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ABSTRACT

Since Keynes first argued that backwardation was the normal state of affairs on futures markets, there have been several theoretical explanations for its existence. In particular, Fort and Quirk have shown that the "Houthakker effect" can lead to a backwardation equilibrium. In this paper, we consider another argument for backwardation suggested by Houthakker, namely, asymmetric arbitrage. Our conclusions are generally negative, despite its intuitive appeal. Specifically, in a world with an equal number of short and long hedgers, with identical utility functions and densities over cash and futures prices, if the futures market is a forward market, then in a rational expectations framework, asymmetric arbitrage has no effect on the pattern of futures (or cash) prices. If we are dealing with a true futures market, under the above assumptions, arbitrage will act to encourage short hedging and to discourage long hedging only under some restrictive conditions. Moreover, further quantitative restrictions must be imposed in order to derive a backwardation equilibrium under asymmetric arbitrage.

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1. Introduction

It is now over sixty years since Keynes (1930) first argued that the "normal" state of affairs on futures markets was one of backwardation, here interpreted as a situation in which the current price of a futures contract is less than its expected price at maturity of the futures contract. Keynes argued that short hedgers (long in the cash market, short in the futures market) would pay a risk premium to speculators, this premium representing the degree of backwardation in the market. Keynes did not explain why it was that only short hedgers, and not both short and long hedgers, would have to pay such a premium. (Long hedgers are long in the futures market and short in the cash market). Later, Hicks (1965) argued that a preponderance of short over long hedgers was to be expected because purchasers of inputs have more possibilities of substitution available to them than do the producers of a commodity. Kaldor (1939) admitted the possibility of an excess of long over short hedging on the market, in part because of the quantity risks that a producer exposes himself to if he engages in a hedge to avoid price risks. In the more recent literature, backwardation and the preponderance of short over long hedging has been attributed to information asymmetries (Danthine (1978)), to a highly elastic demand for the final good (Macminn,

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Morgan and Smith (1984)) or to the fact that futures contracts provide poor consumption hedges (Richard and Sundaresan (1981)), and sometimes backwardation is simply imposed ad hoc as a condition of the model describing the futures market (Baesel and Grant (1982)).

In this paper, we explore an explanation for backwardation advanced by Houthakker (1959), namely the idea that arbitrage on the futures market is asymmetric in such a way as to favor short hedgers over long hedgers. The idea here is that at any point in time, the futures price cannot exceed the cash price plus carrying costs to the maturity date of the futures contract, since otherwise there is a riskless profit to be earned by selling a futures contract, buying cash and storing to deliver on the futures. Arbitrage thus provides an upper limit on the amount by which the futures price can exceed the cash price, but there is no corresponding arbitrage operation available to limit the amount by which the cash price can exceed the futures price.

Actually, Houthakker suggested two explanations for backwardation in his seminal work of the 1950s and 1960s, the second being the tendency for the delivery options admissible under a futures contract to be better substitutes for one another at low than at high cash prices. In a recent paper, Fort and Quirk (1984) show that under an appropriate specification of such a "Houthakker effect," a backwardation equilibrium can be constructed, even when there is an equal number of short and long hedgers on the market, with identical utility functions and densities over cash and futures prices.

Briefly, our results in the present paper are the following. In a world with an equal number of short and long hedgers, with identical utility functions and densities over cash and futures prices, then asymmetric arbitrage has no effect on the pattern of cash and futures prices when the futures market is in fact a forward market, that is, a market in which the cash and futures prices are identically equal at maturity of the futures contract. In such a world, under rational expectations, the resulting equilibrium is a martingale equilibrium in the futures market (current price of the futures contract equals its expected price next period), with the current futures price equal to the current cash price plus carrying costs to maturity of the futures contract.

The situation is different in a true futures market, that is, a market in which there are two or more delivery options admissible under the futures contract, the options being less than perfect substitutes for one another.

In a true futures market, the effect of asymmetric arbitrage on a previous martingale equilibrium is indeterminate in the general case; it might be to produce a backwardation equilibrium, or a contango, or no change at all. Given an arbitrary symmetric joint density over the cash and futures prices and given an arbitrary concave utility function for traders, the introduction of asymmetric arbitrage does not even necessarily encourage short hedging and discourage long hedging, despite the intuitive appeal of Houthakker's argument. For one very special case, that of a uniform density with the utility function satisfying constant or decreasing absolute risk aversion, we show that Houthakker's conjecture concerning the effect of aymmetric arbitrage on hedging patterns holds, so long as cash and futures commitments are technical complements for one another. But even in this special case, additional restrictions need to be imposed to guarantee a backwardation equilibrium. Moreover, imposing a rational expectations framework on the model of the futures market implies that given a T-period futures contract, the effects of asymmetric arbitrage show up only in the futures markets for periods T-1 and T while in earlier periods, the futures market behaves like a forward market. In effect, rational expectations, by precluding the possibility of capital gains by traders in earlier periods, rules out speculation as a market force during those periods.

The upshot of all this is that, despite its intuitive appeal, Houthakker's argument for backwardation based on asymmetry of arbitrage has no standing when the market is a forward market, and is at best highly conjectural when applied to a true futures market.

