A SUPPORT PRICE THEOREM FOR THE CONTINUOUS TIME MODEL OF CAPITAL ACCUMULATION

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1. INTRODUCTION

In this paper, we shall consider a model of capital accumulation and prove the existence of a support price path for the optimal path of capital accumulation. The considered economic model is a continuous time model of infinite horizon.

Under some assumptions of differentiability, we can obtain a dual path for the optimal path by the Euler equation, or by the maximum principle of Pontryagin [1962]. (See, for example, Halkin [1974] and Haurie [1976].) In this paper, however, we shall not make any differentiability assumptions. Instead, we shall assume the appropriate convexity of the model, which is more natural in economics than differentiability. Thus, our problem is, so to speak, the "convex" problem of optimal control without differentiability.

The convex problem of optimal control has been studied by Rockafellar [1971] and Halkin [1972]. In non-differentiable and convex models of finite horizon, they proved the existence of a dual path for the optimal path which "supports" the Hamiltonian function. It is difficult to compare our argument directly with their argument, since their formulations are much different from ours. However, our results are more general and useful in the following sense: First of all, the model considered in this paper is of infinite horizon. Second, our optimality criterion is a general one, that is, the so-called overtaking criterion

originally introduced by von Weizsacker [1965] and Gale [1967]. Third, we shall prove the existence of a dual price path which supports the value function as well as the Hamiltonian function. This property of the support price path was established by Benveniste and Scheinkman [1977] in a differentiable model with a somewhat stronger "interiority" assumption on the optimal path. The fact that a price path supports both the value function and the Hamiltonian function is particularly useful in proving the "turnpike" property of the optimal paths (see McKenzie [1976]).

The main result in the paper is Theorem I in section 4, in which the existence of a dual price path for the optimal path is proved. This theorem is a counterpart of the support price lemma proved by McKenzie [1976, L.1] in a discrete time model. One of the key lemmas in our argument is Lemma II in section 7, which exactly corresponds to the "induction" argument by Weitzman [1973] and McKenzie [1974 and 1976]. Of course, since our model is in continuous time, their induction procedure cannot be applied directly in our case. However, even in the case of continuous time models, their method is quite useful and actually makes the proof simpler and more elementary. A proof which is similar to ours is found in Halkin [1972], but his method seems to be effective only for finite horizon models.

2. MATHEMATICAL NOTATION

Let N be the set of all positive integers. For each $n \in N$, R^n denotes the n-dimensional Euclidean space. When $n = 1$, we write R instead of R^1. For any $x, y \in R^n$, the inner product of x and y is denoted by $x \cdot y$. The Euclidean norm of any $x \in R^n$ is denoted by $\|x\|$, i.e., $\|x\| = \sqrt{x \cdot x}$. For any subset U of R^n, int U denotes the interior of U in R^n and co U denotes the convex hull of U.

For any concave (or convex) function $f\colon U \to R \cup \{-\infty\} \cup \{+\infty\}$ defined on a convex subset U of R^n, symbol $\partial f(x)$ denotes the set of all subgradients of function f at $x \in U$, i.e.,

$$\partial f(x) = \{p \in R^n \mid f(x) - p \cdot x \geqq (\text{or} \leqq)\ f(y) - p \cdot y \text{ for all } y \in U\}$$

A mapping $F\colon U \to 2^{R^\ell}$ defined on a subset U of R^n to the family of all non-empty subsets of R^ℓ is called a <u>correspondence</u>. Correspondence F is called <u>lower semi-continuous</u> at $x_o \in U$ if, for any $y_o \in F(x_o)$ and any sequence $\{x_i\}_{i \in N}$ in U converging to x_o, there exists a sequence $\{y_i\}_{i \in N}$ converging to y_o such that $y_i \in F(x_i)$ for all $i \in N$. The correspondence F is called <u>lower semi-continuous</u> if F is lower semi-continuous at all $x \in U$.

A function $f\colon E \to R^n$ defined on a closed interval $E \subset R$ to R^n is called <u>absolutely continuous</u> if the restriction of f on any compact interval is absolutely continuous in the usual sense. Also, the derivative of f is denoted by $\dot{f}$.

Any definitional term from measure theory, such as "integrable," "measurable," and "almost every" should be interpreted in the sense of Lebesgue.

3. THE MODEL

Let $m \in N$ be the number of different commodities (capitals) in the economy. The technology of the economy is described by a correspondence $Y\colon [0,\infty) \to 2^{R^m \times R^m}$ mapping $t \in [0,\infty)$ to a subset $Y(t)$ of $R^m \times R^m$. The notation $(x,y) \in Y(t)$ means that at time t if we have amount x of commodities (capitals), we can increase the amount of the commodities by y. Namely, the pair (x,y) is a technologically possible combination of the amount of capital stock and the level of investment at time t.

Define a correspondence $X\colon [0,\infty) \to 2^{R^m}$ by

$$X(t) = \{x \in R^m \mid (x,y) \in Y(t) \text{ for some } y \in R^m\}$$

<u>Assumption I</u>:

(i) The correspondence Y is lower semi-continuous and convex-valued, i.e., $Y(t)$ is convex for all $t \in [0,\infty)$.

(ii) int $X(t) \neq \phi$ for all $t \in [0,\infty)$.

Social welfare at any point in time is represented by the instantaneous utility function $u\colon G_Y \to R$, where G_Y is the "graph" of the correspondence Y, i.e.,

$$G_Y = \{(x,y,t) \in R^m \times R^m \times [0,\infty) \mid (x,y) \in Y(t)\}.$$

Namely, for each $(x,y,t) \in G_Y$, $u(x,y,t)$ is interpreted as the maximum level of social satisfaction that can be attained at time t if the amount of capital stock is x and the level of investment is y.

Assumption II:

The function u is a continuous function such that, for each $t \varepsilon [0,\infty)$, $u(x,y,t)$ is a concave function in $(x \cdot y)$.

Remark 3.1:

Allowing $u(x,y,t)$ to assume the value $-\infty$ on the boundary of $Y(t)$ (where the boundary is taken relative to the smallest affine set containing $Y(t)$) would not be a more general assumption since setting $u(x,y,t)$ equal to $-\infty$ is equivalent to excluding (x,y) from $Y(t)$. We can always perform this latter operation because $Y(t)$ is not necessarily closed. (Note that such an operation does not destroy the convexity of $Y(t)$ because of the concavity of $u(x,y,t)$.)

An absolutely continuous function $k: [0,\infty) \to R^m$ is said to be a feasible path between time r and time s, where $r,s \varepsilon [0,\infty)$ and $r \leqq s$, if $(k(t),\dot{k}(t)) \varepsilon Y(t)$ for almost every $t \varepsilon [r,s]$. An absolutely continuous function $k: [0,\infty) \to R^m$ is called a feasible path from time r, where $r \varepsilon [0,\infty)$, if $(k(t),\dot{k}(t)) \varepsilon Y(t)$ for almost every $t \varepsilon [r,\infty)$. For each $x \varepsilon R^m$ and $r \varepsilon [0,\infty)$, let $A(x,r)$ denote the set all feasible paths k from time r such that $k(r) = x$.

