	$\bar{\Lambda}_H \cap \Lambda_O \cap \bar{\Lambda}_C$	$\bar{\Lambda}_H \cap \bar{\Lambda}_O \cap \Lambda_C$	$\bar{\Lambda}_H \cap \Lambda_O \cap \Lambda_C$	$\Lambda_H \cap \bar{\Lambda}_O \cap \Lambda_C$	$\Lambda_H \cap \Lambda_O \cap \bar{\Lambda}_C$	$\Lambda_H \cap \Lambda_O \cap \Lambda_C$
$\Omega(x, A)$	$\{1,2\}, N$	$\{1,2\}, N$	$\{1,2\}, N$	$\{1,2\}, \{1,3\}, N$	2^N	$\{1\}, \{1,2\}$ $\{1,3\}, N$	$\{1,2\}, N$	$\Omega(x, X) = \phi$ $\forall x, X : x \in X \in \mathcal{A}$
$\Omega(x, \{x,y\})$	$\{1,3\}, N$	$\{1,2\}, \{1,3\}, N$	2^N	$\{1,3\}, N$	2^N	$\{1,2\}, N$	2^N	—
$\Omega(x, \{x,z\})$	$\{1,2\}, N$	$\{1,2\}, N$	$\{1,3\}, N$	N	$\{1,3\}, N$	$\{1,3\}, N$	$\{1,2\}, \{1,3\}, N$	—

Table 5

II \ I	Λ_H	Λ_C	Λ_O	$\Lambda_H \cap \Lambda_O$	$\Lambda_H \cap \Lambda_O$	$\Lambda_C \cap \Lambda_O$	$\Lambda_H \cap \Lambda_C \cap \Lambda_O$
Λ^{BR}	—	—	1) Λ^{PU}	—	2) Λ^{PU}	3) $\Lambda^{\mathcal{D}V}$	4) $\Lambda^{\mathcal{D}V}$
$\Lambda^{BR} \cap \Lambda^{3^\circ\text{-}a}$	5) $\Lambda^{UI(X)}$	6) $\Lambda^{PI(X)}$	7) Λ^{PU}	8) $\Lambda^{PI(X)}$	9) Λ^{PU}	10) $\Lambda^{\mathcal{D}V}$	11) $\Lambda^{\mathcal{D}V}$
Λ^{CR} $\Lambda^{BR} \cap \Lambda^{3^\circ}$	12) Λ^{UI}	13) Λ^{PI}	14 Λ^{PU}	15) Λ^{PI}	16) Λ^{PU}	17) $\Lambda^{\mathcal{D}V}$	18) $\Lambda^{\mathcal{D}V}$
Λ^{SCR}	19) Λ^{xP}	20) U	21) V	22) U	23) V	24) ϕ	25) ϕ

Table 6

II \ I	Λ_H	Λ_C	Λ_O	$\Lambda_H \cap \Lambda_C$	$\Lambda_H \cap \Lambda_O$	$\Lambda_C \cap \Lambda_O$	$\Lambda_H \cap \Lambda_O \cap \Lambda_C$
$\Lambda^{2^\circ} \cap \Lambda^{3^\circ}$	1) $\Lambda^{\widetilde{UI}}$	2) $\Lambda^{\widetilde{PI}}$	3) $\Lambda^{\widetilde{PU}}$	4) $\Lambda^{\widetilde{PI}}$	5) $\Lambda^{\widetilde{PU}}$	6) $\Lambda^{\widetilde{\mathcal{D}V}}$	7) $\Lambda^{\widetilde{\mathcal{D}V}}$
$\Lambda^{2^\circ} \cap \Lambda^{3^\circ} \cap \cap \Lambda^{4^\circ}$	8) $\Lambda^{\widetilde{kP}}$	9) $U, 0, 1$	10) $V, 0, 1$	11) $U, 0, 1$	12) $V, 0, 1$	13) $0, 1$	14) $0, 1$

Table 7

1	$\hat{K}$	decisive voter $C^*(X) = C_{i^*}(X)$	weak orders	dictator $G^* = G_{i^*}$
2	$\hat{H} \cap \hat{C} \cap \hat{O}$ $(C \cap O)$	decisive voter $C^*(X) = C_{i^*}(X)$	partial orders	oligarchy $G^* = \bigcap_{i \in \omega} G_i$
3	$H \cap C$ (C)	partial intersection $C^*(X) = \bigcap_{i \in \omega} C_i(X)$	all binary relations	federation $G^* = \bigcup_{j=1}^{\ell} \bigcap_{i \in \omega_j} G_i$
4	$H \cap O$ (O)	partial union $C^*(X) = \bigcup_{i \in \varepsilon} C_i(X)$		
5	H	union of intersections $C^*(X) = \bigcup_{j=1}^{\ell} \bigcap_{i \in \omega_j} C_i(X)$ or intersection of unions $C^*(X) = \bigcap_{j=1}^{t} \bigcup_{i \in \varepsilon_j} C_i(X)$		

