

Cognitive Hierarchies in Extensive Form Games*

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Abstract

The cognitive hierarchy (CH) approach posits that players in a game are heterogeneous with respect to levels of strategic sophistication. A level- k player believes all other players in the game have lower levels of sophistication distributed from 0 to $k-1$, and these beliefs correspond to the truncated distribution of a “true” distribution of levels. We extend the CH framework to extensive form games, where these initial beliefs over lower levels are updated as the history of play in the game unfolds, providing information to players about other players’ levels of sophistication. For a class of centipede games with a linearly increasing pie, we fully characterize the dynamic CH solution and show that it leads to the game terminating earlier than in the static CH solution for the centipede game in reduced normal form.

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“What surprised me was how bad they played.”

—Beth Harmon, *The Queen’s Gambit*

1 Introduction

In many situations, people interact with one another over time, in a multi-stage environment, such as playing chess or bargaining with alternating offers. The standard approach to studying these situations is to model them as extensive form games where equilibrium theory is applied, usually with refinements such as subgame perfection or other notions of sequential rationality. However, sequential rationality is a strong empirical assumption, perhaps implausibly strong, since behavior in many laboratory experiments systematically violates it (see, for example, [Camerer \(2003\)](#)).

In response to these anomalous findings, researchers have proposed a variety of models that relax the full rationality assumptions embodied in standard equilibrium concepts, such as Nash equilibrium. The focus of this paper is the “level- k ” family of models, which assume a hierarchical structure of strategic sophistication among the players, where level- k sophisticated players can think k strategic steps and believe everyone else is less sophisticated in the sense that they think fewer than k strategic steps. The *standard* level- k model assumes level- k players believe all other players are level- $(k-1)$ (see [Nagel \(1995\)](#)).

However, applications of the level- k approach have been limited almost exclusively to the analysis of games in strategic form, where all players make their moves simultaneously, and the theory has not been formally developed for the analysis of general games in extensive form.¹ To apply the standard level- k model to extensive form games, one would assume that at each decision node, a level- k player will choose an action that maximizes the continuation value of the game, assuming all other players are level- $(k-1)$ players in the continuation game. As a result, each player’s belief about other players’ level is *fixed* at the beginning. However, as the game proceeds, this fixed belief can lead to a logical conundrum, as a level- k player can be “surprised” by an opponent’s previous move that is not consistent with the strategy of a level- $(k-1)$ player.

If one closely examines this problem, the incompatibility derives from two sources that imply players cannot learn: (1) each level of player’s prior belief about the other players’ levels is degenerate, i.e., a singleton; and (2) players ignore the information contained in the history of the game. To solve both of these problems at the same time, as an alternative to the standard level- k approach, we use the cognitive hierarchy (CH) version of level- k , as proposed by [Camerer et al. \(2004\)](#), and extend it to games in extensive form. Like the standard level- k model, the CH framework posits that players are heterogeneous with respect to levels of strategic sophistication and believe that other players are less sophisticated. However, their beliefs are not degenerate. A level- k player believes all other players have lower levels distributed anywhere from level 0 to $k-1$.

Furthermore, the CH framework imposes a partial consistency requirement that ties the players’ prior beliefs on the level-type space to the true underlying distribution of levels.

¹Some special cases have been studied, which we discuss below.

Specifically, a level- k player’s beliefs are specified as the truncated true distribution of levels, conditional on levels ranging from 0 to $k-1$, i.e., players have “truncated rational expectations.” This specification has the important added feature, relative to the standard level- k model, that more sophisticated players also have beliefs that are closer to the true distribution of levels, and very high level types have approximate rational expectations about the behavior of the other players. Thus, the CH approach blends aspects of purely behavioral models and equilibrium theory.

In our extension of CH to games in extensive form, a player’s prior beliefs over lower levels are updated as the history of play in the game unfolds, revealing information about the distribution of other players’ levels of sophistication. These learning effects can be quite substantial as we illustrate later in the paper. Hence, the main contribution of this paper is to propose a new CH framework for the general analysis of games in extensive form and, in doing so, provide new insights beyond those offered by the original CH model.