Assumption III:

If k is a feasible path from time r, then

$$\int_r^s u(k(t),\dot{k}(t),t)dt < +\infty \text{ for all } s \varepsilon [r,\infty).$$

The above assumption enables us to define a criterion of optimality for feasible paths. A feasible path k_* from time r is said to be overtaken by a feasible path $k \varepsilon A(k_*(r),r)$ if there exist $\varepsilon > 0$ and $s_o \geqq r$ such that

$$\int_r^s u(k(t),\dot{k}(t),t)dt > \int_r^s u(k_*(t),\dot{k}_*(t),t)dt + \varepsilon$$

for all $s > s_o$. A feasible path k_* from time r is called an optimal path from time r if k_* is not overtaken by any $k \varepsilon A(k_*(r),r)$.

Remark 3.2:

This kind of optimality criterion was introduced by von Weizsacker [1965] and Gale [1967]. An optimal path as defined here is commonly called a "weakly maximal" path by Brock [1970] and McKenzie [1976].

4. NECESSARY CONDITIONS FOR THE OPTIMAL PATHS

Let k_* be an optimal path from time 0. Then, we can define a function $\bar{u}$: $G_Y \to R$ by

$$\bar{u}(x,y,t) = u(x,y,t) - u(k_*(t),\dot{k}_*(t),t) \quad \text{for each } (x,y,t) \in G_Y. \tag{4.1}$$

If $\int_r^s u(k_*(t),\dot{k}_*(t),t)dt > -\infty$ for all $r,s \in [0,\infty)$ with $r \leqq s$, then we can define a function V: $R^m \times [0,\infty) \to R \cup \{-\infty\} \cup \{+\infty\}$ by

$$V(x,r) = \sup_{k\in A(x,r)} [\liminf_{s\to\infty} \int_r^s \bar{u}(k(t),\dot{k}(t),t)dt] \quad \text{for each } (x,r) \in R^m \times [0,\infty). \tag{4.2}$$

For each $r \in [0,\infty)$, the "effective domain" of function $V(.,r)$ is deonted by $D(r)$, i.e.,

$$D(r) = \{x \in R^m | V(x,r) > -\infty\}. \tag{4.3}$$

Here, we should note that the above (4.1), (4.2), and (4.3) are defined for a particular optimal path k_* from time 0, and that they depend on the optimal path.

Remark 4.1:

The above-defined function V is commonly called the value function, which was introduced by McKenzie [1976] in the framework of overtaking-optimality criterion. We can easily check that the function V has the following properties:

(i) For each $r \in [0,\infty)$, $V(x,r)$ is a concave function over all $x \in D(r)$.

(ii) $V(k_*(t),t) = 0$, and $k_*(t) \in D(t)$ for all $t \in [0,\infty)$. In particular, $D(t) \neq \phi$ for all $t \in [0,\infty)$.

(iii) If k is a feasible path between time r and time s, then

$$V(k(r),r) \geqq \int_r^s \bar{u}(k(t),\dot{k}(t),t)dt + V(k(s),s).$$

Although the function u is continuous by Assumption II, the function $\bar{u}$ may not be continuous since $\dot{k}_*$ is not necessarily continuous. Therefore, we cannot identify the function $\bar{u}$ with the function u.

Assumption IV:

(i) $\int_r^s u(k_*(t),\dot{k}_*(t),t)dt > -\infty$ for all $r,s \in [0,\infty)$ with $r \leqq s$.

(ii) $k_*(t) \in \text{int } X(t)$ for all $t \in [0,\infty)$.

(iii) $\partial V(k_*(0),0) \neq \phi$, where $\partial V(.,0)$ denotes the set of all subgradients for function $V(.,0)$.

Theorem I:

Let k_* be an optimal path from time 0 satisfying Assumption IV. Then, under Assumption I, II, and III, for any $p \in \partial V(k_*(0),0)$ there exists an absolutely continuous function q_*: $[0,\infty) \to R^m$ with the following properties:

(i) $q_*(0) = p$.

(ii) $q_*(t) \in \partial V(k_*(t),t)$ for all $t \in [0,\infty)$

(iii) $-(\dot{q}_*(t),q_*(t)) \in \partial u(k_*(t),\dot{k}_*(t),t)$ for almost every $t \in [0,\infty)$.

In the above, for each $t \in [0,\infty)$, symbols $\partial V(.,t)$ and $\partial u(.,.,t)$ denote the sets of all subgradients for functions $V(.,t)$ and $u(.,.,t)$ respectively.

A proof of this theorem will be given later. The theorem presented here is a counterpart of the theorem which was proved by McKenzie [1976, L 1] in a discrete time model.

There are some new features in our theorem which are not found in the usual duality theorems for continuous time models. First, we have replaced the usual assumption of finiteness of the utility integral over the infinite horizon for all feasible paths by the weaker set -- Assumptions III and IV (i), (iii).

Second, condition (i) of our theorem says that we can choose any point in $\partial V(k_*(0),0)$ as an initial price for the support price path. That is, for any point in $\partial V(k_*(0),0)$, there exists a price path which starts from the point and supports the optimal path.

Third, the theorem says that conditions (ii) and (iii) hold at the same time. In other words, the price path q_* supports the value function $V(.,t)$ as well as the utility function $u(.,.,t)$ at every time t. The existence of a price path with such a property is not obvious in non-differentiable models.

Our theorem can be restated by using the Hamiltonian equation. Define a function $H: R^m \times R^m \times [0,\infty) \to R \cup \{-\infty\} \cup \{+\infty\}$ by $H(p.x,t) = \text{Sup}\,\{u(x,y,t) + p \cdot y \mid (x,y) \in Y(t)\}$, for each $(p,x,t) \in R^m \times R^m \times [0,\infty)$.

Remark 4.2:

The function H is commonly called the Hamiltonian function. It is well known that for each $t \in [0,\infty)$, $H(p,x,t)$ is a convex function in p and is a concave function in x.

Theorem I':

Let k_* be an optimal path from time 0 satisfying Assumption IV. Then, under Assumptions I, II, and III, for any $p \in \partial V(k_*(0),0)$ there exists an absolutely continuous function $q_*: [0,\infty) \to R^m$ with the following properties:

(i) $q_*(0) = p$.

(ii) $q_*(t) \in \partial V(k_*(t),t)$ for all $t \in [0,\infty)$.

(iii) $H(q_*(t),k_*(t),t) = u(k_*(t),\dot{k}_*(t),t) + q_*(t)\cdot\dot{k}_*(t)$ for almost every $t \in [0,\infty)$.

(iv) $\dot{k}_*(t) \in \partial_1 H(q_*(t),k_*(t),t)$ for almost every $t \in [0,\infty)$.

(v) $-\dot{q}_*(t) \in \partial_2 H(q_*(t),k_*(t),t)$ for almost every $t \in [0,\infty)$.