2. The Model

We consider a world in which there is a futures market as well as cash markets in the commodity options deliverable under the futures contract. This is a T period ($t = 0,1,2,\dots,T$) world. There is one futures contract available, maturing at time T. Traders on the futures market are long (L) hedgers, short (S) hedgers, and speculators. All traders are assumed to have the same strictly

concave utility function over income, and the same probability beliefs concerning futures and cash prices for periods in the future.

Let p_t^c denote the cash price at time t of a commodity option deliverable under the futures contract. Let p_t^f denote the price of the futures contract at time t. W_t^S and W_t^L are the cash commitments at time t of each short and long hedger, and V_t^S, V_t^L are their futures commitments at time t. π_t^S, π_t^L denote the profits of short and long hedgers over the period beginning at time t-1 and ending at time t, given by

$$\begin{aligned}\pi_t^S &= (p_t^c - p_{t-1}^c)W_{t-1}^S + (p_{t-1}^f - p_t^f)V_{t-1}^S - k_{t-1}(W_{t-1}^S) + \pi_t^* \\ \pi_t^L &= (p_{t-1}^c - p_t^c)W_{t-1}^L + (p_t^f - p_{t-1}^f)V_{t-1}^L + k_{t-1}(W_{t-1}^L) + \pi_t^*\end{aligned} \qquad (1)$$

In these expressions, short hedgers are viewed as elevator operators and long hedgers are viewed as millers. Thus short hedgers buy cash wheat to store it, and sell futures contracts to hedge their cash commitments. Long hedgers are assumed to undertake commitments to deliver flour in the future at a wheat-equivalent price equal to the current cash price plus carrying costs. They buy the cash wheat required for milling at the time that flour is to be delivered, hedging these projected wheat purchases by current purchases of futures. Both long and short hedgers are assumed to terminate their cash and futures positions at time t, the end of the period. $k_{t-1}(\cdot)$ is a strictly concave function representing the "carrying costs" associated with cash commitments, including convenience yield as well as interest, warehousing, insurance and the like, associated with carrying a unit of the commodity during the t^{th} period. For symmetry, the $k_{t-1}(\cdot)$ function is assumed to be the same for both short and long hedgers. π_t^* is profits from activities not directly related to the cash and futures commitments, and again for symmetry is assumed to be the same for both short and long hedgers.

The objective functions for the hedgers are then given by

$$Eu^S = \sum_{t=1}^{T} \delta^t Eu(\pi_t^S)$$

$$Eu^L = \sum_{t=1}^{T} \delta^t Eu(\pi_t^L) \tag{2}$$

where δ is a discount factor.

The timing of decisions is the following. At time t = 0, p_0^f and p_0^c are known and the cash and futures commitments W_0^S, W_0^L, V_0^S, V_0^L are undertaken. At time t = 1, the cash and futures markets reopen and p_1^f, p_1^c are known at the time that W_1^S, W_1^L, V_1^S, V_1^L are undertaken. The process continues for t = 2,...,T − 1. Finally, at time t = T, the horizon ends with the futures contract maturing and with the cash and futures market again open.

It is assumed that the commodity in question is a seasonal good. Time t = 0 can be thought of as the harvest time, with no harvest occurring again until after time t = T. Thus all of the commodity available for use at time t = 1 to t = T is represented by the cash commitments of short hedgers (elevator operators) at time t = 0. Similarly, it is assumed that all of the commitments for consumption at t = 1 to t = T are represented by the cash commitments

of long hedgers (millers) at time t = 0. Assuming an equal number of identical short and long hedgers, we have the following market clearing conditions.

<u>Cash Markets</u>:

$$W_t^S = W_t^L, \ t = 0, 1, \ldots, T - 1 \tag{3}$$

<u>Futures Markets</u>:

$$V_t^S = V_t^L + V_t^{spec}, t = 0,1,\ldots,T - 1 \tag{4}$$

where V_t^{spec}, t = 0,1,2,...,T − 1 is the amount of purchases of futures contracts by pure speculators. Speculators buy futures whenever expected profits from purchases are positive ($Ep_t^f > p_{t-1}^f$) and sell futures whenever expected profits from sales are positive ($Ep_t^f < p_{t-1}^f$). We assume that the aggregate (excess) demand functions for futures by speculators are of less than infinite elasticity.

In describing the pattern of prices on the futures market, we use the following terminology. The futures market attains a <u>martingale equilibrium</u> at time t − 1 if the market clearing prices p_{t-1}^f, p_t^f satisfy the condition:

$$E(p_t^f | p_{t-1}^f) = p_{t-1}^f. \tag{5}$$

The futures market is said to exhibit <u>backwardation</u> at time t − 1, if

$$E(p_t^f | p_{t-1}^f) > p_{t-1}^f. \tag{6}$$

Similarly, the futures market exhibits a <u>contango</u> at time t − 1 if

$$E(p^f_t | p^f_{t-1}) < p^f_{t-1}. \tag{7}$$

In analyzing the effect of asymmetric arbitrage on the pattern of prices on the futures market, it is helpful to distinguish between two cases, the case of a forward market, and the case of a "true" futures market. A forward market is one where there is only one option deliverable under the futures contract, so that $p^c_T = p^f_T$ is known to be the relationship that will hold at time T between the market clearing prices on the cash and futures markets. This is the case where "perfect hedges" occur, and is the case typically studied in the theoretical literature dealing with futures markets (e.g., see Anderson and Danthine (1983)). In contrast, if two or more options are deliverable under a futures contract, with the options being less than perfect substitutes for one another, then we have the case of a "true" futures market.

