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SOUR NOTES ON THE THEORY OF VOTE TRADING*

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The recent literature on logrolling or vote trading has been quite long on intuitive argument and carefully constructed examples, and short on general theorems. This state of affairs is not too surprising since for all the scholarly attention the subject has recently enjoyed, there is remarkably little agreement on concepts or definitions. As a result most arguments are carried out in an ambiguous setting, and authors appear to arrive at quite different conclusions about the outcomes of vote trading in legislatures. Just to provide some orientation for those who have not plowed through the literature recently, I shall provide a brief review of the literature on the subject.

The literature on vote trading has of course grown out of that on voting institutions and, more generally, out of the study of aggregation procedures or mechanisms of social choice (Black [1958], Arrow [1963]). In an early study Buchanan and Tullock [1962] introduced a distinction between explicit logrolling (in which legislators make agreements among themselves to exchange votes) and implicit logrolling (in which parties or candidates make up platforms out of the whole set of issues and legislators choose among them). They analyze explicit logrolling in their famous "road repairing" example and show that voters have an incentive to make bargains among themselves

so that an excessive amount of investment takes place in road repair. Evidently one feature of the situation Buchanan and Tullock examine is that under majority rule over all possible road repair packages, there is no "equilibrium" social choice. That is, each platform (under implicit logrolling) is defeated in majority vote by some other platform.

James Coleman [1966] provided a somewhat different set of concepts to analyze vote exchanges. He argued that logrolling permits the expression of "intensity" of preference and that this possibility allows circumvention of Arrow's paradox. While this latter statement rests on a misconception, Coleman did introduce some tools that seem useful for studying vote trading. In particular his model explicitly recognizes the dependence of voter behavior on expectations or beliefs about what the other legislators may do. Further, Coleman argues that with a sufficiently large number of issues and legislators, a vote-trading system ". . . approaches in one sense a market situation . . ." [1966, p. 1117]. Further ". . . just as a free market with pure competition can be conceived in economic exchange, and used as a theoretical model from which actual systems can be studied, a similar model of pure competition can be conceived in collective decisions." [1966, p. 1118]. As we shall subsequently see, there are rather severe limitations on how far one can fruitfully carry this view. Nevertheless, it seems useful to ask to what extent concepts from the theory of exchange can help us explain legislative phenomena.

Coleman's paper elicited a rather lively response and in a comment R.E. Park [1967] established that (where voters' preferences are represented by additive utility functions) a vote-trading equilibrium is just what would have occurred without vote trading. Mueller [1967] emphasized the costs of forming coalitions and policing defections which would prevent a vote-trading system from approximating an

exchange economy. Mueller concluded that ". . . when voters are able to make and keep vote-trading agreements, their welfare will be greater than if no agreements were made" [1967, p. 1310]. It is unclear just what concept of "welfare" Mueller is invoking in his conclusion.

If he means that vote trading can produce Pareto improvements, then we shall see that the veracity of his remarks depends on the form of the voters' preferences. In particular, if legislators have what are called "separable" preferences, Mueller's assertion is false since in this case a version of Park's theorem applies: if there is a vote-trading equilibrium, it is identical to the "sincere voting" outcome in the case where no trade is allowed. Of course Mueller may be using some other notion of welfare in which case it is not possible to decide the question of whether vote trading can produce welfare gains.

In his reply to Park's and Mueller's papers, Coleman [1967] seems to realize that a "market for votes" contains some fundamental imperfections. In particular, he argues that even if votes are fully exchangeable, if legislators are permitted to learn what trades actually take place among others, there may fail to be a price equilibrium. He also insists that as long as the legislators' subjective beliefs about what trades are occurring elsewhere do not change "too much" that there will be an equilibrium in the trading process.

In a recent contribution Tullock [1970] argued that explicit logrolling is "stable" and leads to an expected increase in welfare of the members of the legislature. It seems that unless he has hidden some strong assumptions about what coalitions are permitted to form, or is using a curious concept of stability, that explicit logrolling is not generally stable. I believe that the most natural stability concepts in this case are these:

1. Each legislator has a vote intention for each bill such that it would not profit any of them to change their vote intention on any issue.

2. Each legislature has a vote intention on each bill and no majority has the incentive to collectively change its vote intentions.

By either of these tests, Tullock's example is not stable. Perhaps he is employing a somewhat different concept which is unapparent to me.

Along a somewhat different line, Wilson [1969] produced a market model for vote trading and gives a set of axioms that limit the set of "admissible" decision rules that legislators may adopt in choosing among alternative packages of votes on issues. Among his results is a theorem that states that the legislators will buy votes on issues in proportion to the "utility gain" they experience in the issue. In particular it will not happen that if I care most about issue one and much less about issues two, three, four, etc. (though I still care) that I will spend all my votes on issues two through n to procure votes on issue one. Generally speaking I will hold votes on issues of less importance to me under Wilson's assumptions. Wilson's model is peculiar in that the decision rule chosen seems not to depend explicitly on the expected behavior of the other legislators.