In the above, for each $t \in [0,\infty)$, symbols $\partial_1 H(.,k_*(t),t)$ and $\partial_2 H(q_*(t),;,t)$ denote the sets of all subgradients for functions $H(.,k_*(t),t)$ and $H(q_*(t),.,t)$ respectively.

Remark 4.3:

Theorem I and Theorem I' are equivalent to each other. In order to show the equivalence, it suffices to prove that condition (iii) of Theorem I implies conditions (iii), (iv), and (v) of

Theorem I', and conversely that conditions (iii) and (v) of Therorem I' imply condition (iii) of Theorem I. Although the verification is not entirely trivial, we shall not include it since the equivalence is a well-known fact.

The following theorem outlines a relation between the value function and the utility function, which was proved under somewhat stronger assumptions by Benveniste and Scheinkman [1971, Prop. 1].

<u>Theorem II</u>:

Let k_* be an optimal path from time 0 satifsying Assumption IV. Then, under Assumptions I, II, and III, the following holds:

$\partial V(k_*(t),t) \subset - \partial_2 u(k_*(t),k_*(t),t)$ for almost every $t \in [0,\infty)$,

where symbol $\partial_2 U(k_*(t),.,t)$ denotes the set of all subgradients for function $u(k_*(t),.,t)$ for each $t \in [0,\infty)$.

This theorem can be proved by using Theorem I. The proof will be given in a following section.

5. THE OUTLINE OF THE PROOF OF THEOREM I

In order to prove Theorem I, it suffices to show that the following auxiliary theorem is true.

<u>Auxiliary Theorem</u>:

Let k_* be an optimal path from time 0 satisfying Assumption IV. Then, under Assumptions I, II, and II, for any $p \in \partial V(k_*(0),0)$ there exists an absolutely continuous function $q_1 : [0,1] \to R^m$ with the following properties:

(i) $q_1(0) = p$

(ii) $q_1(t) \in \partial V(k_*(t),t)$ for all $t \in [0,1]$.

(iii) $-(\dot{q}_1(t),q_1(t)) \in \partial u(k_*(t),k_*(t),t)$ for almost every $t \in [0,1]$.

The auxilary theorem implies that since k_* is also an optimal path from time 1, there exists an absolutely continuous function $q_2 : [1,2] \to R^m$ with the following properties:

$q_2(1) = q_1(1)$.

$q_2(t) \in \partial V(k_*(t),t)$ for all $t \in [1,2]$.

$-(\dot{q}_2(t),q_2(t)) \in \partial u(k_*(t),k_*(t),t)$ for almost every $t \in [1,2]$.

By repeating the same argument and constructing such a function $q_n : [n-1,n] \to R^m$ for each $n \in N$, we can obtain an absolutely continuous function $q_* : [0,\infty) \to R^m$, which is defined by

$$q_*(t) = q_n(t) \text{ when } t \in [n-1,n].$$

Obviously, by construction, function q_* satisfies all the conditions required in Theorem I. Thus, we know that the Auxiliary Theorem implies Theorem I.

Furthermore, we can easily show that the following two propositions imply the Auxiliary Theorem.

Proposition I:

For all $t_o \in [0,\infty)$, there exist two numbers $r, s \in [0,\infty)$ with $r \leqq t_o < s$ ($r = t_o$ only when $t_o = 0$) such that there exist feasible paths k_i between time r and time s, $i = 0, 1, \ldots, m$, with the following properties:

(i) $k_*(t) \in \text{int co}\{k_0(t), k_1(t), \ldots, k_m(t)\}$ for all $t \in [r,s]$.

(ii) $\left|\int_r^s u(k_i(t), \dot{k}_i(t), t)dt\right| < \infty$ for all $i = 0,1,\ldots,m$.

Proposition II:

Suppose that there exist feasible paths k_i between time r and times, $i = 0,1,\ldots,m$, satisfying conditions (i) and (ii) in Proposition I. Then, for any $p \in \partial V(k_*(r),r)$ there exists an absolutely continuous function $q : [r,s] \to R^m$ with the following properties:

(i) $q(r) = p$.

(ii) $q(t) \in \partial V(k_*(t),t)$ for all $t \in [r,s]$.

(iii) $-(\dot{q}(t),q(t)) \in \partial u(k_*(t),\dot{k}_*(t),t)$ for almost every $t \in [r,s]$.

In fact, since $[0,1]$ is compact, Proposition I implies that there exist finitely many pairs $\{r_i, s_i\}$ with $r_i < s_i$, $i = 1,2,\ldots,\ell$, such that $[0,1] \subset \bigcup_{i=1}^{\ell} [r_i,s_i]$, and such that each pair $\{r_i, s_i\}$ has the desirable properties of the pair $\{r,s\}$ in the proposition. Without loss of generality, we can assume that

$$0 = r_1 < s_1 = r_2 < s_2 = r_3 < \ldots < s_{\ell-1} = r_\ell < s_\ell = 1.$$

Since $p \in \partial V(k_*(0),0)$ by assumption, by applying Proposition 2 to each pair $\{r_i,s_i\}$ successively from $i = 1$ to ℓ, we can construct the function $q_1 : [0,1] \to r^m$ desired in Auxiliary Theorem.

Thus, all we have to do is to prove Propositions I and II. This will be done in the following two sections.

Remark 5.1:

Proposition II may be called "the local existence theorem of a support price path. The proposition shows a sufficient condition for the existence of such a support price path, while Proposition I insures that the sufficient condition is indeed satisfied.

6. Proof of Proposition I

The following is one of the fundamental lemmas in our argument.

Lemma I:

For any $(x_o, y_o, t_o) \in G_Y$ with $x_o \in \text{int } X(t_o)$, there exist two numbers $r, s \in [0, \infty)$ with $r \leqq t_o < s$ ($r = t_o$ only when $t_o = 0$) such that there exists an absolutely continuous function $h: [r, s] \to R^m$ with the following properties:

(i) $(h(t), \dot{h}(t)) \in Y(t)$ for almost every $t \in [r, s]$.

(ii) The derivative $\dot{h}$ is a continuous function.

(iii) $(h(t_o), \dot{h}(t_o)) = (x_o, y_o)$.

Proposition I can be easily proved by this lemma. In fact, since $k_*(t_o) \in \text{int } X(t_o)$, there exist vectors $v_0, v_1, \ldots, v_m \in \text{int } X(t_o)$ such that $k_*(t_o) \in \text{int co } \{v_0, v_1, \ldots, v_m\} \subset \text{int } X(t_o)$. Therefore, by Lemma I, for each $i = 0, 1, \ldots, m$, there exist two numbers $r_i, s_i \in [0, \infty)$ with $r_i \leqq t_o < s_i$ ($r_i = t_o$ only when $t_o = 0$) such that there exists an absolutely continuous function $h_i: [r_i, s_i] \to R^m$ with the following properties:

(6.1) $(h_i(t), \dot{h}_i(t)) \in Y(t)$ for almost every $t \in [r_i, s_i]$.

(6.2) The derivative $\dot{h}_i$ is a continuous function.