Because choice of the option to deliver under the futures contract is up to the seller, buyers and sellers in a true futures market know that what will be delivered under the futures contract will be that option with the lowest cash price at time T. Hence arbitrage ensures that the relationship between equilibrium prices of the futures and any delivery option at time t = T in a true futures market takes the less restrictive form $p^f_T \leq p^c_T$; hedges now become "imperfect" and there is a nondegenerate joint pdf over p^f_T, p^c_T that must be analyzed in examining the time pattern of cash and futures prices.

Moreover, at any time $t < T$, arbitrage imposes additional

constraints on the futures or forward price, through the relationship $p^f_t \leq p^c_t + \sum_{\tau=t}^{T-1} k'_\tau$, where k'_τ is the marginal cost of carrying a unit of the commodity over the $(\tau + 1)$st period. If this constraint were violated, then there would be a riskless profit that could be earned by selling a futures, buying a delivery option on the cash market, and then holding this option to deliver at time T under the futures contract. Because arbitrage acts only to impose an upper (but not a lower) bound on p^f_t, arbitrage is asymmetric. We first investigate the effect of asymmetric arbitrage on a forward market.

3. Price Patterns in a Forward Market

Since $p^c_T = p^f_T$ is the equilibrium condition at time T in a forward market, thus at $t = T - 1$, cash and futures commitments of short and long hedgers are chosen under the degenerate joint density $f(p^c_T)(\equiv f(p^f_T))$, held in common by all traders. First order conditions for a short hedger are given by

$$\begin{aligned} &\int_0^\infty u'(\pi^S_T)[p^c_T - p^c_{T-1} - k'_{T-1}(W^S_{T-1})]f(p^c_T)dp^c_T = 0 \\ &\int_0^\infty u'(\pi^S_T)[p^f_{T-1} - p^c_T]f(p^c_T)dp^c_T = 0 \end{aligned} \tag{8}$$

Similarly, first order conditions for the long hedger are given by

$$\begin{aligned} &\int_0^\infty u'(\pi^L_T)[p^c_{T-1} + k'_{T-1}(W^L_{T-1}) - p^c_T]f(p^c_T)dp^c_T = 0 \\ &\int_0^\infty u'(\pi^L_T)[p^c_T - p^f_{T-1}]f(p^c_T)dp^c_T = 0 \end{aligned} \tag{9}$$

Consider as a possible candidate for equilibrium in the $t = T - 1$ cash and futures market the following price and commitment pattern:

$p^f_{T-1} = p^c_{T-1} + k'_{T-1}(W_{T-1})$, where $W_{T-1} = W^S_{T-1} = W^L_{T-1}, V^S_{T-1} = V^L_{T-1} = W_{T-1}$, and with $E(p^f_T | p^f_{T-1}) = p^f_{T-1}$, $E(p^c_T | p^c_{T-1}) = p^c_{T-1} + k'_{T-1}(W_{T-1})$. Here $k'_{T-1}(W_{T-1})$ represents the marginal carrying cost of carrying a unit of inventory from time $t = T - 1$ to time $t = T$.

Note that combining the two first order conditions in (8) and (9) we have

$$[p^f_{T-1} - p^c_{T-1} - k'_{T-1}(W^S_{T-1})] = 0.$$
$$[p^f_{T-1} - p^c_{T-1} - k'_{T-1}(W^L_{T-1})] = 0.$$

Since k_{T-1} is strictly concave, we have $W^S_{T-1} = W^L_{T-1} = W_{T-1}$, which satisfies the cash market equilibrium condition in (3).

Further, integrate the first integral in (8) by parts to obtain

$$u'(\pi^S_T)\int_0^{p^c_T}[x - p^c_{T-1} - k'_{T-1}(W^S_{T-1})]f(x)dx \Big|_0^\infty \tag{10}$$
$$-(W^S_{T-1} - V^S_{T-1})\int_0^\infty u''(\pi^S_T)\int_0^{p^c_T}[x - p^c_{T-1} - k'_{T-1}(W^S_{T-1})]f(x)dx\,dp^c_T = 0$$

Given that $E(p^c_T | p^c_{T-1}) = p^c_{T-1} + k'_{T-1}(W_{T-1})$, strict concavity of u implies that $W^S_{T-1} = V^S_{T-1}$. A similar development establishes that $W^L_{T-1} = V^L_{T-1}$. Hence we satisfy the market clearing condition $V^S_{T-1} = V^L_{T-1}$ for the futures market.

Consider next equilibrium in the time $t = T - 2$ market. Suppose it is common knowledge at $t = T - 2$ that all traders have identical utility functions and identical probability beliefs about time $t = T - 1$ and time $t = T$ cash and futures prices. Then we claim that a rational expectations equilibrium at time $t = T - 2$ is one such that $p^f_{T-2} = p^c_{T-2} + k'_{T-2}(W_{T-2}) + k'_{T-1}(W_{T-1})$, with $E(p^f_{T-1} | p^f_{T-2}) = p^f_{T-2}$ and $E(p^c_{T-1} | p^c_{T-2}) = p^c_{T-2} + k'_{T-2}(W_{T-2})$. Here $W_{T-2} = W^S_{T-2} = W^L_{T-2}$ such that $k'_{T-2}(W^S_{T-2}) = k'_{T-2}(W^L_{T-2})$, and $V^S_{T-2} = V^L_{T-2} = W_{T-2}$.