A recent set of papers by Kadane [1972], Bernholz [1973, 1974] and Koehler [1973] seem to follow much more closely in the tradition of work inaugurated by Park. In particular, using Buchanan and Tullock's concepts these authors demonstrate that under certain conditions, if implicit logrolling has a stable platform then the set of issues passed in explicit logrolling is the same platform. Secondly if such a stable platform exists it is characterized as the set of bills that

would pass if everyone voted their true preferences one issue at a time. Thirdly, if the implicit logrolling process has no stable platform then neither does the explicit process.

After reviewing the relevant literature I am struck by a series of disagreements among the various authors. First, those who have dealt with vote trading seem neatly divided on the question of prevalence. Buchanan and Tullock, and Haefele [1971] seem to believe that much vote trading occurs in legislative institutions. Buchanan and Tullock remark that there are ". . . certain relatively rare institutional situations in which logrolling will not be likely to occur. . . ." [1962, p. 132]. Riker and Brams [1973] and Kramer [1972] on the other hand appear to believe that actual vote trading is not a frequent occurrence in most legislatures.

A second division one finds in the literature concerns the degree to which vote trading in a legislature approximates an exchange economy. Coleman [1966] suggests that the approximation is close. Riker and Brams, and Haefele caution against making too much of the apparent similarities. Riker and Brams remark:

> In general . . . the market for legislative votes is quite different from the market for private goods and it is not wise to draw analogies from one market to the other. [1973, p. 1236]

Third, there is the traditional question of the "social benefit" of permitting logrolling. Haefele, Buchanan and Tullock, Coleman, and others argue that the introduction of vote trading can increase the "welfare" of legislatures. But Riker and Brams dispute this claim by demonstrating that logrolling may make everyone worse off than they would have been without it.

Finally, it seems to me that the most fundamental division in the literature lies between those who believe that vote trading ought to be modeled as a cooperative game and those who choose to model it as a noncooperative process. Coleman, in his reply to Park and Mueller

is quite explicit on this point: "If a number of persons communicate, this upsets the equilibrium of vote prices. Thus any description of the perfect system of vote exchanges . . . is valid only in the case where individuals act wholly individualistically." [1967, p. 1316] Also Coleman's model is characterized by the fact that legislators make probability judgments about each other, in effect, treating each other's behavior as independent of one's own. Wilson's model is basically noncooperative as is Mueller's.

On the other hand, the recent work of Kadane, Bergholz, and others as well as the contributions by Buchanan and Tullock contain cooperative analyses. That is, legislators are assumed to communicate and form coalitions. In between these two approaches are those by Riker and Brams, Haefele, and to some extent Coleman, that might be called partially cooperative. That is, certain types of coalitions may form while others may not.

It appears that Kadane and others have pretty well described the situation if the cooperative assumptions are approximated in a legislature. I am skeptical enough about the possibilities of coalition formation in legislatures that I prefer to examine the effects of restrictions on coalition formation and so the models in this paper are of the noncooperative and partially cooperative sorts.

In this paper I want to explore the ways logrolling can actually lead to results that are unrelated to the presence or absence of a majority winner. I argue that the modifications that are introduced into the "cooperative" models of Kadane, Koehler, and others are departures in the direction of realistic descriptions and therefore that the determinate arbitrariness I obtain may occur with some regularity in legislative bodies. To the extent that this is true, logrolling may be employed by skilled (or otherwise privileged) legislators to secure distributive (as opposed to welfare) gains.

I shall proceed by setting up a simple model of a legislature and consider the various arguments within its confines. There is a

set of legislators $N = \{1, 2, \ldots, n\}$ and a finite set of binary issues, $M = \{1, 2, \ldots, m\}$. The set of social states is represented by the cartesian product $\{0,1\}^m$. Each legislator has a binary preference relation R_i which is a weak order. Bills are assumed to come up for votes in a fixed order known in advance. Each legislator is endowed with a single vote on each issue and decisions are made by simple majority rule one issue at a time.

I. VOTE TRADING AND EQUILIBRIA

In spite of the substantial literature on the subject, little attention has been given to actual trading processes. Those theorists who argue that legislative vote trading approximates an exchange economy postulate conditions which, presumably, would determine prices for votes, and trading is assumed to take place at the equilibrium prices. In this section a wholly different approach is taken. A trade may take place only if it improves the welfare of the traders and there is no reference to equilibrium exchange ratios. Such trading procedures are sometimes called "nontâtonnement" processes in economics. The appeal of such a model in legislative circumstances is, of course, that we need not posit the existence and operation of a set of markets (that have not been observed) in order to analyze vote-trading phenomena.