(6.3) $h_i(t_o) = v_i$.

From (6.3), it follows that $k_*(t_o) \in \text{int co } \{h_0(t_o), h_1(t_o), \ldots, h_m(t_o)\}$. Therefore, since $h_0, h_1, \ldots, h_m$, and k_* are continuous functions, there exist two numbers $r, s \in [0, \infty)$ with $r_i \leqq r \leqq t_o < s \leqq s_i$ for all $i = 0, 1, \ldots, m$ ($r = t_o$ only when $t_o = 0$) such that

(6.4) $k_*(t) \in \text{int co } \{h_0(t), h_1(t), \ldots, h_m(t)\}$ for all $t \in [r, s]$.

For each $i = 0, 1, \ldots, m$, define a function $k_i: [0, \infty) \to R^m$ by

$$k_i(t) = \begin{cases} h_i(r) & \text{for each } t \in [0, r) \\ h_i(t) & \text{for each } t \in [r, s] \\ h_i(s) & \text{for each } t \in (s, \infty). \end{cases}$$

Then, by (6.1), $k_0, k_1, \ldots, k_m$ are feasible paths between time r and time s, and, by (6.4), satisfy condition (i) of Proposition I. Also, by (6.2) and Assumption II, for each $i = 0, 1, \ldots, m$, $u(k_i(t), \dot{k}_i(t), t)$ can be regarded as a continuous function of $t \in [r, s]$, and its integral exists. Thus, by definition of $\bar{u}$ and Assumptions III and IV(i), condition (ii) of Proposition I is proved. This completes the proof of Proposition I.

In order to prove Lemma I, we need the following three sublemmas.

Sublemma 6.1:

The correspondence $X: [0, \infty) \to 2^{R^m}$ is lower semi-continuous and convex-valued.

Proof: This sublemma is straightforward from Assumption I (i).

Q.E.D.

Sublemma 6.2:

For any $x_o \in R^m$ and $t_o \in [0, \infty)$ with $x_o \in \text{int } X(t_o)$, there exist a compact neighborhood U of x_o and two numbers $r, s \in [0, \infty)$ with $r \leqq t_o < s$ ($r = t_o$ only when $t_o = 0$) such that

$(x, t) \in U \times [r, s]$ implies $x \in \text{int } X(t)$.

Proof: Suppose that this sublemma is not true. Then, there exists a sequence $\{(x_n, t_n)\}_{n\in N}$ in $R^m \times [0, \infty)$ converging to a point (x_o, t_o) with $x_o \in \text{int } X(t_o)$ such that $x_n \notin \text{int } X(t_n)$ for all $n \in N$. Since $x_o \in \text{int } X(t_o)$, we can find vectors $v_0, v_1, \ldots, v_m \in X(t_o)$ such that

(6.5) $\quad x_o \in \text{int co } \{v_0, v_1, \ldots, v_m\}$.

Since the correspondence X is lower semi-continuous by Sublemma 6.1, for each $i = 0, 1, \ldots, m$, we have a sequence $\{v_i^n\}_{n\in N}$ converging to v_i such that $v_i^n \in X(t_n)$ for all $n \in N$. Therefore, from (6.5), it follows that $x_n \in \text{int co } \{v_0^n, v_1^n, \ldots, v_m^n\}$ for all sufficiently large $n \in N$. Since $X(t_n)$ is convex by Sublemma 6.1, this implies that $x_n \in \text{int } X(t_n)$ for all sufficiently large $n \in N$. This is a contradiction.

Q.E.D.

Let G_X denote the "graph" of the correspondence X, i.e.,

$$G_X = \{(x, t) \in R^m \times [0, \infty) \mid x \in X(t)\}.$$

Define a correspondence $F: G_X \to 2^{R^m}$ by

$$F(x, t) = \{y \in R^m \mid (x, y) \in Y(t)\} \text{ for each } (x, t) \in G_X.$$

Sublemma 6.3:

The correspondence F is convex-valued and lower semi-continuous at any $(x_o, t_o) \in G_X$ with $x_o \in \text{int } X(t_o)$.

Proof: Suppose that $x_o \in \text{int } X(t_o)$, $y_o \in F(x_o, t_o)$, and that a sequence $\{(x_n, t_n)\}_{n\in N}$ in G_X coverges to (x_o, t_o). Since $x_o \in \text{int } X(t_o)$, there are $(v_0, w_0), (v_1, w_1), \ldots, (v_m, w_m) \in Y(t_o)$ such that

(6.6) $\quad x_o \in \text{int co } \{v_0, v_1, \ldots, v_m\}$.

Since the correspondence Y is lower semi-continuous by Assumption I (i), for each $i = 0, 1, \ldots, m$, we have a sequence $\{(v_i^n, w_i^n)\}_{n\in N}$ converging to (v_i, w_i) such that $(v_i^n, w_i^n) \in Y(t_n)$ for all $n \in N$. Also, since $(x_o, y_o) \in Y(t_o)$, for the same reason, we have a sequence $\{(x_n', y_n')\}_{n\in N}$ converging to (x_o, y_o) such that $(x_n', y_n') \in Y(t_n)$ for all $n \in N$.

By (6.6), we know that there is a number $\epsilon_o > 0$ such that, for all sufficiently large $n \in N$,

(6.7) $\quad \|x - x_o\| < \epsilon_o$ implies $x \in \text{int co } \{v_0^n, v_1^n, \ldots, v_m^n\} \subset X(t_n)$.

Also, obviously, for all sufficiently large $n \in N$, we have

(6.8) $\quad \|x_n - x_o\| < \epsilon_o/3$ and $\|x_n' - x_o\| < \epsilon_o/3$.

Therefore, in proving the lower semi-continuity of F, we can assume without loss of generality that (6.7) and (6.8) are true for all $n \in N$.

For each $n \in N$ with $x_n' \neq x_n$, pick a point x_n'' such that $\epsilon_o/3 < \|x_n'' - x_o\| < \epsilon_o$ and $x_n = \theta_n x_n' + (1-\theta_n)x_n''$ for some $0 \leqq \theta_n \leqq 1$. And for each $n \in N$ with $x_n' = x_n$, let $x_n'' = x_n'$ and $\theta_n = 1$. Then, in any case, $x_n = \theta_n x_n' + (1-\theta_n)x_n''$ for all $n \in N$. Clearly, θ_n goes to 1 as n goes to ∞, since x_n and x_n' converge to x_o.

Moreover, for each $n \in N$, pick a point y_n'' such that $(x_n'', y_n'') \in Y(t_n)$ and $y_n'' \in \text{co } \{w_0^n, w_1^n, \ldots, w_m^n\}$. This is possible, since $\|x_n'' - x_o\| < \epsilon_o$, i.e., by (6.7), $x_n'' \in \text{int co } \{v_0^n, v_1^n, \ldots, v_m^n\}$ for all $n \in N$. Clearly, $\{y_n''\}_{n\in N}$ is a bounded sequence.