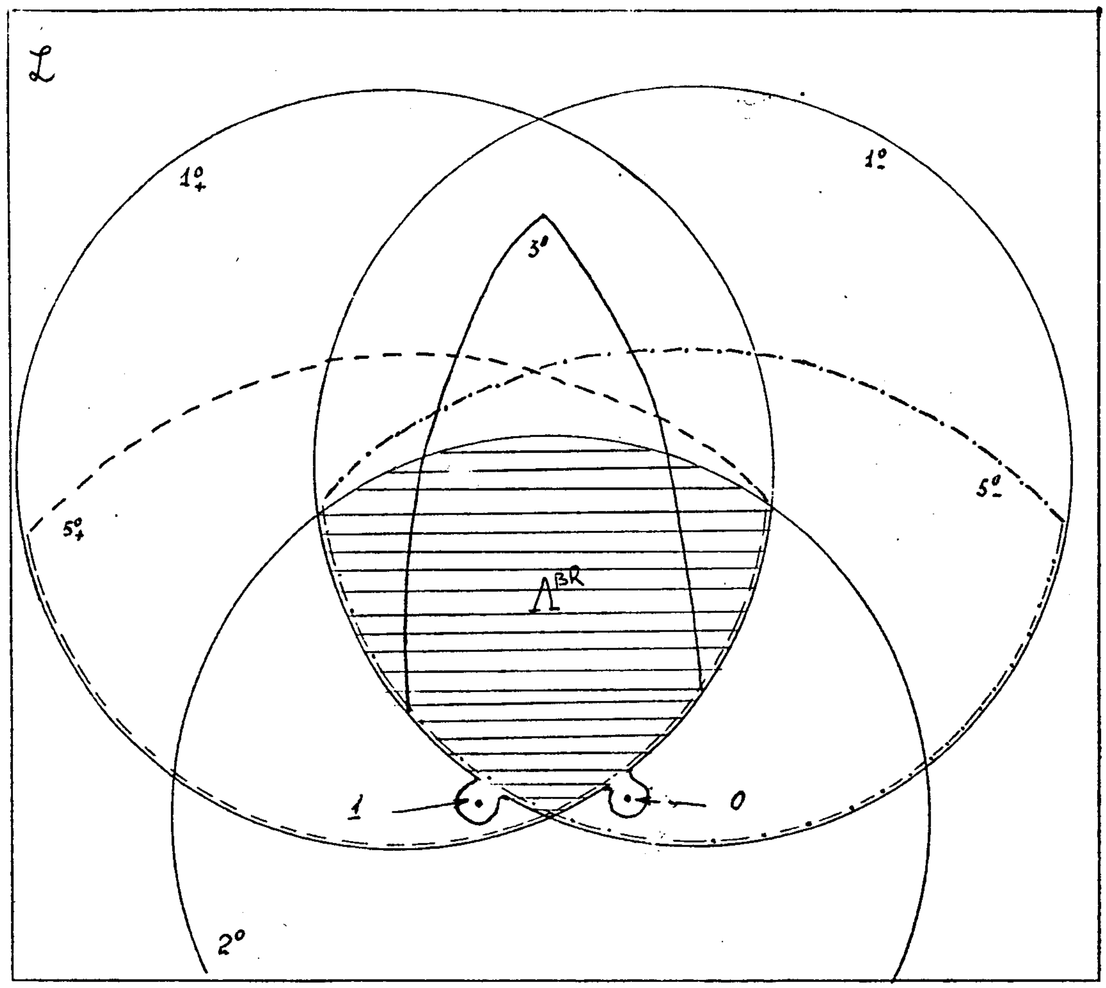

Fig. 1

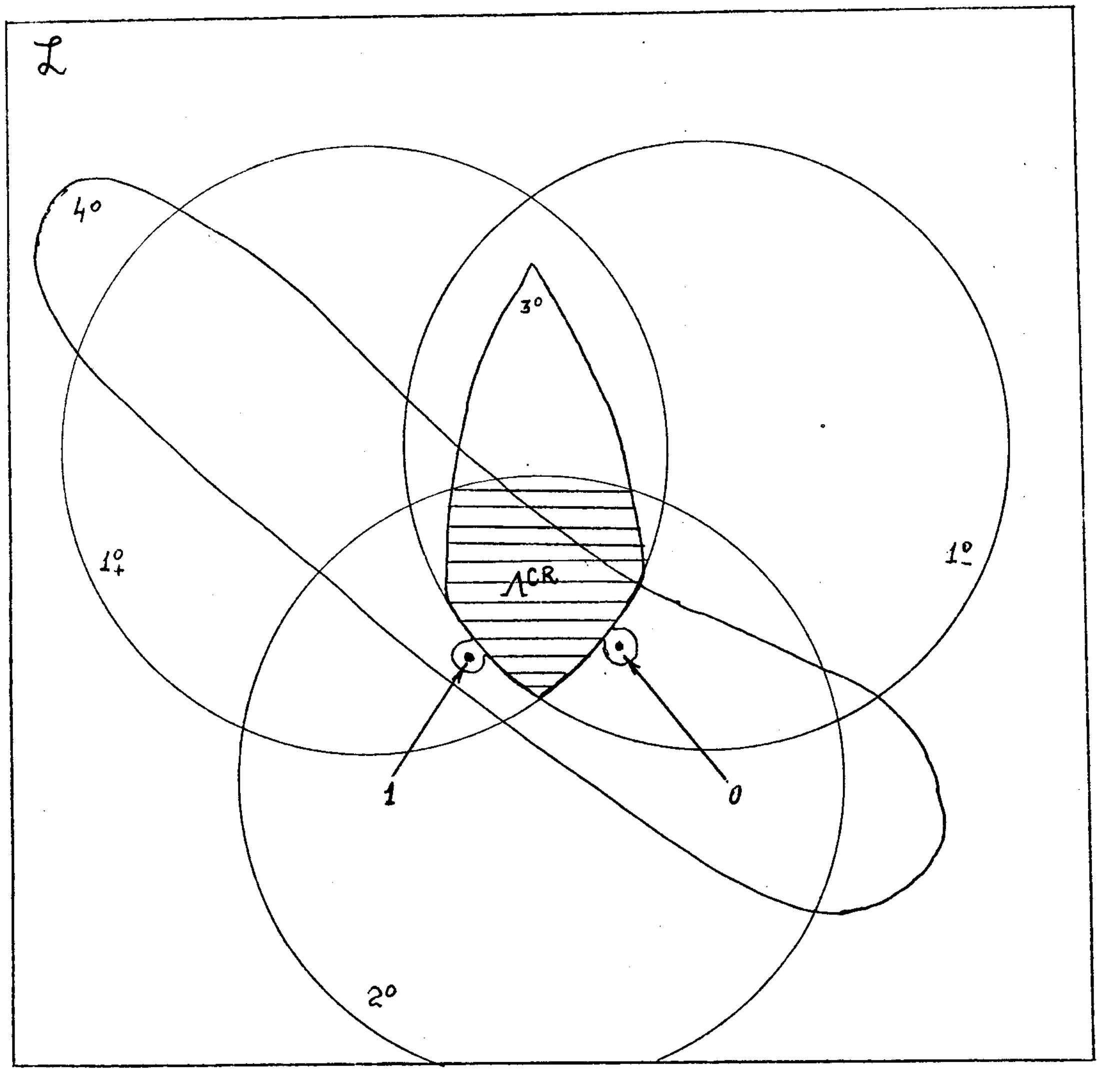

Fig. 2

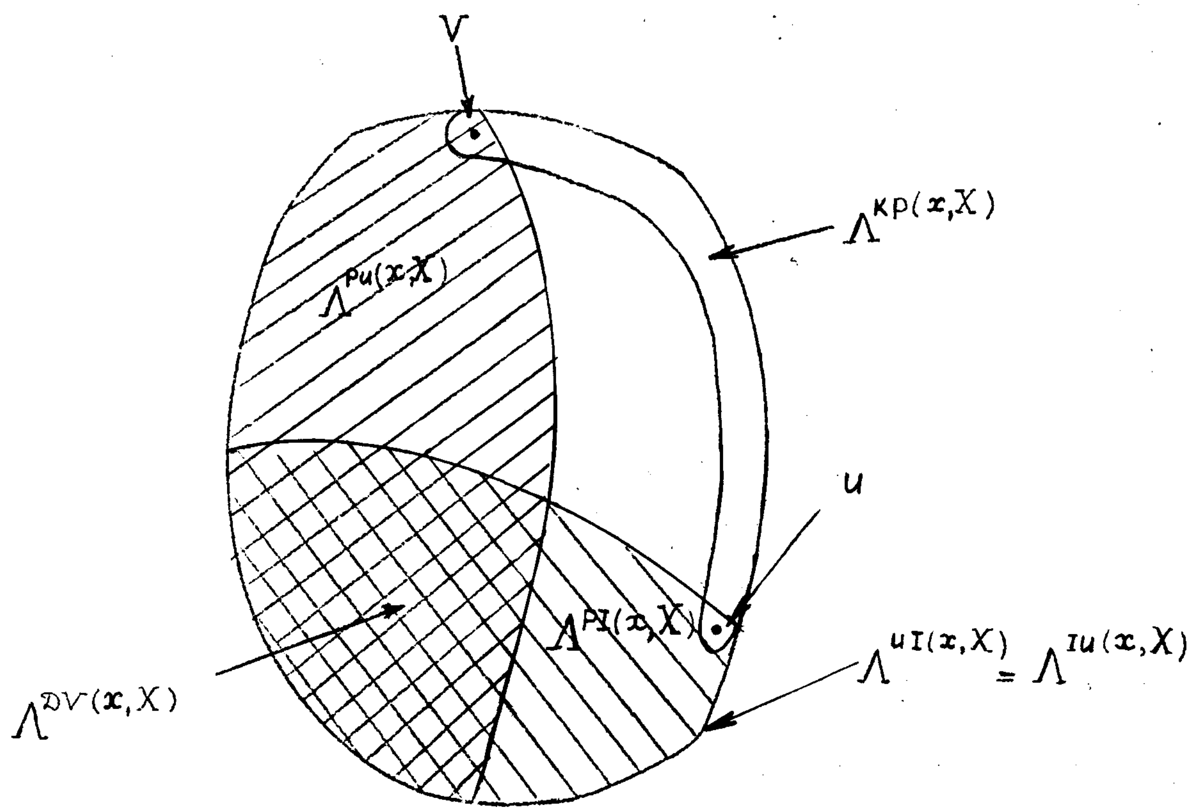
V
$\Lambda^{KP(x,X)}$
$\Lambda^{Pu(x,X)}$
u
$\Lambda^{PI(x,X)}$
X
$\Lambda^{uI(x,X)} = \Lambda^{Iu(x,X)}$
$\Lambda^{DV(x,X)}$