Perhaps the simplest model is one where each actor knows everyone's preferences and all the trades that have occurred. Each is assumed to vote as he has agreed. In this case he knows that if the votes are taken at the current time, exactly which social state $x \in \{0,1\}^m$ will be chosen. We require that two traders (say Mr. 1 and Mr. 2) will exchange votes only if, given that no one else makes any trades, the two actors can change the resulting social state from x to y and yP_1x and yP_2x. Let's also require that all trades be between pairs of actors. This trading procedure is essentially the same as that given by Riker and Brams and Haefele.

In the first part of this section I make an additional assumption on the form of the individual preference orderings.

<u>Definition.</u> A preference ordering R on the set of social states $\{0,1\}^m$ is called <u>separable</u> if for any $Q \subseteq M$ and $x_i = y_i$ $\forall i \in Q$ and xRy then if $\bar{x}_i = \bar{y}_i$ $\forall i \in Q$ and $\bar{x}_j = x_j$ and $\bar{y}_j = y_j$ $\forall j \in M - Q$ then $\bar{x}R\bar{y}$.

The idea of this definition is that the legislator's preference on any subset of bills can be unambiguously determined and does not depend on which bills have already passed. This assumption is found throughout the literature on vote trading in various guises. In particular, separability is necessarily satisfied if legislators have additive utility functions on the bills. If preferences satisfy separability, we can naturally define vote intentions as follows.

<u>Definition.</u> $S_i = \{S_{i1}, \ldots, S_{im}\}$ is a vector indicating how Mr. i casts his vote on issue j.

$$S_{ij} = \begin{cases} 1 \\ -1 \end{cases}$$

Given the vote intentions of each legislator S_i, one can now determine which social state would be chosen if no aggrements are made. We say that given $S_1, \ldots, S_n$, $x \in \{0,1\}^m$ is <u>chosen</u> if and only if

$$x_j = \begin{cases} 1 & \text{if } \sum_i S_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}$$

For clarity we may sometimes write $x(S_1, S_2, \ldots, S_n)$. Now, two legislators may find it desirable to make an exchange. We now formalize what the set of feasible trades looks like for any pair of legislators.

Definition. If $x(S_1, \ldots, S_n)$ is chosen, then we say that $z \in A_{ij}(x) \Longleftrightarrow \exists$ vote intentions $\bar{S}_i$, $\bar{S}_j$ such that $z = z(S_i, \ldots, \bar{S}_i, \ldots, \bar{S}_j, \ldots, S_n)$.

That is $A_{ij}(x)$ is the set of social states that voters i and j could cause to be chosen by simply making an agreement to alter only their own vote strategies from what they were when x was chosen and assuming that everyone else will cast S_k $\forall k \neq i, j$. In a sense $A_{ij}(x)$ characterized the power of the coalition of i and j. Evidently if on the balloting that produced x, all issues were decided by more than two votes, then $A_{ij}(x) = x$ for all i, j.

Definition. $\exists$ a feasible trade between i and j at x if and only if $\exists z \in A_{ij}(x)$ and zP_ix and zP_jx.

Definition. $x \in \{0,1\}^m$ is an equilibrium with respect to pairwise trades if and only if $\forall i, j \in N$ $\nexists z \in A_{ij}(x)$, with the property that zP_ix and zP_jx.

This notion of equilibrium is intended to capture the notion that $x(S_1, \ldots, S_n)$ is stable if and only if if any pair of voters has the power to change the social state from x to z, they do not both desire to do so. In a sense, a state is in equilibrium if all mutual gains from trade have been exhausted.

From the point of view of what Buchanan and Tullock call implicit logrolling, a new binary relation is introduced. The interpretation of this relation is as follows. xDy if and only if x defeats y in a simple majority vote. Notice that in defining this "majority dominance" relation, voting is not on an issue-by-issue basis as before. Rather x is pitted against y in a direct contest.

Definition. Let D be a binary relation on $\{0,1\}^m$ defined as follows

$$xDy \Longleftrightarrow n(xP_iy) > n(yP_ix).$$

Definition. x^0 is a majority winner $\Longleftrightarrow \forall y \in \{0,1\}^m$ x^0Dy.

Several authors (Bernholz, Koehler, and Kadane have argued that if there is an "incentive to logroll" then there is no majority winner and conversely. The first theorem we present demonstrates that if by an "incentive to logroll" we mean that only Riker-Brams or Haefele type trades can take place, then the converse is false.

Theorem 1. There exists a legislature with no majority winner and in which there is an equilibrium with respect to pairwise trading.

Proof. Assume there are two binary issues and five voters with the following separable preferences.

1, 2	3	4	5
(1, 0)	(0, 1)	(1, 1)	(1, 1)
(0, 0)	(0, 0)	(1, 0)	(0, 1)
(1, 1)	(1, 1)	(0, 1)	(1, 0)
(0, 1)	(1, 0)	(0, 0)	(0, 0)

One can check that there is no majority winner.