Let $y_n = \theta_n y_n' + (1-\theta_n)y_n''$ for each $n \in N$. Then, $(x_n, y_n) \in Y(t_n)$, that is, $y_n \in F(x_n, t_n)$ for all $n \in N$. Furthermore, y_n goes to y_o as n goes to ∞, since y_n' converges to y_o, θ_n converges to 1, and $\{y_n''\}_{n\in N}$ is bounded. This proves the lower semi-continuity of correspondence F.

Moreover, correspondence F is easily shown to be convex-valued, since correspondence Y is convex-valued.

Q.E.D.

Proof of Lemma I: Since $x_o \in \text{int } X(t_o)$, by Sublemma 6.2, we have a compact neighborhood U of x_o and two numbers r', $s' \in [0, \infty)$ with $r' \leqq t_o < s'$ ($r' = t_o$ only when $t_o = 0$) such that $(x, t) \in U \times [r', s']$ implies $x \in \text{int } X(t)$.

Define a correspondence $F': U \times [r', s'] \to 2^{R^m}$ by

$$F'(x, t) = \begin{cases} \{y_o\} & \text{for } (x, t) = (x_o, t_o) \\ F(x, t) & \text{for } (x, t) \neq (x_o, t_o). \end{cases}$$

By Sublemma 6.3, we can easily prove that correspondence F' is convex-valued and lower semi-continuous. Therefore, by a continuous selection theorem in Michael [1956, Th. 3.1'''], we have a continuous function $f: U \times [r', s'] \to R^m$ such that $f(x, t) \in F'(x, t)$ for all $(x, t) \in U \times [r', s']$. Hence, by a well-known theorem on the existence of solutions for ordinary differential equations (for example, see Filippov [1964, Th.4]), we have two numbers r, $s \in [r', s']$ with $r \leqq t_o < s$ ($r = t_o$ only when $t_o = r'$) and an absolutely continuous function $h: [r, s] \to R^m$ such that $h(t_o) = x_o$ and $\dot{h}(t) = f(h(t), t)$ for almost every $t \in [r, s]$. (When $r' = t_o$, we cannot apply such a theorem directly to function f, but to a continuous extension f' of f defined by

$$f'(x, t) = \begin{cases} f(x, t) & \text{for } (x, t) \in U \times [r', s'] \\ f(x, r') & \text{for } (x, t) \in U \times [r'-1, r'). \end{cases}$$

Therefore, our argument is true even in the case of $r' = t_o$.)

By construction of function f, we have conditions (i) and (iii) of Lemma I. Also, $\dot{h}(t) = f(h(t), t)$ is continuous since f is continuous. Namely, we have condition (ii) of the lemma.

Q.E.D.

7. Proof of Proposition II

The following lemma will play a central role in our argument. The lemma corresponds to the "induction" procedure by Weitzman [1973] and McKenzie [1974 and 1976] in discrete time models.

Lemma II:

Suppose that there exist feasible paths k_i between time r and time s, $i = 0, 1, \ldots, m$, satisfying conditions (i) and (ii) in Proposition I. Then, for any $t', t'' \in [r, s]$ with $t' \leqq t''$ and any $p' \in \partial V(k_*(t'), t')$, there exists $p'' \in \partial V(k_*(t''), t'')$ such that

$$\int_{t'}^{t''} u(k_*(t), \dot{k}_*(t), t)dt - p'.k_*(t') + p''.k_*(t'')$$

$$\geqq \int_{t'}^{t''} u(k(t), \dot{k}(t), t)dt - p'.k(t') + p''.k(t'')$$

for all feasible path k between time t' and t".

Proof: By definition of the value function V, we have

$$V(k(t'), t') \geqq \int_{t'}^{t''} \bar{u}(k(t), \dot{k}(t), t)dt + V(k(t''), t'')$$

for all feasible path k between time t' and time t". Also, since $p' \in \partial V(k_*(t'), t')$, we have

$$V(k_*(t'), t') - p'.k_*(t') \geqq V(x, t') - p'.x \quad \text{for all } x \in R^m.$$

The above two inequalities imply that

$$(7.1) \qquad \int_{t'}^{t''} \bar{u}(k_*(t), \dot{k}_*(t), t)dt + V(k_*(t''), t'') - p'.k_*(t')$$

$$\geqq \int_{t'}^{t''} \bar{u}(k(t), \dot{k}(t), t)dt + V(k(t''), t'') - p'.k(t')$$

for all feasible path k between time t' and time t".

Let α_* denote the left-hand side of inequality (7.1). Define two subsets C_1 and C_2 of R^{m+1} by

$$C_1 = \{(\alpha, x) \in R \times R^m \mid x = k(t'') \text{ and } \alpha > \alpha_* - \int_{t'}^{t''} \bar{u}(k(t), \dot{k}(t), t)dt + p'.k(t') \text{ for some feasible path k between time t' and time t''}\}$$

and

$$C_2 = \{(\alpha, x) \in R \times R^m \mid x \in D(t'') \text{ and } \alpha \leqq V(x, t'')\}$$

We can easily check that both C_1 and C_2 are non-empty and convex. Also, from (7.1), it follows that they are disjoint. Therefore, by a well-known separation theorem, we have a non-zero vector $(\pi, - p'') \in R \times R^m$ such that

$$(7.2) \qquad \pi[\alpha_* - \int_{t'}^{t''} \bar{u}(k(t), \dot{k}(t), t)dt + p'.k(t')] - p''.k(t'')$$

$$\geqq \pi V(x, t'') - p''.x$$

for all $x \in D(t'')$ and all feasible path k between time t' and time t" with $|\int_{t'}^{t''} \bar{u}(k(t), \dot{k}(t), t)dt| < \infty$.

Put $k = k_*$ in (7.2). Then,

$$(7.3) \qquad \pi V(k_*(t''), t'') - p''.k_*(t'') \geqq \pi V(x, t'') - p''.x \quad \text{for all } x \in D(t'').$$

Also, put $x = k_*(t'')$ in (7.2). Then,

$$(7.4) \qquad \pi\left[\int_{t'}^{t''} \bar{u}(k_*(t), \dot{k}_*(t), t)dt - p'.k_*(t')\right] + p''.k_*(t'')$$

$$\geqq \pi\left[\int_{t'}^{t''} \bar{u}(k(t), \dot{k}(t), t)dt - p'.k(t')\right] + p''.k(t'')$$

for all feasible path k between time t' and time t'' with

$$\left|\int_{t'}^{t''} \bar{u}(k(t), \dot{k}(t), t)dt\right| < \infty.$$

We can easily see that the particular forms of C_1 and C_2 imply $\pi \geqq 0$. Suppose that $\pi = 0$. Then, it follows from (7.4) that $p''.k_*(t'') \geqq p''.k_i(t'')$ for all $i = 0, 1, \ldots, m$, where $k_0, k_1, \ldots, k_m$ are functions assumed to exist in this lemma. Therefore, since $k_0, k_1, \ldots, k_m$ satisfy condition (i) of Proposition I, we can conclude that $p'' = 0$. However, this is a contradiction to the premise that $(\pi, -p'') \neq 0$. Thus, we have proved that $\pi > 0$.