The argument is much like the one above establishing the martingale property in the time $t = T - 1$ markets. Given the common knowledge assumption, each trader knows that the equilibrium price pattern in the time $t = T - 1$ markets is one such that $p^f_{T-1} = p^c_{T-1} + k'_{T-1}(W_{T-1})$. Since $p^f_{T-1} = p^c_{T-1} + k'_{T-1}(W_{T-1})$, again we can describe the probability beliefs of traders in terms of a degenerate density over p^c_{T-1} only, say $g(p^c_{T-1})$.

First order conditions for the short hedger are then given by

$$\int_0^\infty u'(\pi^S_{T-1})[p^c_{T-1} - p^c_{T-2} - k'_{T-2}(W^S_{T-2})]g(p^c_{T-1})dp^c_{T-1} = 0 \tag{11}$$
$$\int_0^\infty u'(\pi^S_{T-1})[p^f_{T-2} - p^c_{T-1} - k'_{T-1}(W^S_{T-1})]g(p^c_{T-1})dp^c_{T-1} = 0$$

The long hedger's first order conditions are

$$\int_0^\infty u'(\pi^L_{T-1})[p^c_{T-2} + k'_{T-2}(W^L_{T-2}) - p^c_{T-1}]g(p^c_{T-1})dp^c_{T-1} = 0 \tag{12}$$
$$\int_0^\infty u'(\pi^L_{T-1})[p^c_{T-1} + k'_{T-1}(W^L_{T-1}) - p^f_{T-2}]g(p^c_{T-1})dp^c_{T-1} = 0$$

Using the earlier approach, it immediately follows from (11) and (12) that if $W^S_{T-2} = W^L_{T-2} = W_{T-2}$ such that $k'_{T-2}(W^S_{T-2}) = k'_{T-2}(W^L_{T-2})$, then market clearing prices in the $t = T - 2$ markets satisfy

$$p^f_{T-2} = p^c_{T-2} + k'_{T-2}(W_{T-2}) + k'_{T-1}(W_{T-1}) \text{ with } E(p^f_{T-1}|p^f_{T-2}) = p^f_{T-2},$$
$$E(p^c_{T-1}|p^c_{T-2}) = p^c_{T-2} + k'_{T-2}(W_{T-2}).$$

Similarly, the above arguments applied to $t = T - 3, T - 4, \ldots, 0$. Thus we have established the following.

<u>Proposition 1</u>. Given a forward market with an equal number of short and long hedgers, each with identical utility functions and densities over cash and futures prices, there exists a rational expectations equilibrium which is also a martingale equilibrium, satisfying

$$p^f_t = p^c_t + \sum_{\tau=t}^{T-1} k'_\tau(W_\tau), \quad t = 0,1,\ldots,T - 1,$$
$$p^f_T = p^c_T \quad \text{with}$$
$$E(p^f_t|p^f_{t-1}) = p^f_{t-1}, \quad t = 1,\ldots,T$$
$$E(p^c_t|p^c_{t-1}) = p^c_{t-1} + k'_t(W_t), \quad t = 1,2,\ldots,T$$
$$W^S_t = W^L_t = W_t, \; V^S_t = V^L_t = W_t, \quad t = 0,1,\ldots,T - 1.$$

One thing to note about this rational expectations martingale equilibrium is that there is no role for Houthakker's "asymmetric arbitrage" to play in influencing the configuration of equilibrium prices, or the decisions taken by short or long hedgers. In fact, with a forward market, the futures prices at all times $t = 0,1,2,\ldots,T$ are set at the maximum levels permitted by arbitrage (futures price equals the cash price plus marginal carrying cost).

4. <u>Price Patterns on a True Futures Market</u>

The situation is quite different once we move to a true futures market, with two or more delivery options available under the futures contract. In a true futures market, asymmetric arbitrage can impose a binding constraint on the joint pdf over the cash and futures prices, and hence can have an impact on the decisions of hedgers concerning their cash and futures commitments, which in turn has an effect on the pattern of the market clearing prices in the cash and futures markets.

Recall that in a true futures market, arbitrage ensures that $p^f_T \le p^c_T$, and $p^f_t \le p^c_t + \sum_{\tau=t}^{T-1} k'_\tau$, $t = 0,1,\ldots,T - 1$, but there are no corresponding constraints limiting the amount by which the cash price can exceed the futures price at any point in time.

Consider now a futures market in which arbitrage is not permitted to occur. Let $h(p^c_t, p^f_t)$ denote the joint density over the cash and futures prices at time t in such a situation, held by all traders at time $t - 1$. Our approach is to first construct an equilibrium for the case where arbitrage is not permitted to occur, and then to contrast the resulting pattern of market clearing prices with that which obtains under arbitrage.