Our first result establishes that every player will update their belief about each of the opponent’s levels independently (Proposition 1). Second, we show that when the history of play in the game unfolds, players become more certain about the opponents’ levels of sophistication, in a specific way. Formally, the support of their beliefs shrinks as the history gets longer (Proposition 2). Third, we show that the probability of paths with strictly dominated strategies being realized converges to zero as the distribution of levels increases (Proposition 3). Nonetheless, solution concepts based on iterated dominance, such as forward induction, can be violated even at the limit when the average level of sophistication converges to infinity. Relatedly, even though the players fully exploit the information from the history, it is not guaranteed that high-level players will use strategies that are consistent with the subgame perfect equilibrium of the game. In fact, behavior of the most sophisticated players can be inconsistent with backward induction, even at the limit when the level of sophistication of all players is arbitrarily high.

Although backward induction is a cornerstone principle of game theory, laboratory experiments reveal systematic behavioral deviations even in very simple games of perfect information. One prominent class of games where observed behavior is grossly inconsistent with backward induction is the increasing-pie “centipede game.” This is an alternating move two-person game, where, in turn, each player can either “take” the larger of two pieces of the current pie, which terminates the game and leaves the other player with the smaller piece of the current pie, or “pass,” which increases the size of the pie and allows the other player to take or pass. The game continues for a predetermined maximum number of turns. The subgame perfect equilibrium of this game of perfect information is solved by backward induction. Payoffs are such that it is optimal to take at the last stage, and both players have an optimal strategy to take if they expect the opponent will take at the next stage. Thus, backward induction implies that the game should end immediately.²

Starting with [McKelvey and Palfrey \(1992\)](#), several laboratory and field experiments

²[Rosenthal \(1981\)](#) introduced the centipede game to demonstrate how backward induction can be challenging and implausible to hold in some environments due to logical issues about updating off-path beliefs. His example is a ten-node game with a linearly increasing pie. Later on a shorter variant with an exponentially increasing pie, called “share or quit,” is studied by [Megiddo \(1986\)](#) and [Aumann \(1988\)](#). The name centipede was coined by [Binmore \(1987\)](#), and named for a 100-node variant.

have reported experimental data from centipede games in a range of environments, such as different lengths of the game (see McKelvey and Palfrey (1992) and Fey et al. (1996)), different subject pools (see Palacios-Huerta and Volij (2009), Levitt et al. (2011), and Li et al. (2021)), different payoff configurations (see García-Pola et al. (2020b), Fey et al. (1996), Zauner (1999), Kawagoe and Takizawa (2012), and Healy (2017)) and different experimental methods (Nagel and Tang (1998), Bornstein et al. (2004), García-Pola et al. (2020a), and Rapoport et al. (2003)). Although standard game theory predicts the game should end in the first stage, such behavior is rarely observed. To this end, we study a family of centipede games with a linearly increasing pie where our dynamic CH theory makes clear predictions about the evolution of beliefs as the game unfolds. Our particular interest in the analysis is the finding that dynamic CH implies a *representation effect* that predicts specific violations of strategic invariance. Consider the following two strategically equivalent extensive form representations of the centipede game. In the first (the usual representation used in centipede experiments), the game is played as an alternate move game: that is, first player 1 decides to take or pass. If they take the game ends; if they pass, it is player 2’s turn to pass or take, and so forth. In the second representation, first player 1 decides at which node to take if the game gets that far, or always pass; the second player then makes the same decision without observing the first player’s choice. We show that dynamic CH implies that centipede games played according to the first extensive form representation of the game will end *earlier, with lower payoffs to the players*, compared to the game played according to the second (strategically equivalent) version, where each player independently chooses a stopping strategy, without observing the choice of their opponent (Theorem 1).

Our theoretical analysis of the representation effect in centipede games shows how dynamic CH can provide new insights related to an unresolved debate in experimental methodology: whether or not the *direct-response method* is behaviorally equivalent to the *strategy method* (see Brandts and Charness (2011)). Under the direct-response method of obtaining behavioral data for a game, the game is played sequentially according to its extensive form representation. Thus, each player can observe the actions taken by others in the previous subgames and update their beliefs about other players’ levels. In contrast, under the strategy method, the game is played simultaneously according to the reduced normal form of the game. Thus, players take actions in hypothetical situations without observing any moves of the other players, so each player’s choices are guided solely by their prior beliefs. Our dynamic CH model will generally predict different patterns of behavior and outcomes under the direct response method despite the fact that the two methods are strategically equivalent.