Without loss of generality, we can put $\pi = 1$. Therefore, by (7.3), we have $p'' \in \partial V(k_*(t''), t'')$. Also, since $\pi = 1$, in (7.4) we can ignore the condition of $\left|\int_{t'}^{t''} \bar{u}(k(t), \dot{k}(t), t)dt\right| < \infty$. Moreover, by definition of $\bar{u}$, we can replace $\bar{u}$ in (7.4) by u. This completes the proof of Lemma II.

Q.E.D.

Now let us begin to prove Proposition II. Pick $p \in \partial V(k_*(r), r)$. For each $n \in N$, define a finite subset T_n of $[r, s]$ by

$$T_n = \{t \in [r, s] \mid t = r + \frac{i(s-r)}{2^n}, i = 0, 1, \ldots, 2^n\}.$$

Apply Lemma II to each pair $\{r + \frac{(i-1)(s-r)}{2^n}, r + \frac{i(s-r)}{2^n}\}$ successively from $i = 1$ to 2^n. Then we have $(2^n + 1)$-tuple of vectors denoted by $\{p_n(t) \mid t \in T_n\}$, where $p_n(r) = p$, such that

$$(7.5) \qquad p_n(t) \in \partial V(k_*(t), t) \quad \text{for all } t \in T_n \text{ and}$$

$$(7.6) \qquad \int_{t'}^{t''} u(k_*(t), \dot{k}_*(t), t)dt - p_n(t').k_*(t') + p_n(t'').k_*(t'')$$

$$\geqq \int_{t'}^{t''} u(k(t), \dot{k}(t), t)dt - p_n(t').k(t') + p_n(t'').k(t'')$$

for all $t', t'' \in T_n$ with $t' \leqq t''$ and all feasible path k between time t' and time t''.

We can prove the following:

(7.7) Set $\{p_n(t) \mid n \in N \text{ and } t \in T_n\}$ is bounded.

Suppose that this is not true. Then, there is an infinite subset N_o of N such that for each $n \in N_o$ we can pick up $t_n \in T_n$ and $\|p_n(t_n)\|$ goes to ∞ as $n \in N_o$ goes to ∞. Without loss of generality, we can assume that $\lim_{n\to\infty} t_n = t_o$ and $\lim_{n\to\infty} \frac{p_n(t_n)}{\|p_n(t_n)\|} = p_o \neq 0$. On the other hand, by (7.6) we have

$$\left[\int_r^{t_n} u(k_*(t), \dot{k}_*(t), t)dt - p.k_*(r) + p_n(t_n).k_*(t_n)\right]/\|p_n(t_n)\|$$

$$\geqq \left[\int_r^{t_n} u(k(t), \dot{k}(t), t)dt - p.k(r) + p_n(t_n).k(t_n)\right]/\|p_n(t_n)\|$$

for all $n \in N_o$ and all feasible path k between time r and time s.

Therefore, in the limit, $p_o.k_*(t_o) \geqq p_o.k(t_o)$ for all feasible path k between time r and time s. By assumption of the existence of functions $k_0, k_1, \ldots, k_m$, satisfying condition (i) of Proposition I, we can conclude that $p_o = 0$. This is a contradictioon. Thus, (7.7) is proved.

Let $T = \bigcup_{n \in N} T_n$. We can prove the following:

(7.8) There is a bounded function q_o: $T \to R^m$ with the following properties:

(i) $q_o(r) = p$.

(ii) $q_o(t) \in \partial V(k_*(t), t)$ for all $t \in T$.

(iii) $$\int_{t'}^{t''} u(k_*(t), \dot{k}_*(t), t)dt - q_o(t').k_*(t') + q_o(t'').k_*(t'')$$

$$\geqq \int_{t'}^{t''} u(k(t), \dot{k}(t), t)dt - q_o(t').k(t') + q_o(t'').k(t'')$$

for all $t', t'' \in T$ with $t' \leqq t''$ and all feasible path k between time t' and time t".

For each $t \in T_1$, we have a sequence $\{p_n(t) \mid n \geqq 1 \text{ and } n \in N\}$. Since T_1 is a finite set, by (7.7) we can find an infinite subset N_1 of N such that for any $t \in T_1$, sequence $\{p_n(t) \mid n \in N_1\}$ converges to a point, say $q_o(t)$. Then, for each $t \in T_2$, we have a sequence $\{p_n(t) \mid n \geqq 2$ and $n \in N_1\}$. Again, since T_2 is a finite set, by (7.7) we have an infinite subset N_2 of N_1 such that for any $t \in T_2$, sequence $\{p_n(t) \mid n \in N_2\}$ converges to a point, say $q_o(t)$. (Although $T_1 \subset T_2$, this notation is consistent since $N_1 \supset N_2$.) By repeating this procedure, we have $N_1 \supset N_2 \supset$

$N_3 \supset, \ldots$ such that for any $i \in N$ and any $t \in T_i$, sequence $\{p_n(t) \mid n \in N_i\}$ converges to a point $q_o(t)$. Therefore, by picking up a number n_i from each N_i, we have an infinite subset of N denoted by $N_* = \{n_1, n_2, n_3, \ldots\}$ such that if $t \in T_i$ for some $i \in N$, then sequence $\{p_n(t) \mid n \geqq i$ and $n \in N_*\}$ converges to $q_o(t)$. In this way, we can define a function q_o: $T \to R^m$, which is bounded because of (7.7). Obviously, condition (i) of (7.8) holds, since $p_n(r) = p$ for all $n \in N$. If $t \in T$, i.e, $t \in T_i$ for some $i \in N$, then (7.5) is true for all $n \in N_*$ with $n \geqq i$. Since set $\partial V(k_*(t), t)$ is closed, condition (ii) of (7.8) holds in the limit. Also, if $t', t'' \in T$ and $t' \leqq t''$, then $t', t'' \in T_j$ for some $j \in N$. Therefore, (7.6) is true for all $n \in N_*$ with $n \geqq i$. Thus, condition (iii) of (7.8) holds in the limit. This completes the proof of (7.8).

Suppose that function q_o is not continuous. Then, since function q_o is bounded, there are sequences $\{t'_n\}_{n \in N}$ and $\{t''_n\}_{n \in N}$ converging to a point t_o such that $t'_n \leqq t_o \leqq t''_n$ for all $n \in N$ and $\lim_{n\to\infty} (q_o(t''_n) - q_o(t'_n)) = \bar{p} \neq 0$. By condition (iii) of (7.8), we have $\bar{p}.k_*(t_o) \geqq \bar{p}.k(t_o)$ for all feasible path k between time r and time s. By assumption of the existence of functions $k_0, k_1, \ldots, k_m$ satisfying condition (i) of Proposition I, we can conclude that $\bar{p} = 0$. This is a contradiction. Thus, function q_o is proved to be a continuous function. Hence, since T is a dense subset of $[r, s]$, function q_o can be uniquely extended to a continuous function, say q: $[r, s] \to R^m$.