Fig. 3

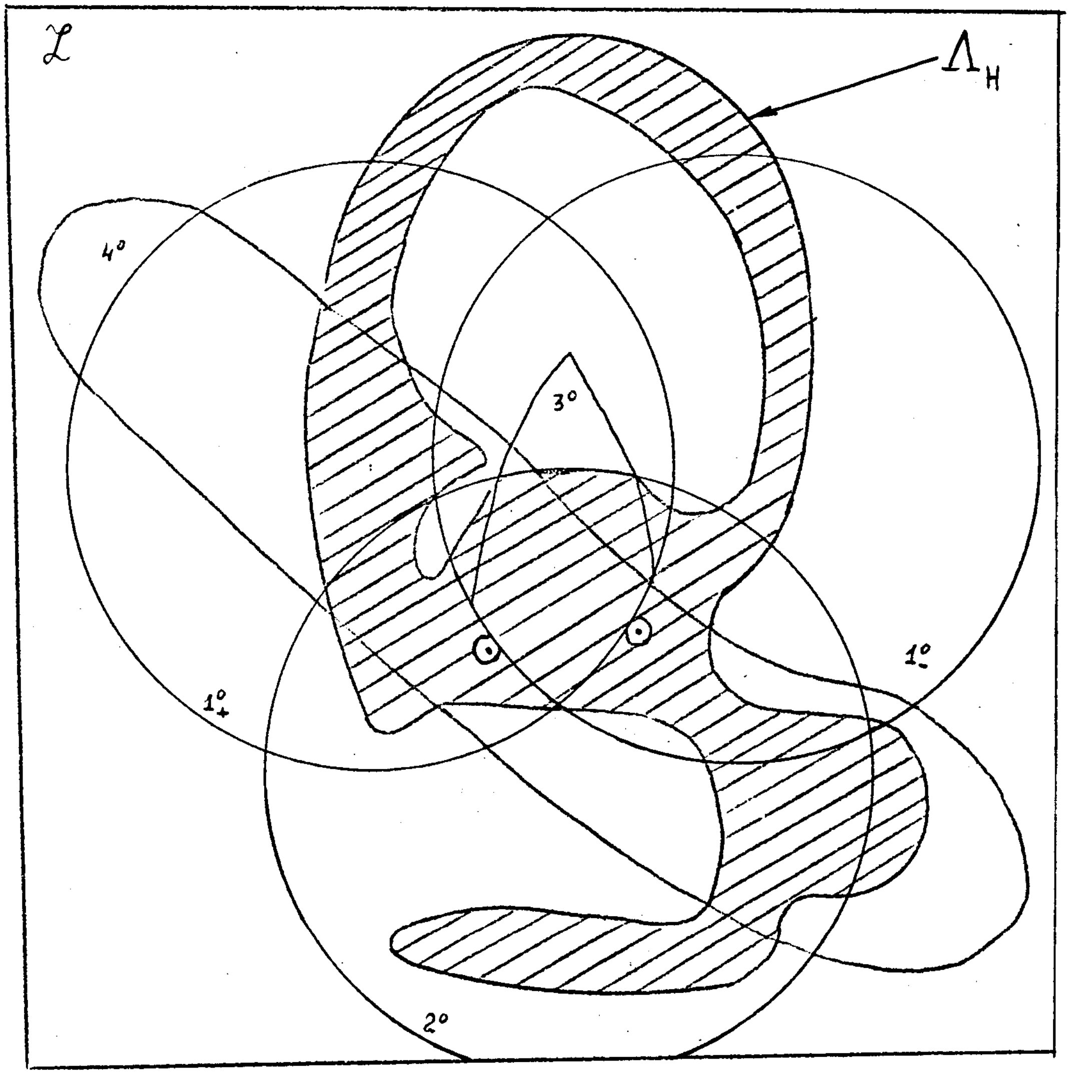

Fig. 4

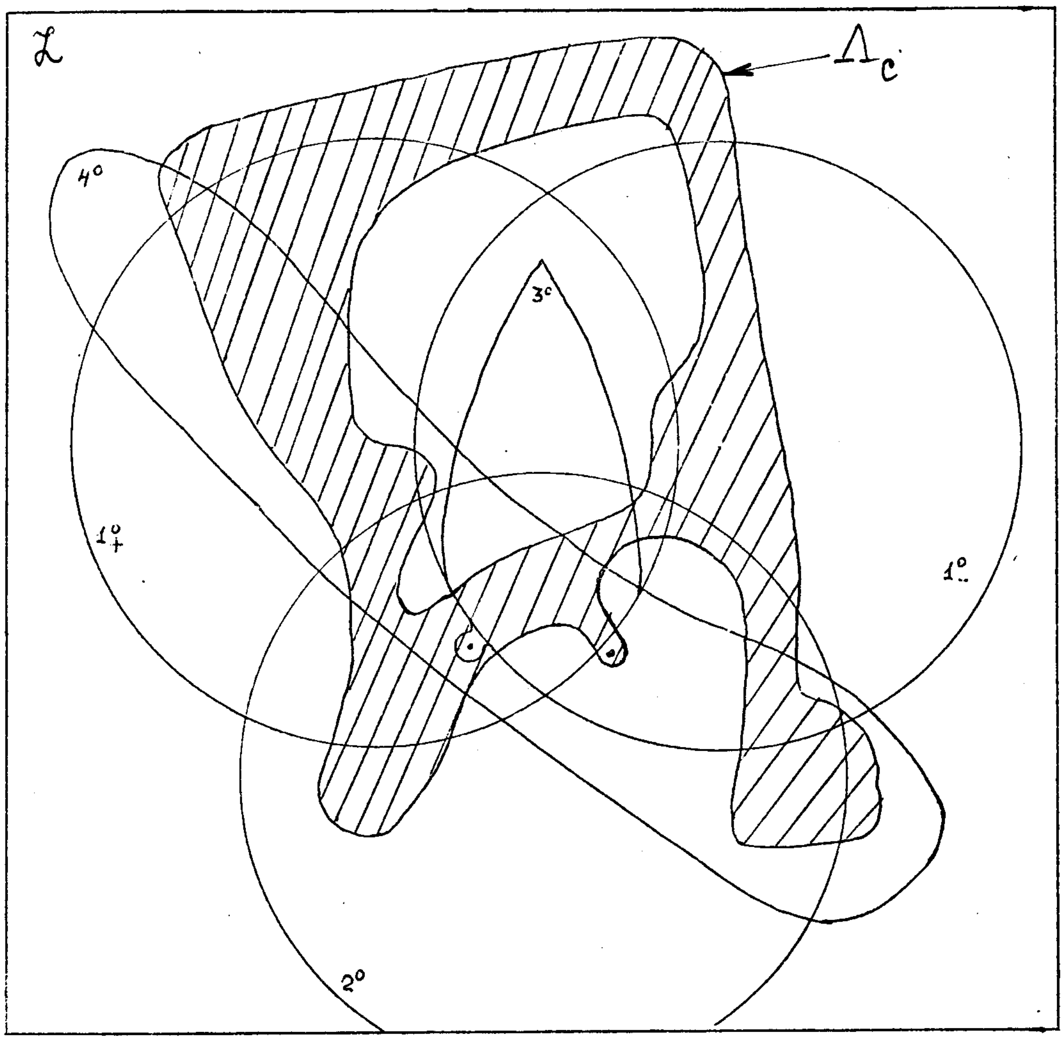

Fig. 5

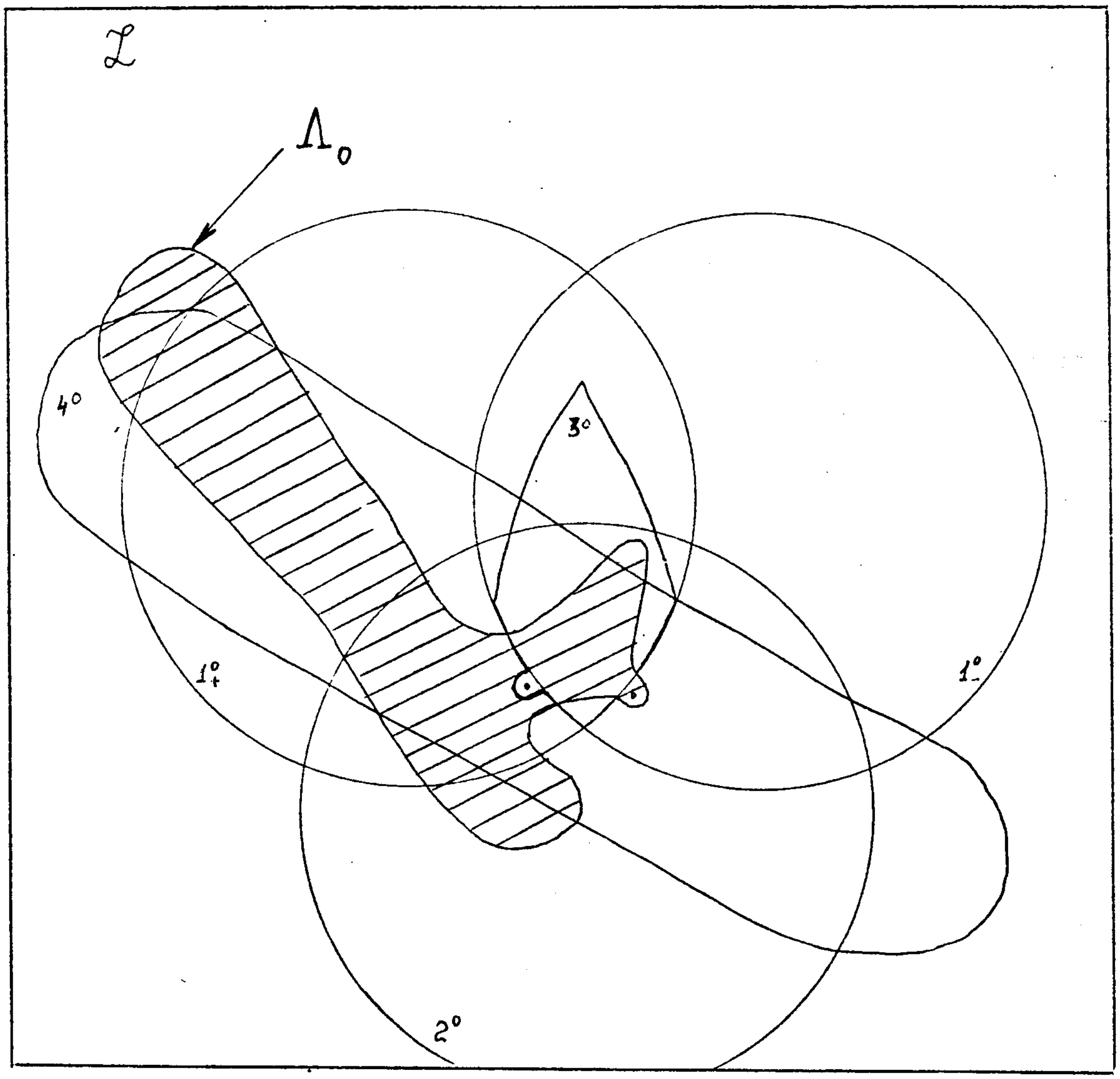

Fig. 6