While the direct-response method is the most commonly used method to implement centipede game experiments, there are a few exceptions. Nagel and Tang (1998) is the first paper to report the results from a centipede game experiment conducted as a simultaneous move game, the reduced normal form. In their 12-node centipede games, each player has seven available strategies that correspond to an intended “take-node” or always passing, and they make their decisions simultaneously. Pooling the data over many repetitions, they find that only 0.5% of outcomes coincide with the equilibrium prediction, suggesting that the non-equilibrium behavior in the centipede games cannot solely be attributed to the violation of backward induction. However, as the authors remarked, the results may be confounded

with the effect of reduced normal form: “...There might be substantial differences in behavior in the extensive form game and in the normal form game...” (Nagel and Tang (1998), p. 357). One of our contributions is to show that dynamic CH provides a theoretical rationale for the existence of this representation effect.³

The paper is organized as follows. The related literature is discussed in the next section. Section 3 sets up the model. Section 4 establishes properties of the belief-updating process and explores the relationship between our model and subgame perfect equilibrium with several examples. In Section 5, the representation effect is explored in a detailed analysis of centipede games with a linearly increasing pie. We conclude in Section 6.

2 Related Literature

The idea of limited depth of reasoning in games of strategy has been proposed and studied by economists and game theorists for at least thirty years (see, for example, Binmore (1987, 1988), Selten (1991, 1998), Aumann (1992), Stahl (1993), and Alaoui and Penta (2016, 2018)). On the empirical side, Nagel (1995) conducts the first laboratory experiment explicitly designed to study hierarchical reasoning in simultaneous move games, using the “beauty contest” game. Each player chooses a number between 0 and 100. The winner is the player whose choice is closest to the average of all the chosen numbers discounted by a parameter $p \in (0, 1)$. To analyze the data, Nagel (1995) assumes level-0 players choose randomly. Level-1 players believe all other players are level-0 and best respond to them by choosing $50p$. Following the same logic, level- k players believe all other players are level- $(k-1)$ and best respond to them with $50p^k$.

This iterative definition of hierarchies has been applied to a range of different environments. For instance, Ho et al. (1998) also analyze the beauty contest game while Costa-Gomes et al. (2001) and Crawford and Iriberri (2007a) consider the strategic levels in a variety of simultaneous move games. Costa-Gomes and Crawford (2006) study the “two-person guessing game,” a variant of the beauty contest game. Finally, the level- k approach has also been applied to games of incomplete information. Crawford and Iriberri (2007b) apply this approach to reanalyze auction data, and Cai and Wang (2006) and Wang et al. (2010) use the level- k model to organize empirical patterns in experimental sender-receiver games. All these studies assume level- k players best respond to degenerate beliefs of level- $(k-1)$ players.

This standard level- k model has been extended in a number of ways. One such approach is that each level of player best responds to a mixture of all lower levels. Stahl and Wilson (1995) are the first to construct and estimate a specific mixture model of bounded rationality in games where each level of player best responds to a mixture between lower levels and equilibrium players. Camerer et al. (2004) develop the CH framework, where level- k players best respond to a *mixture* of the behavior of all lower level behavioral types from 0 to $k-1$. In addition, players have correct beliefs about the relative proportions of these lower levels, so it includes a consistency restriction on beliefs in the form of truncated rational expectations.

A second direction is to endogenize the strategic levels of players, using a cost benefit

³Such a representation effect is also a feature of QRE. See Goeree et al. (2016), pp. 67-72 and 80-85.

approach. [Alaoui and Penta \(2016\)](#) develop a model of endogenous depth of reasoning, where each player trades off the benefit of additional levels of sophistication against the cost of doing so. Players can have different benefit and cost functions, depending on their beliefs and strategic abilities, respectively. A model is developed for two-person games with complete information and calibrated against experimental data. [Alaoui et al. \(2020\)](#) provide some additional analysis and a laboratory experiment that further explores the implications of this model.

Third, [De Clippel et al. \(2019\)](#) study the implications of the standard level-k approach to mechanism design. They establish a form of the revelation principle for level-k implementation and obtain conditions on the implementability of social choice functions under a range of assumptions about level-0 behavior.