Because we wish to explore the effects of asymmetric arbitrage under as simple conditions as possible, it is convenient to begin with a set of assumptions under which the equilibrium (without arbitrage) is a martingale equilibrium. In particular, assume that the density held by traders at $t = T - 1$ is symmetric about Ep^c_T, Ep^f_T, and consider as a candidate for equilibrium in the T - 1 markets the price and commitment relationships:

$$p^f_{T-1} = p^c_{T-1} + k'_{T-1}(W_{T-1}).$$

$$E(p^f_T | p^f_{T-1}) = p^f_{T-1},\ E(p^c_T | p^c_{T-1}) = p^c_{T-1} + k'_{T-1}(W_{T-1})$$

with $W^S_{T-1} = W^L_{T-1} = W_{T-1}$ satisfying $k'_{T-1}(W^S_{T-1}) = k'_{T-1}(W^L_{T-1})$,

and with $V^S_{T-1} = V^L_{T-1}$.

At $t = T - 1$ the first order conditions for the short hedger are

$$\frac{\partial E_h u^S}{\partial W^S_{T-1}} = \int_0^\infty\int_0^\infty u'(\pi^S_T)[p^c_T - p^c_{T-1} - k'_{T-1}(W^S_{T-1})]h(p^c_T,p^f_T)dp^c_T dp^f_T = 0$$

$$\frac{\partial E_h u^S}{\partial V^S_{T-1}} = \int_0^\infty\int_0^\infty u'(\pi^S_T)[p^f_{T-1} - p^f_T]h(p^c_T,p^f_T)dp^c_T dp^f_T = 0 \tag{13}$$

Similarly, the first order conditions for the long hedger are

$$\frac{\partial E_h u^L}{\partial W^L_{T-1}} = \int_0^\infty\int_0^\infty u'(\pi^L_T)[p^c_{T-1} + k'_{T-1}(W^L_{T-1}) - p^c_T]h(p^c_T,p^f_T)dp^f_T dp^c_T = 0$$

$$\frac{\partial E_h u^L}{\partial V^L_{T-1}} = \int_0^\infty\int_0^\infty u'(\pi^L_T)[p^f_T - p^f_{T-1}]h(p^c_T,p^f_T)dp^f_T dp^c_T = 0 \tag{14}$$

Suppose that $W^S_{T-1} = W^L_{T-1} = W_{T-1}$ satisfies $k'_{T-1}(W^S_{T-1}) = k'_{T-1}(W^L_{T-1})$ and assume that $V^S_{T-1} = V^L_{T-1} = V_{T-1}$. Let $x = p^c_T - Ep^c_T$, $y = p^f_T - Ep^f_T$. Then by symmetry, $h(Ep^c_T + x, Ep^f_T + y) = h(Ep^c_T - x, Ep^f_T - y)$ for all x,y. We also have $\pi^S_T(x,y) = W_{T-1}x - V_{T-1}y + \pi^*_T = \pi^L_T(-x,-y)$.

Rewriting the first order conditions (13) and (14), we have

$$\frac{\partial E_h u^S}{\partial W^S_{T-1}} = \int_{-Ep^c_T}^\infty\int_{-Ep^f_T}^\infty u'(\pi^S_T(x,y))xh(Ep^c_T + x, Ep^f_T + y)dydx = 0$$

$$\frac{\partial E_h u^L}{\partial W^L_{T-1}} = -\int_{-Ep^c_T}^\infty\int_{-Ep^f_T}^\infty u'(\pi^L_T(x,y))xh(Ep^c_T + x, Ep^f_T + y)dydx = 0 \tag{15}$$

$$\frac{\partial E_h u^S}{\partial V^S_{T-1}} = \int_{-Ep^c_T}^\infty\int_{-Ep^f_T}^\infty u'(\pi^S_T(x,y))yh(.,.)dydx = 0$$

$$\frac{\partial E_h u^L}{\partial V^L_{T-1}} = -\int_{-Ep^c_T}^\infty\int_{-Ep^f_T}^\infty u'(\pi^L_T(x,y))yh(.,.)dydx = 0$$

Clearly, by substituting (-x,-y) for (x,y) in the second and fourth equations, these reduce to the first and third. Hence market clearing in both the cash and futures markets is consistent with the first order conditions in the $t = T - 1$ markets.

We might note that in contrast to the $t = T - 1$ equilibrium in the case of a forward market, here there is no guarantee that all cash commitments will be hedged; all we know is that $V^S_{T-1} = V^L_{T-1}$.

Consider now the $t = T - 2$ markets. Again invoking a common knowledge assumption, all traders know that the equilibrium pattern of prices on the $t = T - 1$ markets will satisfy $p^f_{T-1} = p^c_{T-1} + k'_{T-1}(W_{T-1})$. Using the line of reasoning employed earlier, we can show that a rational expectations equilibrium exists on the $t = T - 2$ markets such that $p^f_{T-2} = p^c_{T-2} + k'_{T-2}(W_{T-2}) + k'_{T-1}(W_{T-1})$ with $E(p^f_{T-1}|p^f_{T-2}) = p^f_{T-2}$, and $E(p^c_{T-1}|p^c_{T-2}) = p^c_{T-2} + k'_{T-2}(W_{T-2})$, with $W^S_{T-2} = W^L_{T-2} = W_{T-2}$ satisfying $k'_{T-2}(W^S_{T-2}) = k'_{T-2}(W^L_{T-2})$, and with $V^S_{T-2} = V^L_{T-2} = W_{T-2}$. Note that we do not require symmetry of the density over time $t = T - 1$

prices, since the rational expectations assumption reduces the $t = T - 1$ market to a forward market. Similarly, the same argument applies to $t = T - 3, T - 4, \ldots, 0$. We formalize this as follows.