If the voting is done one issue at a time and each legislator votes sincerely, we have that position 1 beats position 0 on issue one via the group $\{1, 2, 4, 5\}$ and that position 1 beats position 0 on issue two via the group $\{3, 4, 5\}$.

Given that the status quo is (1, 1) there is no feasible trade between a pair of legislators. Thus (1, 1) is an equilibrium with respect to pairwise trading.

Q.E.D.

This simple result indicates that if trades are for some reason difficult to organize, vote-trading processes may have equilibria even though the underlying majority dominance relation is cyclic. Of course the basic phenomenon here is that no one member can by changing his vote change the outcome on issue one from what it would have been under issue-by-issue sincere voting.

One need not look far to find legislative situations which bear some similarity to that characterized in the proof of theorem one. For example, a legislature considering a number of pork-barrel projects, each contained in the district of a single member, would have vote-trading equilibria in this sense with no projects constructed even though the majority dominance relation is cyclic.

It is possible to establish a somewhat more surprising theorem. Assuming only pairwise trading is allowed, the majority dominance relation on $\{0,1\}^m$ may have a majority winner and yet, from a given status quo, pairwise trading may not lead to it. Some care must be taken in choosing the status quo in a natural way. We simply let the status quo be the position chosen in issue-by-issue sincere voting, beginning with the state (0, 0). Given this definition we say that a social state $x \in \{0,1\}^m$ is <u>unreachable in pairwise trades</u> if and only if starting at the status quo there is no sequence of feasible trades leading to x.

<u>Theorem 2.</u> Assuming legislators hold weak orders there is a legislature with a majority winner that is unreachable in pairwise trades.

<u>Proof.</u> Consider a legislature in which all the legislators hold the following preferences (which violates separability).

11
00
01
10

Then 11 is the majority winner and in pairwise contests 00 beats 01 and 10. If we interpret 00 as the status quo clearly no pairwise trades are possible if n is at least 5. Q.E.D.

It is already well known [Kadane, 1972] that if the legislator's preferences are separable then if $x \in \{0,1\}^m$ is the majority winner, it is simply the vector of majority winners in issue-by-issue voting. This is simply our notion of the status quo and so in this case the majority winner is trivially reachable in pairwise trades.

Theorems one and two indicate that once restrictions are placed on the way in which trading can occur, vote trading may produce results that are largely independent of the principle of majority rule. Permitting only pairwise trading may create equilibria where none existed previously or fail to reach equilibria which do exist. Of course similar theorems can be proved when somewhat larger trades may be organized though if the trades are permitted to become "large enough" the vote-trading process will begin to bear a "nice" relation to majority rule. In particular if any size trade is feasible under the condition that the welfare of all traders is improved, and if preferences are separable, then we have the Kadane-Park-Bergholz situation. If there is a majority winner, it is trivially reachable. If there is no majority winner the trading process has no equilibria.

I would argue that these two theorems, simple as they are, may be quite relevant to the operation of legislatures. The argument is of course that restrictions on the size of feasible trades arise in very natural ways in legislative institutions. In the absence of fungible votes the bookkeeping involved in large trades is likely to be difficult. Secondly, the possibility of trades induces members to be deceitful about how they would cast their untraded vote. It seems to me the larger the trade the more difficult it is for traders to police this sort of thing. These

observations would seem to be behind the stringent Riker-Brams pairwise trading requirements. To the extent that they are descriptive of legislative situations, one must expect that observed equilibria will bear little relation to the majority winner if one exists.

One may reasonably ask whether or not reachable equilibria of the pairwise trading process always exist. We give an example to show that no general existence theorem is possible.

Theorem 3. There exist legislatures where members have separable preferences which have no reachable equilibria in pairwise trading.

Proof. Consider a legislature of five members with the following configuration of preferences:

Member	1	2	3	4	5
	11	00	01	10	11
	01	01	00	00	10
	10	10	11	11	01
	00	11	10	01	00
S_{ij}^1	(1,1)	(-1,-1)	(-1,1)	(1,-1)	(1,1)

Evidently if everyone votes sincerely $x^1 = (1,1)$.