Fourth, there have been several papers that model how an individual’s strategic level evolves when the same game is repeated multiple times. The standard level-k model is ideally suited to understanding how naive individuals behave when they encounter a game for the first time. This is a limitation since, in most laboratory experiments in economics and game theory, subjects play the same game with multiple repetitions, in order to gain experience and to facilitate convergence to equilibrium behavior. It is also a limitation since many games studied by economists and other social scientists are aimed at understanding strategic interactions between highly experienced players (oligopoly, procurement auctions, legislative bargaining, for example), where some convergence to equilibrium would be natural to expect.

In this vein, [Ho and Su \(2013\)](#) and [Ho et al. \(2021\)](#) propose a modification of CH that allows for learning across repeated plays of the same sequential game, in a different way than in [Stahl \(1996\)](#), but in the same spirit. In their setting, an individual player repeatedly plays the same game (such as the guessing game) and updates his or her beliefs about the distribution of levels *after* observing past outcomes of earlier games, but holding fixed beliefs during each play of the game. In addition to updating beliefs about other players’ levels, a player also *endogenously* chooses a new level of strategic sophistication for themselves, in the spirit of [Stahl \(1996\)](#), for the next iteration of the game. This is different from our dynamic CH framework where each player updates their beliefs about the levels of other players after each move within a single game. Moreover, because players are forward-looking in dynamic CH, they are *strategic learners*—i.e., they correctly anticipate the evolution of their posterior beliefs in later stages of the game—which leads to a much different learning dynamic compared with naive adaptive learning models.

All of these extensions add significantly to the literature on level-k behavior for games in strategic form, by allowing for a richer set of heterogeneous beliefs, by incorporating cognitive costs into the model, and by showing how the model can be used to address classic mechanism design problems, but all of them are limited by a restriction to simultaneous move games. In extensive form games, the timing structure is crucial, and beliefs evolve as the game unfolds and players have an opportunity to adjust their beliefs in response to past actions, which is the focus of this paper. Our dynamic CH provides an extension in this direction, under the assumption that players are forward looking about the actions of their opponents in the entire game tree. [Rampal \(2022\)](#) develops an alternative approach for multistage games of perfect information. He models levels of sophistication by assuming that players have limited

foresight in the sense of a rolling horizon; that is, players only look forward a fixed number of stages. This creates a hierarchy of strategic sophistication that depends on the length of a player’s rolling horizon. In addition, players are uncertain about their opponents’ foresight. The baseline game of perfect information is then transformed into a game of incomplete information, with specific assumptions about players’ beliefs about payoffs at non-terminal nodes that correspond to the current limit of their horizon in the game.

At a more conceptual level, our dynamic generalization of CH is related to other behavioral models in game theory. There is a connection between dynamic CH and misspecified learning models (see, for example, [Hauser and Bohren \(2021\)](#)) in the sense that level- k players wrongly believe all other players are less sophisticated. However, in contrast to categorical types of players in misspecified learning models, dynamic CH provides added structure to the set of types in a systematic way, such that higher-level types have a more accurate belief about opponents’ rationality at the aggregate level.

In the context of social learning, application of our model to the investment game is related to [Eyster and Rabin \(2010\)](#) and [Bohren \(2016\)](#) who model the updating process when there exist some behavioral types of players in the population. Our model is also related to the Agent Quantal Response Equilibrium (AQRE) ([McKelvey and Palfrey, 1998](#)), where the updating process about opponents’ types is combined with stochastic choice, as with the level-0 CH players. An important difference is that in QRE, players have fully rational expectations.

3 The Model

This section formally develops the dynamic cognitive hierarchy model for extensive form games. In section 3.1, we introduce notation for extensive form games. Next, we define the dynamic CH updating process in section 3.2, specifying how players’ beliefs about other players’ levels evolve from the history of play. This leads to a definition of the *dynamic CH solution* of a game.

3.1 Extensive Form Games

Let $N_0 = \{0, 1, \dots, n\} \equiv \{0\} \cup N$ be a finite set of *players*, where player 0 is called “chance.” Let H be a finite set of *histories*, and let \prec be a partial order on H representing precedence, with \preceq being the corresponding weak order. There exists a unique element of H called the *initial history* h_\emptyset with the property that $\neg \exists h \in H : h \prec h_\emptyset$. For every element $h' \in H \setminus \{h_\emptyset\}$, there exists a unique *predecessor* $h = \alpha(h')$ with the property that $h \prec h'$. Define the set of *actions* by $A = \{(h', h) \mid h \in H \setminus \{h_\emptyset\}, h' = \alpha(h)\}$. A history $z \in Z \subset H$ is a *terminal history* if $\neg \exists h \in H : z \prec h$; otherwise, it is a *non-terminal history*. For every non-terminal history h , let $Z_h = \{z \in Z \mid h \prec z\}$ be the set of terminal histories after h . Each player $i \in N$ has a *payoff function* (in von Neumann-Morgenstern utilities) $u_i : Z \rightarrow \mathbb{R}$.