Proposition 2. Given an equal number of short and long hedgers, each with identical utility functions and density functions over futures and cash prices, and with the density over $t = T$ prices symmetric about the mean cash and futures prices, there exists a rational expectations equilibrium which is also a martingale equilibrium, satisfying

$$p_t^f = p_t^c + \sum_{\tau=t}^{T-1} k_\tau', \qquad t = 0,1,2,\ldots,T-1.$$

$$E(p_t^f \mid p_{t-1}^f) = p_{t-1}^f, \qquad t = 1,2,\ldots,T$$
$$E(p_t^c \mid p_{t-1}^c) = p_{t-1}^c + k_{t-1}', \qquad t = 1,2,\ldots,T.$$

We next examine the effects of asymmetric arbitrage on the cash and futures commitments of traders. A natural way to incorporate asymmetric arbitrage into the picture is to assume that if $h(p_t^c, p_t^f)$ is the density when arbitrage is not permitted, and $f(p_t^c, p_t^f)$ is the density when arbitrage can occur, then

$$f(p_t^c, p_t^f) = \begin{cases} h(p_t^c, p_t^f) & \text{for } p_t^f < p_t^c + \phi(t) \\ \int_{p_t^c+\phi(t)}^{\infty} h(p_t^c, p_t^f)\,dp_t^f & \text{for } p_t^f = p_t^c + \phi(t) \\ 0 & \text{for } p_t^f > p_t^c + \phi(t) \end{cases} \tag{16}$$

where $\phi(t) = \sum_{\tau=t}^{T-1} k_\tau'$.

Thus the effect of arbitrage is to concentrate at $(p_t^c, p_t^c + \phi(t))$ all the probability weight assigned under h to (p_t^c, p_t^f) for higher values of p_t^f. Given this specification of f, it immediately follows that h stochastically dominates f in the sense of first degree stochastic dominance (see Quirk and Saposnik (1963)), since, for any p_t^c, we have

$$\int_0^{p_t^f} h(p_t^c, v)\,dv \leq \int_0^{p_t^f} f(p_t^c, v)\,dv$$

for all p_t^f, with strict inequality for some values of p_t^f. By the well known property of dominating distributions, $E_h u > E_f u$ if u is monotone increasing in p_t^f, and $E_h u < E_f u$ if u is monotone decreasing in p_t^f. Hence we have the following.

Proposition 3. Arbitrage acts to increase expected utility for short hedgers, and to decrease expected utility for long hedgers.

Proof: For every W,V, $E_h u(\pi^S(W,V)) < E_f u(\pi^S(W,V))$ since π^S is monotone decreasing in p_t^f while u is monotone increasing in π^S. Let W^*, V^* maximize $E_h u(\pi^S)$ and let W^{**}, V^{**} maximize $E_f u(\pi^S)$. Then $E_h u(\pi^S(W^*,V^*)) < E_f u(\pi^S(W^*,V^*)) \leq E_f u(\pi^S(W^{**},V^{**}))$. A similar argument establishes the proposition for long hedgers.

When arbitrage is permitted, the first order conditions for short hedgers are given by

$$\frac{\partial E_f u^S}{\partial W^S_{T-1}} = \int_0^\infty\int_0^{p^c_T} u'(\pi^S_T)[p^c_T - p^c_{T-1} - k'_{T-1}(W^S_{T-1})]f(p^c_T,p^f_T)dp^f_T dp^c_T = 0$$

$$\frac{\partial E_f u^S}{\partial V^S_{T-1}} = \int_0^\infty\int_0^{p^c_T} u'(\pi^S_T)[p^f_T - p^f_{T-1}]f(p^c_T,p^f_T)dp^f_T dp^c_T = 0 \tag{17}$$

Let $\tilde{W}_{T-1}^{S},\tilde{V}_{T-1}^{S}$ denote the optimal choices of the short hedger under arbitrage, satisfying (17), and let $\overline{W}^S_{T-1},\overline{V}^S_{T-1}$ denote the choices of the short hedger when arbitrage is not permitted, satisfying (13). Evaluate the first order conditions in (17) at $\overline{W}^S_{T-1},\overline{V}^S_{T-1}$ and consider

$$\frac{\partial E_h u^S}{\partial W^S_{T-1}} - \frac{\partial E_f u^S}{\partial W^S_{T-1}},\ \frac{\partial E_h u^S}{\partial V^S_{T-1}} - \frac{\partial E_f u^S}{\partial V^S_{T-1}},\text{ evaluated at }\overline{W}^S_{T-1},\overline{V}^S_{T-1}.$$