$A_{34}(x^1) = \{(1,1), (1,0), (0,1), (0,0)\}$ and we find that there is an $x^2 = (0,0)$ such that $x^2 \in A_{34}(x^1)$ and $x^2 P_3 x^1$, and $x^2 P_4 x^1$

S_{ij}^2	(1,1)	(-1,-1)	(-1,-1)	(-1,-1)	(1,1)

$A_{13}(x^2) = \{(0,0), (0,1), (1,0), (1,1)\}$

and $x^3 = (0,1) \in A_{13}(x^2)$ with $x^3 P_1 x^2$, $x^3 P_3 x^2$

S_{ij}^3	(1,1)	(-1,-1)	(-1,1)	(-1,-1)	(1,1)

$A_{14}(x^3) = \{(0,1), (0,0), (1,1), (1,0)\}$

$\exists x^4 = (1,1) \in A_{14}(x^3)$, $x^4 P_1 x^3$, $x^4 P_4 x^3$

and $x^4 = x^1$ and $S_{ij}^4 = S_{ij}^1$

Note that when x^2 was the status quo

$A_{45}(x^2) = \{(1,0), (0,1), (1,1), (0,0)\}$

$\hat{x}^3 = (1,0) \in A_{45}(x^2)$ and $\hat{x}^3 P_4 x^2$, $\hat{x}^3 P_5 x^2$

$\hat{S}^3$:	(1,1)	(-1,-1)	(-1,-1)	(1,-1)	(1,1)

$A_{35}(\hat{x}^3) = \{(1,0), (1,1), (0,0), (0,1)\}$

$\hat{x}^4 = (1,1) \in A_{35}(\hat{x}^3)$ and $\hat{S}_{ij}^4 = S_{ij}^1$

This establishes that there is no reachable equilibrium in this situation since for any reachable social situation a feasible pairwise trade exists.

Q.E.D.

Theorem three establishes that a pairwise trading process will sometimes fail to reach an equilibrium. The problem seems to be that if given the status quo any pairwise trade takes place, certain issues will be decided by only one vote. This feature of the Riker-Brams rules allows some legislators the power to alter the social state by changing their voting strategies. All that is required for instability is that some pair of legislators with this power be able to agree to change the social state to something both would prefer. In the Riker-Brams trading procedure, there is no concern on the part of legislators to provide for a "margin of victory."

II. INFORMATION, MONEY, AND VOTE TRADING

To the extent that only small trades are possible, new equilibria may be introduced which are dependent on the historical process by which issues arise. Certain other possibilities would seem to permit potential equilibria to be destroyed. Assume, for example, that each legislator holds an initial quantity of money which he is free to exchange for votes as he pleases. Without going into too much analysis, one can see that a pairwise trade need only make one member better off to be feasible since compensation in money is possible. For example, in the preference distribution given in the proof of theorem 1, Mr. 1 might be willing to pay up to $10 for the outcome to change from (1, 1) to (1, 0), while Mr. 3 might willingly accept any amount over $2 to vote for 0 over 1 on issue two. In such a case, allowing pairwise trading with money, we would not have (1, 1) as an equilibrium.

Of course, in this example, Mr. 3 was pivotal on issue two and so if he switched his vote and everyone else voted the same as before the trade, the outcome would be different. Without much difficulty one can imagine situations in which each issue won by more than two votes and there is no majority winner. In such cases the introduction of money by itself, will not upset pairwise trading equilibria. Consider what would happen if Mr. 3 was willing to pay up to $10 for the defeat of issue one, while Mr. 1 and Mr. 2 would each gladly accept $2 to change their votes. If there is a restriction to pairwise trades, Mr. 3 would not trade with either Mr. 1 or Mr. 2 since neither is pivotal. In this case even with the availability of money (1, 1) may still be a pairwise equilibrium.

A little reflection should convince the reader that the problem in this last example is the requirement that a legislator be unwilling to trade unless he is made "better off" than before as a result of his trade. If we amend this restriction to the more modest requirement that he must expect to be better off, a whole new set of trading possibilities arise.

In the last example, if Mr. B "expects" to be able to obtain votes from both Mr. 1 and Mr. 2 on issue one, he would be willing to engage in an initial pairwise trade with one of them. One can see that depending on what expectations legislators have about trading possibilities, the pairwise trade restriction may not circumscribe outcomes in the manner of theorems one and two at all. Of course, this possibility must depend on the expectations held by the legislators, and so one cannot hope for general results along these lines unless more structure is put on the way legislators gather information about each others' preferences and about which trades have taken place.

In our discussion of trading processes we have so far assumed that when a trader decides to trade he looks only at his own current and future trading possibilities (as best he can). Of course a legislator will generally be aware that his own welfare depends on what future trades the other legislators undertake. If he surrenders his vote on some issue to one of his peers, it may be traded to others with consequences far different from those that would have arisen from the initial trade. Apparently, this situation would be quite difficult to model and as a result a number of questions about it remain unanswered. In particular can we expect any relationship between the majority domination relation and some sort of stability concept in a trading model of this complexity?

Theorems 1 and 2 indicate that we cannot expect any general relationship between these concepts since the model therein contains expectations that are admissible in the present discussion. Nevertheless it is perhaps not too much to hope that appropriate restrictions on how expectations are formed and modified could lead to some positive results. It seems to me that the complications mentioned here are very much present in actual vote-trading situations, and so unless attempts are made to come to grips with actual trading processes, we will be unable to address ourselves to the issues raised by vote trading which are mentioned in the introduction.