Let $P : H \setminus Z \rightarrow N$ and define $H_i = \{h \in H \setminus Z : P(h) = i\}$ to be the set of i ’s *histories*; assume without loss of generality that $H_0 = \{h_\emptyset\}$. For each $h \in H_i$, the set of *actions available to i at h* is $A(h) \equiv \{(h, h') \in A \mid h = \alpha(h')\}$. For each i , a partition, Π_i of H_i defines

i 's information sets. Information set $I_i \in \Pi_i$ specifies a subset of histories contained in H_i that i cannot distinguish from one other, where for any $h \in H_i$, $I_i(h)$ is the element of Π_i that contains h . Furthermore, i 's available actions are the same for all histories in the same information set. Formally, $h' \in I_i(h) \Rightarrow A(h') = A(h)$.⁴

A *behavioral strategy* for player $i = 1, \dots, n$ is a function $\sigma_i : H_i \rightarrow \Delta(A)$ satisfying $\sigma_i(h) \in \Delta(A(h))$ for all $h \in H_i$ and $h' \in I_i(h) \Rightarrow \sigma_i(h') = \sigma_i(h)$. For each $h \in H_i$ and $a \in A(h)$, we use the shorthand $\sigma_{ih}(a) = \sigma_i(h)(a)$ to denote the probability i takes action $a \in A(h)$ at h . Moreover, for any $h \neq h_\emptyset$ and $i = P(\alpha(h))$, we use $\sigma_i(\alpha(h), h)$ to denote the probability that player i moves from $\alpha(h)$ to h . The behavioral strategy for player 0 is exogenously fixed at σ_0 . Let Σ_i denote the set of behavioral strategies for player i and let $\Sigma = \prod_{i \in N} \Sigma_i$ be the set of behavioral strategy profiles. We use the notation $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$ and write elements of Σ as $\sigma = (\sigma_i, \sigma_{-i})$ when we want to focus on a particular player $i \in N$.

An *extensive form game*, Γ , is defined by the tuple $\Gamma = \langle N_0, H, \prec, P, \Pi, u, \sigma_0 \rangle$.

3.2 Cognitive Hierarchies and Belief Updating

Each player i is endowed with a level of sophistication, $\tau_i \in \mathbb{N}_0$, where $\Pr(\tau_i = k) = p_{ik}$ for all $i \in N$ and $k \in \mathbb{N}_0$, and the distribution is *independent* across players. Without loss of generality, we assume $p_{ik} > 0$ for all $i \in N$ and $k \in \mathbb{N}_0$. Let $\tau = (\tau_1, \dots, \tau_n)$ be the level profile and τ_{-i} be the level profile without player i . Each player i has a prior belief about all other players' levels and these prior beliefs satisfy *truncated rational expectations*. That is, for each i and k , a level- k player i believes all other players in the game are at most level- $(k-1)$. For each $i, j \neq i$ and k , let $\mu_{ij}^k(\tau_j)$ be level- k player i 's prior belief about player j 's level, and $\mu_i^k(\tau_{-i}) = (\mu_{ij}^k(\tau_j))_{j \neq i}$ be level- k player i 's prior belief profile. Furthermore, for each i and k , level- k player i believes any other player j 's level is independently distributed according to the lower truncated probability distribution function:

$$\mu_{ij}^k(\kappa) = \begin{cases} \frac{p_{j\kappa}}{\sum_{m=0}^{k-1} p_{jm}} & \text{if } \kappa < k \\ 0 & \text{if } \kappa \geq k. \end{cases} \quad (1)$$

The assumption underlying μ_{ij}^k is that level- k types of each player have a correct belief about the relative proportions of players who are less sophisticated than they are, but maintain the incorrect belief that other players of level $\kappa \geq k$ do not exist. The j subscript indicates that different players can have different level distributions.