We can prove the following:

(7.9) The continuous function $q\colon [r, s] \to R^m$ satisfies the following conditions:

(i) $q(r) = p$.

(ii) $q(t) \in \partial V(k_*(t), t)$ for all $t \in [r, s]$.

(iii) $\int_{t'}^{t''} u(k_*(t), \dot{k}_*(t), t)dt - q(t').k_*(t') + q(t'').k_*(t'')$

$$\geqq \int_{t'}^{t''} u(k(t), \dot{k}(t), t)dt - q(t').k(t') + q(t'').k(t'')$$

for all $t', t'' \in [r, s]$ with $t' \leqq t''$ and all feasible path k between time t' and time t''.

Condition (i) of (7.9) obviously follows from condition (i) of (7.8). Also, since function q is a continuous extension of function q_o and since T is dense in $[r, s]$, condition (iii) of (7.8) implies condition (iii) of (7.9). To prove condition (ii) of (7.9), let $x_o \in \text{int } X(t_o)$ and $t_o \in (r, s]$. Then, by Lemma I, we have an absolutely continuous function $h\colon [r', t_o] \to R^m$, where $r \leqq r' < t_o$, satisfying the following conditions:

$(h(t), \dot{h}(t)) \in Y(t)$ for almost every $t \in [r', t_o]$.

The derivative $\dot{h}$ is a continuous function.

$h(t_o) = x_o$.

Since T is dense in $[r, s]$, we have a sequence $\{t_n\}_{n \in N}$ converging to t_o such that $t_n \in T \cap (r', t_o]$ for all $n \in N$. Therefore, by condition (ii) of (7.8), for all $n \in N$

$$V(k_*(t_n), t_n) - q(t_n).k_*(t_n) \geqq V(h(t_n), t_n) - q(t_n).h(t_n).$$

Namely, by definition of the value function, for all $n \in N$

$$\int_{t_n}^{t_o} \bar{u}(k_*(t), \dot{k}_*(t), t)dt + V(k_*(t_o), t_o) - q(t_n).k_*(t_n)$$

$$\geqq \int_{t_n}^{t_o} \bar{u}(h(t), \dot{h}(t), t)dt + V(h(t_o), t_o) - q(t_n).h(t_n).$$

Thus, in the limit, $V(k_*(t_o), t_o) - q(t_o).k_*(t_o) \geqq V(x_o, t_o) - q(t_o).x_o$. This implies $q(t_o) \in \partial V(k_*(t_o), t_o)$, since $k_*(t_o) \in \text{int } X(t_o)$ by assumption. Also, $q(r) = p \in \partial V(k_*(r), r)$. Thus, condition (ii) of (7.9) is proved.

Now we can prove the following:

(7.10) The function q is absolutely continuous.

By (7.9), we have

$$\int_{t'}^{t''} u(k_*(t), \dot{k}_*(t), t)dt + q(t'').(k_*(t'') - k_*(t'))$$

$$- \int_{t'}^{t''} u(k_i(t), \dot{k}_i(t), t)dt - q(t'').(k_i(t'') - k_i(t'))$$

$$\geqq (q(t'') - q(t')).(k_i(t') - k_*(t'))$$

for all $t', t'' \in [r, s]$ with $t' \leqq t''$ and all $i = 0, 1, \ldots, m$, where $k_0, k_1, \ldots, k_m$ are functions which are assumed to exist in Proposition II. Since functions $k_0, k_1, \ldots, k_m$ satisfy condition (i) of Proposition I, we can easily prove the following facts:

(i) For all $t', t'' \in [r, s]$ with $t' \leqq t''$,

$$\max_{0 \leqq i \leqq m} (q(t'') - q(t')).(k_i(t') - k_*(t')) \geqq 0.$$

(ii) There exists $\lambda > 0$ such that $\|k_i(t) - k_*(t)\| \geqq \lambda$ for all $t \in [r, s]$ and all $i = 0, 1, \ldots, m$.

(iii) There exists $\theta > 0$ such that if $v \in R^m$ and $t \in [r, s]$, then $v.(k_i(t) - k_*(t)) \geqq \theta\|v\|.\|k_i(t) - k_*(t)\|$ for some i.

Also, there exists $\beta > 0$ such that $\|q(t)\| \leqq \beta$ for all $t \in [r, s]$, since function q is continuous. Therefore, we can derive the following inequality:

$$\frac{(m+1)}{\lambda\theta}\Big[\int_{t'}^{t''} |u(k_*(t), \dot{k}_*(t), t)|dt + \beta\|k_*(t'') - k_*(t')\|\Big]$$

$$+ \frac{1}{\lambda\theta}\Sigma_{i=0}^{m}\Big[\int_{t'}^{t''} |u(k_i(t), \dot{k}_i(t), t)|dt + \beta\|k_i(t'') - k_i(t')\|\Big]$$

$$\geqq \frac{1}{\lambda\theta} \max_{0\leqq i\leqq m} \Big[\int_{t'}^{t''} u(k_*(t), \dot{k}_*(t), t)dt + q(t'').(k_*(t'') - k_*(t'))$$

$$- \int_{t'}^{t''} u(k_i(t), \dot{k}_i(t), t)dt - q(t'').(k_i(t'') - k_i(t'))\Big]$$

$$\geqq \frac{1}{\lambda\theta} \max_{0\leqq i\leqq m} (q(t'') - q(t')).(k_i(t') - k_*(t'))$$

$$\geqq \frac{1}{\lambda\theta} \min_{0\leqq i\leqq m} \theta\|q(t'') - q(t')\|.\|k_i(t') - k_*(t')\|$$

$$\geqq \|q(t'') - q(t')\|$$

for all $t', t'' \in [r, s]$ with $r' \leqq t''$.

By the above inequality, since Lebesgue integrals are absolutely continuous and since functions k_*, k_0, k_1, $\ldots$, k_m are absolutely continuous, we can easily show that function q is absolutely continuous.

In order to complete the proof of Proposition II, by virtue of (7.9) and (7.10), we have only to prove the following:

(7.11) $\quad - (\dot{q}(t), q(t)) \in \partial u(k_*(t), \dot{k}_*(t), t)$ for almost every $t \in [r, s]$.

First we should note (see, for example, Natanson [1955, p.255]) that for almost every $t_o \in [r, s)$

$$\lim_{\theta\to 0+} \frac{1}{\theta} \int_{t_o}^{t_o+\theta} u(k_*(t), \dot{k}_*(t), t)dt = u(k_*(t_o), \dot{k}_*(t_o), t_o),$$

$$\lim_{\theta\to 0+} \frac{q(t_o+\theta) - q(t_o)}{\theta} = \dot{q}(t_o), \text{ and}$$

$$\lim_{\theta\to 0+} \frac{k_*(t_o+\theta) - k_*(t_o)}{\theta} = \dot{k}_*(t_o).$$

For such a point $t_o \in [r, s)$, suppose that $(x_o, y_o) \in Y(t_o)$ and $x_o \in \text{int } X(t_o)$. Then, by Lemma I, there exist a number s' with $t_o < s' < s$ and an absolutely continuous function h: $[t_o, s'] \to R^m$ satisfying the following conditions:

$(h(t), \dot{h}(t)) \in Y(t)$ for almost every $t \in [t_o, s']$.