Before proceeding with a discussion of market institutions which may serve to reduce the complexity of belief formation, it seems appropriate to venture a few remarks on a model of a legislature which is based on a somewhat more extreme restriction on trading possibilities than that considered here. In a recent publication Kramer [1972] argued that policing of contracts in a legislature is likely to be so expensive that a noncooperative analysis of legislative voting is likely to produce more realistic results than a model based on cooperative game theory. He demonstrated that if no trades are possible, but if legislators are permitted to vote in a "sophisiticated" manner, under assumptions on preferences given in this paper, there always exists an equilibrium in the NASH sense. That is, each player has a sophisticated voting strategy which, assuming that no one else will change his strategy, he does not find it advantageous to depart from. Kramer does not ask whether this equilibrium bears any relation to a majority winner if one exists, but using tools that are employed in the study of logrolling, we may easily give an answer to this question.

Kramer gives a necessary condition that x be the sophisticated voting outcome (which is unique if preferences are strict). It is that if x' differs from x in only one issue, then it is not true that x' defeats x in majority vote. Let y be the outcome obtained by sincere voting majority vote issue by issue (we can ensure this is unique by assuming that preferences are strict and that there are an odd number of voters). Then by Kadane's [1972] improvement algorithm, we know that if x is any other outcome (in particular the sophisticated voting outcome) there is a sequence of outcomes $x = y^1, y^2, \ldots, y^m = y$ such that y^i and y^{i+1} differ in one issue and such that y^{i+1} defeats y^i in majority vote. If there is a majority winner, it is y and so the sophisticated voting outcome must be y. In fact, we have established a bit more than this:

Theorem 4. If voters have strict, separable preference orders, then the unique sophisiticated voting equilibrium is the outcome that results from sincere voting issue by issue.

Thus we have the nonintuitive result that as long as preferences are strict and separable, and issues are voted on one at a time, and no trading is feasible among (the odd number of) legislators, the unique equilibrium is where the members vote sincerely.

III. MARKETS FOR VOTES

Several theorists have approached vote-trading phenomena in a different manner from that discussed above. Rather than consider nontâtonnement trading procedures, Coleman, Wilson, and others have investigated what might happen if there were equilibrium prices for votes and if all trade took place at such prices. If the legislature is large enough one might think that the presence of such markets would alleviate the need for members to form elaborate expectations about what their peers desire and how they might behave. Instead the legislator is assumed to take prices as fixed independently of the quantity of votes he demands and to maximize his well-being subject to a budget constraint.

While the idea of a competitive market for votes is attractive on several grounds, it turns out that when an analysis of the implications of such an institution is made, some disturbing problems arise. In this section I introduce a market for votes and attempt to define individual maximizing behavior and equilibrium prices. The model given here differs from others found in the literature, but it appears to me that most of the problems that arise for this model are ones that really would occur in a market for votes.

The idea of a vote market is that when faced with a particular set of prices the legislator decides on what package of votes he wants to cast on the issues facing the voting body. That is, he formulates

a "demand function" for votes on each issue which depends on the prices he faces. The easiest way to see the difficulty with such a formulation is to consider the problem of a legislator who faces a price ratio of two between issues A and B. For each vote on A he can get two votes on B. How can he decide whether to make an exchange? On a little reflection it seems clear that his willingness to deal at this price depends on what he thinks the outcome of the post-trade voting will be for each imaginable trade. He will pay more for a pivotal vote than one not so fortuitously placed. In other words our formulation will have to reflect the fact that each legislator's choice is dependent on what he thinks the others will do. This dependence is what makes a vote market peculiar. The following definitions characterize this situation.

Definition. Let V_{ij} denote the number of votes on the $j^{\underline{th}}$ issue held by voter i.

Definition. Let $S_{ij} = +1$ or -1 depending on whether the $i^{\underline{th}}$ voter is for or against passage of the $j^{\underline{th}}$ bill (separability and strict preferences ensures this can be defined).

Definition. For any distribution of votes (V_{ij}) the corresponding social state x is defined as

$$x_j = \begin{cases} 1 \iff \sum_i S_{ij} V_{ij} > 0 \quad \forall j \\ 0 \text{ otherwise} \end{cases}$$

Legislators are assumed to maximize their welfare at given prices $p = (p_j)$, where $p_j > 0$ and $\sum_j p_j = 1$.

In this situation the notion of maximality is a little ambiguous. After all if you decide to sell a vote on issue j that you would have cast in favor of the issue's passage, how do you assume it is cast by its new owner? Likewise if you purchase a vote on an issue, should you

assume that this affects your expectation of the vote holdings? In this model I will make the assumption that the legislator purchases and sells only to his adversaries on an issue so that he can in a natural manner adjust his expectation as to the post-trade outcome of the balloting. This assumption of course violates the usual anonymity properties of markets and in doing so implicitly overlooks the possibilities of deceitfulness in announcing voting intentions. However I do not think it too unnatural as a description of vote-trading possibilities.