Then we have

$$\frac{\partial E_h u^S}{\partial W^S_{T-1}} - \frac{\partial E_f u^S}{\partial W^S_{T-1}} = \int_0^\infty[p^c_T - p^c_{T-1} - k'_{T-1}(W_{T-1})]\{\int_{p^c_T}^\infty[u'(\pi^S_T) - u'(\pi^0)]h(p^c_T,p^f_T)dp^f_T\}dp^c_T \tag{18}$$

$$\frac{\partial E_h u^S}{\partial V^S_{T-1}} - \frac{\partial E_f u^S}{\partial V^S_{T-1}} = \int_0^\infty\int_{p^c_T}^\infty[p^f_{T-1} - p^f_T][u'(\pi^S_T) - u'(\pi^0)]h(p^c_T,p^f_T)dp^f_T dp^c_T \tag{19}$$

where $\pi^0 = \pi^S_T$ evaluated at $p^f_T = p^c_T$, with $W^S_{T-1} = \overline{W}^S_{T-1}, V^S_{T-1} = \overline{V}^S_{T-1}$.

In order to show that arbitrage encourages short hedging, in effect we need to solve a comparative statics problem where the exogenous shift involves the change from the density h to the density f. In turn, to solve the comparative statics problem for an arbitrary density h and an arbitrary concave utility function, the signs of (18) and (19) should be determinate. Using integration by parts, it is straightforward to establish that if the utility function satisfies constant or decreasing absolute risk aversion, then (19) is negative for an arbitrary symmetric density h.

Thus we can write (19) as

$$\int_0^\infty\{[u'(\pi^S_T) - u'(\pi^0)]\int_0^{p^f_T}(p^f_{T-1} - x)h(p^c_T,x)dx\Big|^\infty_{p^c_T}$$

$$- \int_{p^c_T}^\infty(\int_0^{p^f_T}(p^f_{T-1} - x)h(p^c_T,x)dx)[u''(\pi^S_T) - u''(\pi^0)](- V^S_{T-1})dp^f_T\}dp^c_T.$$

Since $\pi^S_T = \pi^0$ when $p^f_T = p^c_T$, the first term under the integral is zero. With the utility function exhibiting constant or decreasing absolute risk aversion, $u''' > 0$, and the second term is negative, so that (19) is negative for an arbitrary symmetric h.

However, the sign of (18) depends on obscure properties of the utility function and the density in the general case. Hence, despite the intuitive appeal of the asymmetric arbitrage argument, it turns out that in the general case, we cannot even show that the presence of arbitrage encourages short hedging (and discourages long hedging), much less that arbitrage leads to a backwardation equilibrium.

Turning to a highly special case, assume h is uniformly distributed and that u is characterized by constant or decreasing absolute risk aversion. Then the term in curved bracket of eq. (18)

will be positive and decreasing in p_T^c. Hence $\frac{\partial E_h u^S}{\partial W_{T-1}^S} - \frac{\partial E_f u^S}{\partial W_{T-1}^S}$ is negative, evaluated at $\bar{W}_{T-1}^S, \bar{V}_{T-1}^S$. Similarly, since $u'(\pi_T^S) - u'(\pi^0)$ is increasing in p_T^f and h is uniformly distributed, $\frac{\partial E_h u^S}{\partial V_{T-1}^S} - \frac{\partial E_f u^S}{\partial V_{T-1}^S}$ is negative, evaluated at $\bar{W}_{T-1}^S$, $\bar{V}_{T-1}^S$.

At a regular maximum (a stable equilibrium), and assuming that W and V are technical complements $(\frac{\partial^2 Eu^S}{\partial W \partial V} > 0)$, it follows that $\tilde{W}_{T-1}^S > \bar{W}_{T-1}^S$ and $\tilde{V}_{T-1}^S > \bar{V}_{T-1}^S$, as shown in Figure 1.

In Figure 1, the solid lines identify the (W,V) combinations that set $\frac{\partial Eu^S}{\partial W_{T-1}^S} = 0$ and $\frac{\partial Eu^S}{\partial V_{T-1}^S} = 0$ given the density h, while the dashed lines identify the combinations that set these expressions equal to zero, given the density f. As indicated, the introduction of arbitrage into a situation in which h is uniform shifts these curves so as to produce an increase in both the cash and futures commitments of short hedgers, given the prices p_{T-1}^f, p_{T-1}^c such that $p_{T-1}^f = p_{T-1}^c + k'_{T-1}$, $E(p_T^f | p_{T-1}^f) = p_{T-1}^f$, $E(p_T^c | p_{T-1}^c) = p_{T-1}^c + k'_{T-1}$. A similar argument establishes that the cash and futures commitments of long hedgers are both reduced by the introduction of arbitrage in this situation.