$\dot{h}$ is a continuous function.

$h(t_o) = x_o$ and $\dot{h}(t_o) = y_o$.

Since functions u and $\dot{h}$ are continuous, we have

$$\lim_{\theta\to 0+} \frac{1}{\theta} \int_{t_o}^{t_o+\theta} u(h(t), \dot{h}(t), t)dt = u(h(t_o), \dot{h}(t_o), t_o)$$

$$= u(x_o, y_o, t_o)$$

and

$$\lim_{\theta\to 0+} \frac{h(t_o+\theta) - h(t_o)}{\theta} = \dot{h}(t_o) = y_o,$$

Furthermore, by condition (iii) of (7.9), for all $\theta > 0$ with $\theta < s'-t_o$,

$$\frac{1}{\theta}\int_{t_o}^{t_o+\theta} u(k_*(t), \dot{k}_*(t), t)dt + \frac{q(t_o+\theta) - q(t_o)}{\theta}.k_*(t_o)$$

$$+ q(t_o+\theta).\frac{k_*(t_o+\theta) - k_*(t_o)}{\theta}$$

$$\geqq \frac{1}{\theta}\int_{t_o}^{t_o+\theta} u(h(t), \dot{h}(t), t)dt + \frac{q(t_o+\theta) - q(t_o)}{\theta}.h(t_o)$$

$$+ q(t_o+\theta).\frac{h(t_o+\theta) - h(t_o)}{\theta}$$

Therefore, in the limit, we have

$$u(k_*(t_o), \dot{k}_*(t_o), t_o) + \dot{q}(t_o).k_*(t_o) + q(t_o).\dot{k}_*(t_o)$$

$$\geqq u(x_o, y_o, t_o) + \dot{q}(t_o).x_o + q(t_o).y_o.$$

This inequality holds for all $(x, y) \in Y(t_o)$, because, by convexity of $Y(t_o)$, any point $(x, y) \in Y(t_o)$ can be represented as a limit point of a sequence $\{(x_n, y_n)\}_{n\in N}$ with $(x_n, y_n) \in Y(t_o)$ and $x_n \in \text{int } X(t_o)$ for all $n \in N$. In fact, let $(x, y) \in Y(t_o)$, $(x_o, y_o) \in Y(t_o)$ and $x_o \in \text{int } X(t_o)$. For each $0 < \theta \leq 1$, define $(x_\theta, y_\theta) = \theta(x, y) + (1-\theta)(x_o, y_o)$. Then, $(x_\theta, y_\theta) \in Y(t_o)$ and $x_\theta \in \text{int } X(t_o)$ for all $0 < \theta < 1$. Thus, we have proved that $-(\dot{q}(t_o), q(t_o)) \in \partial u(k_*(t_o), \dot{k}_*(t_o), t_o)$, i.e., (7.11).

This completes the proof of Proposition II.

8. Proof of Theorem II

First we should note (see, for example, Natanson [1955, p.255]) that for almost every $t_o \in [0, \infty)$

$$\lim_{\theta\to 0+} \frac{1}{\theta}\int_{t_o}^{t_o+\theta} u(k_*(t), \dot{k}_*(t), t)dt = u(k_*(t_o), \dot{k}_*(t_o), t_o) \text{ and}$$

$$\lim_{\theta\to 0+} \frac{k_*(t_o+\theta) - k_*(t_o)}{\theta} = \dot{k}_*(t_o).$$

Let t_o be such a point and $(k_*(t_o), y_o) \in Y(t_o)$. Since $k_*(t_o) \in \text{int } X(t_o)$, by Lemma I there exist a number $s > t_o$ and an absolutely continuous function $h: [t_o, s] \to R^m$ with the following properties:

$(h(t), \dot{h}(t)) \in Y(t)$ for almost every $t \in [t_o, s]$.

$\dot{h}$ is a continuous function.

$h(t_o) = k_*(t_o)$ and $\dot{h}(t_o) = y_o$.

Since functions $\dot{h}$ and u are continuous, we have

$$\lim_{\theta\to 0+} \frac{1}{\theta}\int_{t_o}^{t_o+\theta} u(h(t), \dot{h}(t), t)dt = u(h(t_o), \dot{h}(t_o), t_o)$$

$$= u(k_*(t_o), y_o, t_o)$$

and

$$\lim_{\theta\to 0+} \frac{h(t_o+\theta) - h(t_o)}{\theta} = \dot{h}(t_o) = y_o.$$

Furthermore, since k_* is also an optimal path from time t_o, by Theorem I, for any $p_o \in \partial V(k_*(t_o), t_o)$ there exists an absolutely continuous function $q_o: [t_o, \infty) \to R^m$ such that

$q_o(t_o) = p_o$ and

$-(\dot{q}_o(t), q(t)) \in \partial u(k_*(t), \dot{k}_*(t), t)$ for almost every $t \in [t_o, \infty)$.

Therefore, for almost every $t \in [t_o, s]$

$$u(k_*(t), \dot{k}_*(t), t) + \dot{q}_o(t).k_*(t) + q_o(t).\dot{k}_*(t)$$

$$\geqq u(h(t), \dot{h}(t), t) + \dot{q}_o(t).h(t) + q_o(t).\dot{h}(t).$$

By integrating this inequality, for all $\theta > 0$ with $\theta \leqq s-t_o$ we have

$$\int_{t_o}^{t_o+\theta} u(k_*(t), \dot{k}_*(t), t)dt - q_o(t_o).k_*(t_o) + q_o(t_o+\theta).k_*(t_o+\theta)$$

$$\geqq \int_{t_o}^{t_o+\theta} u(h(t), \dot{h}(t), t)dt - q_o(t_o).h(t_o) + q_o(t_o+\theta).h(t_o+\theta).$$

i.e.,

$$\frac{1}{\theta}\int_{t_o}^{t_o+\theta} u(k_*(t), \dot{k}_*(t), t)dt + q_o(t_o+\theta).\frac{k_*(t_o+\theta) - k_*(t_o)}{\theta}$$

$$\geqq \frac{1}{\theta}\int_{t_o}^{t_o+\theta} u(h(t), \dot{h}(t), t)dt + q_o(t_o+\theta).\frac{h(t_o+\theta) - h(t_o)}{\theta}.$$

Thus, in the limit, we have

$$u(k_*(t_o), \dot{k}_*(t_o), t_o) + p_o.\dot{k}_*(t_o) \geqq u(k_*(t_o), y_o, t_o) + p_o.y_o.$$

Namely, $- p_o \in \partial_2 u(k_*(t_o), \dot{k}_*(t_o), t_o)$. Hence, we have

$$\partial V(k_*(t_o), t_o) \subset - \partial_2 u(k_*(t_o), \dot{k}_*(t_o), t_o).$$

This completes the proof of Theorem II.

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