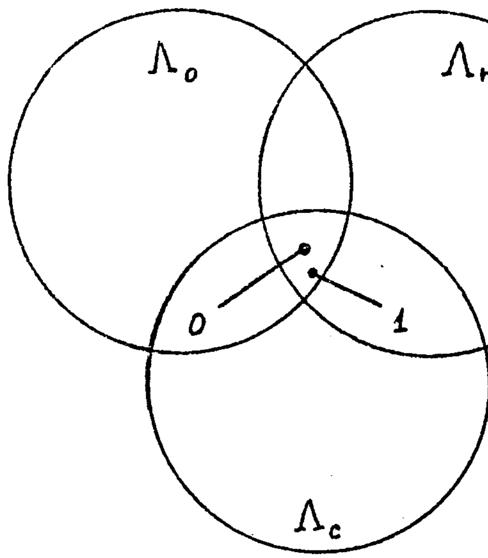

Fig. 7

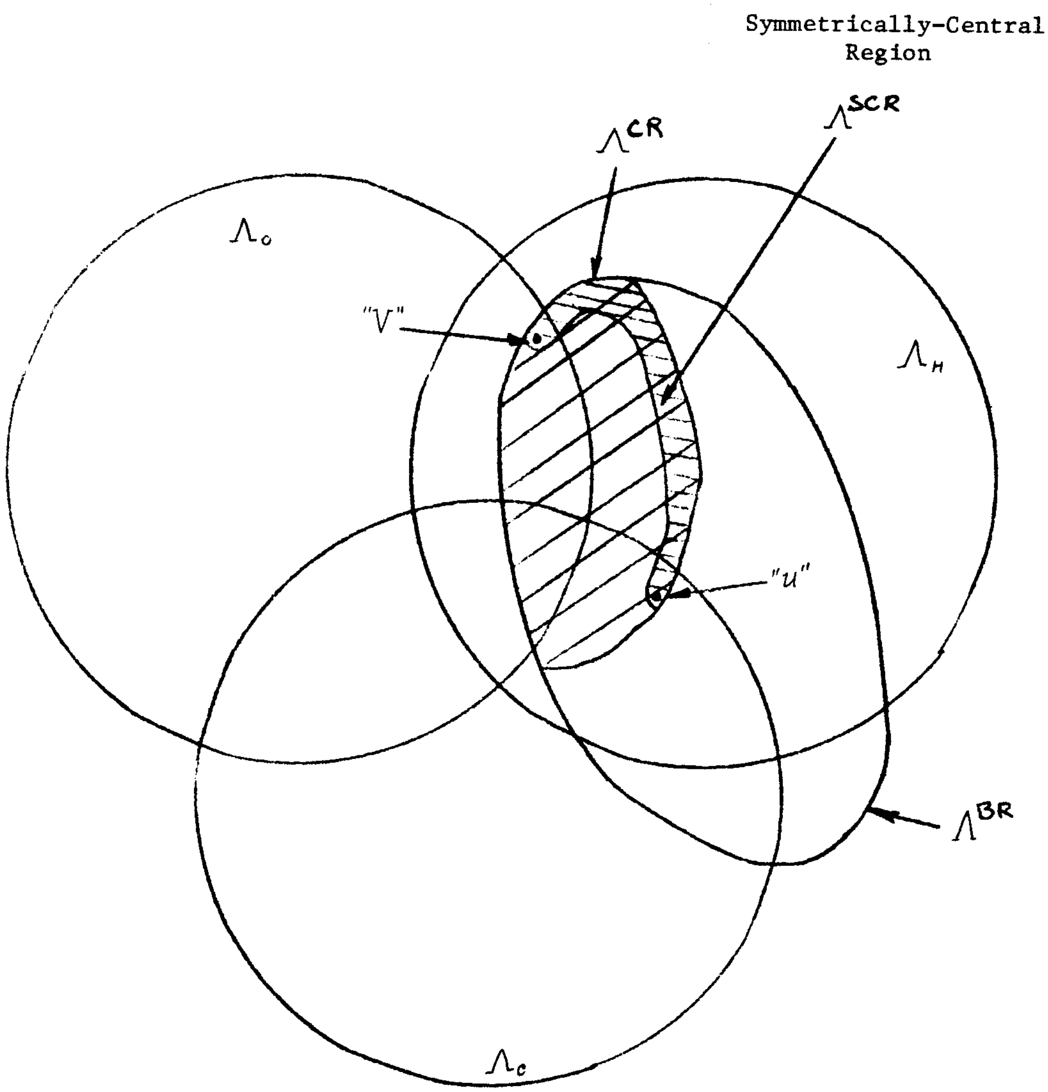
Symmetrically-Central
Region
Λ^{SCR}
Λ^{CR}
Λ_o
"V"
Λ_H
"u"
Λ^{BR}
Λ_c