Definition: We say that V_i is maximal with respect to

$$p, V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_n, S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n$$

for the $i^{\underline{th}}$ voter if and only if for any other strategy W_i, let

$$Z_{ij} = V_{ij} + 2(W_{ij} - V_{ij})$$

and

$$y_j = \begin{cases} 1 \iff S_{1j}V_{1j} + \ldots + S_{i-1j}V_{i-1j} + S_{ij}Z_{ij} + S_{i+1j}V_{i+1j} \\ \qquad\qquad + \ldots + S_{nj}V_{nj} > 0 \quad \forall j \\ 0 \text{ otherwise} \end{cases}$$

and yP_ix, then $\sum_j p_j W_{ij} > 1$. That is, V_i is maximal for legislator i if and only if at the given prices p he cannot change his vote holdings unilaterally in such a manner that the resulting social states leaves him better off without violating his budget constraint. Note that if the legislator trades away a vote on issue j for a quantity on issue k, a two-vote switch is supposed to occur on each issue.

Definition: $[p, (\overline{V}_{ij})]$ is a competitive equilibrium $\iff$ $\overline{V}_i$ is maximal for each i given p and $\forall i$

$$\sum_i \overline{V}_{ij} = n \quad \forall j$$

Note that each member's choice $\overline{V}_i$ depends not only on price but also on his expectations of what votes his peers hold and how they will cast them. Technically therefore we should parameterize a competitive equilibrium on this information. Also, it seems worth remarking that competitive equilibria may exist even when expectations are inconsistent with it. This possibility seems somewhat bizarre, and a more satisfactory model of vote markets might contain some sort of adjustment mechanism for expectations that would serve to bring them into line with one another at equilibrium.

Evidently, this formulation differs from the classic exchange economy in several ways. First of all the demand functions for votes depend not only on prices but on what the other legislators hold and how they will cast their holdings. The common sense of this is quite obvious. You would be willing to pay quite different amounts for a vote depending on whether that vote was or was not pivotal on an issue. Now what this means of course is that, unlike the case of a classical exchange economy, no great reduction in informational requirements is achieved in a legislature by introducing markets for votes. The legislators still have to form beliefs about what trades will be made and about the preferences of their peers in order to make their own decisions.

As it turns out we may demonstrate that, in general, competitive equilibria do not exist in this very simple formulation.

Theorem 5. There are expectations and preference distributions such that no competitive equilibrium exists.

Proof. Assume that three legislators have additive utility functions on two issues represented as follows.

	1	2	3
	11	00	10
	01	01	11
	10	10	00
	00	11	01
Utilities for each issue:	(4,5)	(4,3)	(2,1)
S_{ij}:	(1,1)	(-1,-1)	(1,-1)

We need only consider the price ratio between issue two and issue one $\frac{p_2}{p_1}$ so we will look at the individual demand functions for all ratios between 0 and ∞. We assume that legislators one and two believe that the others will vote sincerely issue by issue while Mr. three expects that Mr. one will vote (-1, 1) and Mr. two will vote (-1, -1). We may now consider three cases.

Case 1. $0 \leq \frac{p_2}{p_1} < \frac{1}{2}$

$$d_1\left(\frac{p_2}{p_1}\right) = \begin{pmatrix} -(\frac{1}{2} - \epsilon) \\ \frac{p_1}{p_2}(\frac{1}{2} - \epsilon) \end{pmatrix} \quad \text{where } \epsilon \text{ is such that } \frac{p_1}{p_2}(\frac{1}{2} - \epsilon) > 1$$

$$d_2\left(\frac{p_2}{p_1}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$d_3\left(\frac{p_2}{p_1}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note that both Mr. two and Mr. three would like to purchase $\frac{1}{2} + \delta$ votes on issue one but at the prices in this interval they would have to give up more than the one vote they have on issue two. Therefore the market will not clear since there is excess demand at any prices in this range.

<u>Case 2.</u> $\frac{1}{2} \leq \frac{p_2}{p_1} < 2$

$$d_1\left(\frac{p_2}{p_1}\right) = \begin{pmatrix} -x \\ \frac{p_1}{p_2}x \end{pmatrix} \quad \text{where} \quad \frac{p_1}{p_2}x > \frac{1}{2}$$

$$d_2\left(\frac{p_2}{p_1}\right) = \begin{pmatrix} y \\ -\frac{p_1}{p_2}y \end{pmatrix} \quad \text{where } y > \frac{1}{2} \quad \text{and} \quad \frac{p_1}{p_2}y \leq 1$$

$$d_3\left(\frac{p_2}{p_1}\right) = \begin{pmatrix} z \\ -\frac{p_1}{p_2}z \end{pmatrix} \quad \text{where} \quad z > \frac{1}{2} \quad \text{and} \quad \frac{p_1}{p_2}z \leq 1$$

Market clearing implies that

$$y + z \leq x$$

and $$\frac{p_1}{p_2}x \leq \frac{p_1}{p_2}(y + z)$$

or, $x = y + z$. But $y > \frac{1}{2}$, $z > \frac{1}{2} \Longrightarrow x > 1$ so that Mr. one would be forced to give up more votes than he has on issue one. Thus there cannot be an equilibrium in this range.