It follows from this that looking at hedging activities only, introducing asymmetric arbitrage leads to a situation in which there is an excess supply of futures contracts at the martingale equilibrium, and an excess demand for cash holdings of the commodity,

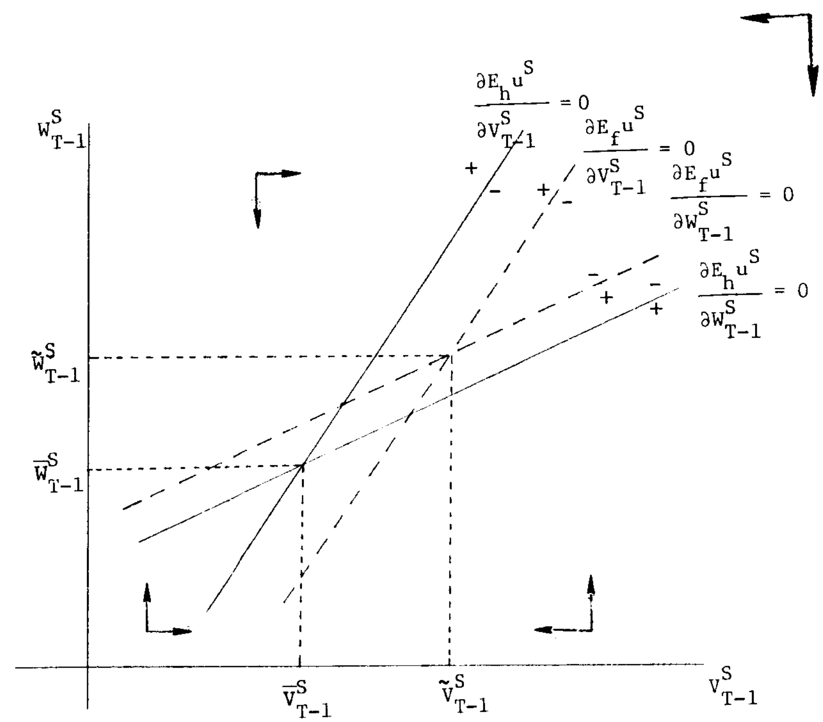

FIGURE 1. Effect of Arbitrage on Short Hedging Commitments

(Uniform density h)

at $t = T - 1$. To clear markets, the futures price p^f_{T-1} falls and the cash price p^c_{T-1} rises.

<u>Proposition 4</u>. Given an equal number of short and long hedgers, all with identical utility functions and densities over cash and futures prices, assume (1) when arbitrage is prohibited, the density $h(p^c_T, p^f_T)$ is uniform; (2) cash and futures commitments are technical complements for one another; (3) the utility function is characterized by constant or decreasing absolute risk aversion. Then the effect of introducing arbitrage is to lower the futures price at $t = T - 1$ and to raise the cash price at $t = T - 1$, both relative to the equilibrium prices when arbitrage is prohibited.

Proposition 4 provides some highly restrictive sufficient conditions for the effect that Houthakker argued was due to arbitrage, with arbitrage encouraging short hedging and discouraging long hedging. Note, however, that even under the highly restrictive conditions of Proposition 4, there is no guarantee that the equilibrium when arbitrage is present is a backwardation equilibrium. The reason is that the introduction of arbitrage makes short hedging more attractive in part because it lowers the expected value of the futures price at time T, since the upper tail of the density h is lopped off by arbitrage. What is required for arbitrage to lead to backwardation is not simply that short hedging be encouraged and long hedging be discouraged; net short hedging must be encouraged enough so that the fall in the futures price at $t = T - 1$ more than compensates

for the fall in the expected value of the futures price at $t = T$. This requires restrictive quantitative conditions on the utility function and on the density, beyond the conditions specified in Proposition 4. It is clear that the presence of asymmetric arbitrage is at best a tenuous argument for a backwardation equilibrium.

One other point should be made about the pattern of futures prices under asymmetric arbitrage, given the rational expectations framework. The common knowledge assumption that underlies rational expectations equilibria guarantees that the only effect that asymmetric arbitrage will have so far as backwardation (or a contango) is concerned is in the $t = T - 1$ market. The reason for this is that whatever is the relationship between the market clearing cash and futures prices on the $t = T - 1$ markets, this relationship will be inferred by all traders at a rational expectations equilibrium at $t = T - 2$. Similar arguments apply to $t = T - 3, T - 4, \ldots, 0$. Backwardation (or a contango) can only occur in the $t = T - 1$ markets. This means that at a rational expectations equilibrium, the upper limit on the futures price imposed by arbitrage does not constrain the equilibrium in any period prior to $T - 1$, and the futures market is reduced to a simple forward market in all such prior periods. The futures price in such periods simply equals the cash price plus carrying costs to maturity of the futures contract, and there is no role for speculation to play, since the futures market attains a martingale equilibrium. This might be viewed as a rationalization of sorts for the widespread use of two period models in the literature on futures markets, or, more correctly perhaps, an argument shedding some doubt as to the use of the rational expectations framework in analyzing a speculative market.

5. Conclusion

In this paper we have explored the implications of asymmetric arbitrage on the pattern of prices on a futures market, and in particular we have looked into the question as to whether asymmetric arbitrage is a force making for backwardation. Our conclusions are generally negative. If the futures market is a forward market, then in a rational expectations framework, asymmetric arbitrage has no effect on the pattern of futures (or cash) prices. If we are dealing with a true futures market, then arbitrage will typically have some effect on the pattern of hedging and hence on the pattern of futures prices. However, there is no clear-cut conclusion that the introduction of arbitrage acts to encourage short hedging and to discourage long hedging; generally this depends on the specific properties of the joint density over cash and futures prices and on the specific properties of the utility function. Furthermore, even when it is known that short hedging increases and long hedging decreases because of the introduction of arbitrage, this does not imply that a martingale equilibrium becomes a backwardation equilibrium; this requires further quantitative restrictions.

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