Fig. 8-a

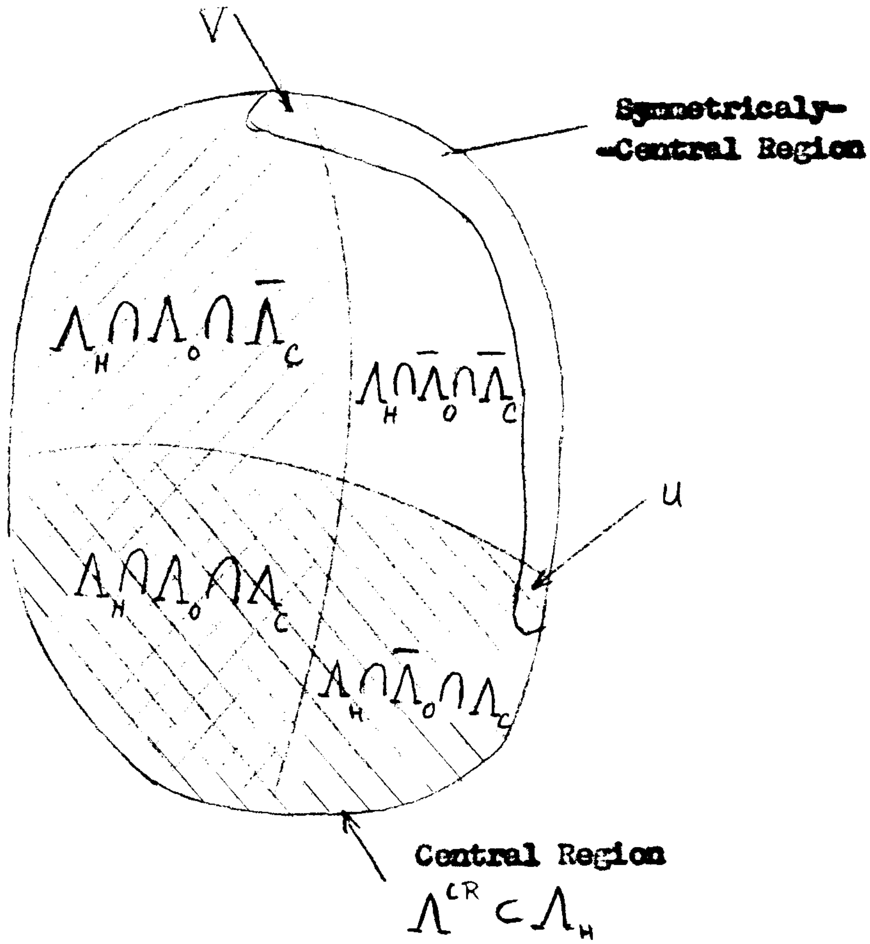
V
Symmetricaly-
-Central Region
$\Lambda_H \cap \Lambda_o \cap \bar{\Lambda}_c$
$\Lambda_H \cap \bar{\Lambda}_o \cap \bar{\Lambda}_c$
u
$\Lambda_H \cap \Lambda_o \cap \Lambda_c$
$\Lambda_H \cap \bar{\Lambda}_o \cap \Lambda_c$
Central Region
$\Lambda^{CR} \subset \Lambda_H$