<u>Case 3.</u> $\frac{p_2}{p_1} \geq 2$

$$d_1\left(\frac{p_2}{p_1}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$d_2\left(\frac{p_2}{p_1}\right) = \begin{pmatrix} y \\ -\frac{p_1}{p_2}y \end{pmatrix} \quad \text{where } y > \frac{1}{2} \quad \text{and} \quad \frac{p_1}{p_2}y < \frac{1}{2}$$

$$d_3\left(\frac{p_2}{p_1}\right) = \begin{pmatrix} z \\ -\frac{p_1}{p_2}z \end{pmatrix} \quad \text{where } z > \frac{1}{2} \quad \text{and} \quad \frac{p_1}{p_2}z < \frac{1}{2}$$

Note that there is excess demand for votes on issue one and so there is no equilibrium in this interval as well.

Q.E.D.

What happens in the proof is, of course, at any set of prices, there is always someone in the market willing to trade who cannot find anyone to trade with. No investigation has yet been made of this phenomenon and so it is difficult to say how ubiquitous it is. It may be worth mentioning that the particular preferences given in the proof produce a cyclic majority dominance relation, and hence no majority winner.

Much more work remains to be done on the subject of the consistency of a competitive analysis of vote trading before firmer conclusions may be drawn. For now I am satisfied to undermine the facile assumption sometimes made that trading for votes may take place at equilibrium prices. This problem of course must encourage a return to the nontâtonnement methods of sections I and II of this paper if we are to obtain answers to the queries advanced in the introduction. Secondly, one must notice that unlike the case of economics,

no informational advantages appear to be held by the market mechanism over actual trading procedures. In both cases each actor is required to formulate beliefs about what their peers desire and what they will do in order to formulate his optimal strategies.

IV. DISCUSSION

The message of this paper is that we really know very little theoretically about vote trading. We cannot be sure about when it will occur, or how often, or what sorts of bargains will be made. We don't know if it has any desirable normative or efficiency properties. In short, we are incapable at present of providing much of a resolution to the divisions among scholars of the subject. A few observations seem worth making nevertheless.

First of all, in the case of trading procedures, it is apparent that if the legislature is modeled as a cooperative game in which any coalition may organize a trade, the social choice process will end up bearing a close relation to the majority domination relation. To the extent that we impose limits (whether institutional, normative, or behavioral) on cooperative behavior, this relation will become attenuated. At the extreme of a completely noncooperative analysis, Kramer's paper indicates that whether or not the majority dominance relation has a majority winner, there is always a noncooperative equilibrium. That is, some definite social choice will be produced no matter what the distribution of preferences. Of course one may not expect to find very strong normative justitication for the social choice in this case, but in the absence of a majority winner, one might argue that this equilibrium is as good as any other. At least Kramer's procedure produces the majority winner if there is one, and this seems to be a desirable property.

The meager results we have on pairwise trading indicate that if the legislature is not too small, then equilibria of such processes are likely to bear little relation to the majority dominance relation. Of

course we do not have an existence theorem comparable to Kramer's in this case so that we cannot claim that such procedures are generally determinate. However, if preferences are separable, if there is a majority winner, then it will be an equilibrium. It would appear that any restriction on coalition formation would be open to the sort of investigation we are discussing here. As of now, existence theorems are yet to be proved, and Theorem 3 must make us despair of finding any very general results along these lines.

Turning now to the analysis of markets for votes, several points suggest themselves. In economic life, markets are organized because some people stand to gain something through their operation. As far as I have been able to determine, such markets probably are not organized in legislatures for the complementary reason. Nothing is gained by anyone through their operation that could not be gained through a direct exchange. The very real indeterminancy of rational behavior in the presence of a price system is a reflection of the interdependency that a voting process imposes on the valuation of votes as commodities. These indeterminancies are in no way reduced by introducing a price system. Of course the fact that a price system fails to accomplish trading efficiencies in this case is compounded by the fact that equilibrium prices may fail to exist.

Even if these difficulties are ignored, it is not known, when competitive equilibria do exist, what properties they have. Since such equilibria will generally depend on expectations, it would be surprising if any general results are to be found. In any event no one seems to have produced any rigorous analysis along these lines.

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