Fig. 8-b

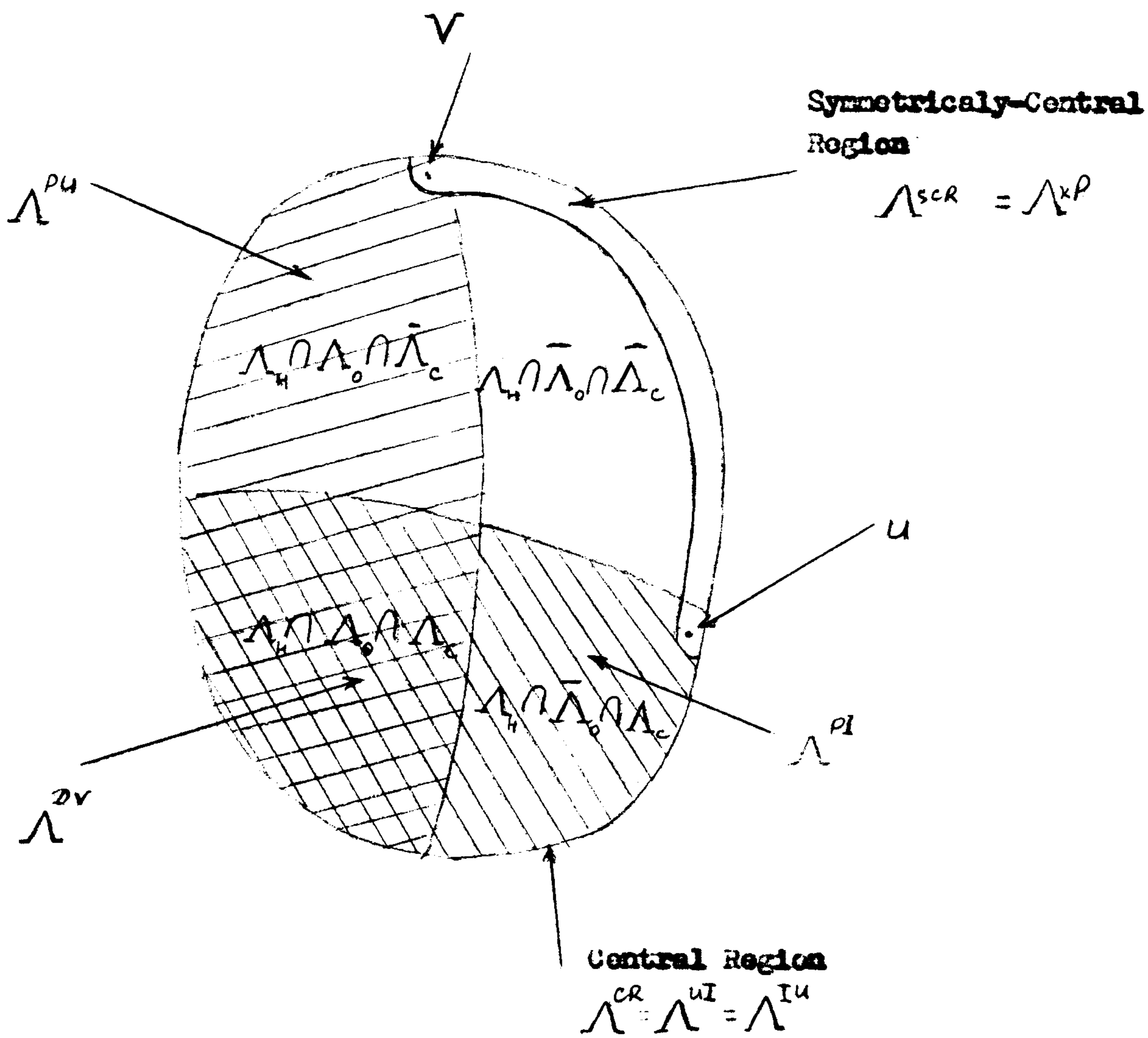
V
Symmetricaly-Central
Region
$\Lambda^{SCR} = \Lambda^{xP}$
Λ^{PU}
$\Lambda_H \cap \Lambda_O \cap \bar{\Lambda}_C$
$\Lambda_H \cap \bar{\Lambda}_O \cap \bar{\Lambda}_C$
u
$\Lambda_H \cap \Lambda_O \cap \Lambda_C$
$\Lambda_H \cap \bar{\Lambda}_O \cap \Lambda_C$
Λ^{PI}
Λ^{DV}
Central Region
$\Lambda^{CR} = \Lambda^{uI} = \Lambda^{Iu}$

Fig. 9