A strategy profile is now a *level-dependent* profile of behavior strategies for each level of each player. Thus, let σ_i^k be the behavioral strategy adopted by level- k player i , where, σ_i^0 uniformly randomizes at each information set. That is, for all $i \in N$, $h \in P_i$ and $a \in A(h)$,

$$\sigma_{ih}^0(a) = \frac{1}{|A(h)|}.$$

In the following we may interchangeably call level-0 players *non-strategic players* and level $k \geq 1$ players *strategic players*. Each strategic player i with level $k > 1$ updates their

⁴We assume that all players in the game have perfect recall. See [Kreps and Wilson \(1982\)](#) for a definition.

beliefs about all other players' levels of sophistication at every history, h . Their posterior beliefs at h depend on the level-dependent strategy profile of the other players, σ_{-i} , and their prior belief about the distribution of player types, μ_i^k .⁵ This updating process is formalized with some additional notation. Let $\sigma_j^{-k} = (\sigma_j^0, \dots, \sigma_j^{k-1})$ be the profile of strategies adopted by the levels below k of player j . In addition, let $\sigma_{-i}^{-k} = (\sigma_1^{-k}, \dots, \sigma_{i-1}^{-k}, \sigma_{i+1}^{-k}, \dots, \sigma_n^{-k})$ denote the strategy profile of the levels below k of all players other than player i .

All strategic players believe every history is possible because $\mu_{ij}^k(0) > 0$ for all i, j, k and $\sigma_{jh}^0(a) > 0$ for all $j \in N$, $h \in P_j$ and $a \in A(h)$. Because all histories are reached with positive probability, given any strategy profile, σ , and prior distribution of levels, μ_i^k , level- k player i can use Bayes' rule to derive the posterior belief about other players' levels. We use $\nu_i^k(\tau_{-i} | h, \sigma_{-i}^{-k})$ to denote level- k player i 's posterior belief about the joint distribution of all other players' levels (lower than k) at $h \in H \setminus Z$ for a given level-dependent strategy profile σ and prior p . Denote level- k player i 's marginal posterior belief about player j 's level at $h \in H \setminus Z$ as $\nu_{ij}^k(\tau_j | h, \sigma_{-i}^{-k})$. Finally, let $\{\nu_i^k(\tau_{-i} | h, \sigma_{-i}^{-k})\}_{h \in H \setminus Z}$ be level- k player i 's *contingent posterior belief* about all other players' levels induced by σ_{-i}^{-k} .

In the dynamic CH model, players correctly anticipate how they will update their posterior beliefs about other players' levels at all future histories of the game. Thus, level- k player i *believes* the other players are using the (normalized) strategy profile, $\tilde{\sigma}_{-i}^{-k} = (\tilde{\sigma}_1^{-k}, \dots, \tilde{\sigma}_{i-1}^{-k}, \tilde{\sigma}_{i+1}^{-k}, \dots, \tilde{\sigma}_n^{-k})$, where for any $j \neq i$:

$$\tilde{\sigma}_j^{-k}(h) = \sum_{\kappa=0}^{k-1} \nu_{ij}^k(\kappa | h, \sigma_{-i}^{-k}) \cdot \sigma_j^\kappa(h).$$

In general, the posterior distribution of levels of other players will be different for different levels of the same player at the same history, since the supports of those distributions will generally differ.⁶ This, in turn, induces different levels of the same player to have different beliefs about the probability distribution over the terminal payoffs that can be reached from that history. For each $i \in N$, $k > 0$, σ , and τ_{-i} such that $\tau_j < k$ for all $j \neq i$, let $\tilde{\rho}_i^k(z | h, \tau_{-i}, \sigma_{-i}^{-k}, \sigma_i^k)$ be level- k player i 's belief about the conditional realization probability of $z \in Z_h$ at history $h \in H \setminus Z$, if the profile of levels of the other players is τ_{-i} and i is using strategy σ_i^k .

Furthermore, level- k of player i uses Bayes' rule to derive the posterior belief over the histories in every information set in the game. For any information set, I , and any $h \in I(h)$, we denote this posterior belief by $\pi_i^k(h)$, where for all i, k , and $h \in H \setminus Z$, $\sum_{h' \in I(h)} \pi_i^k(h') = 1$. Hence, level- k of player i 's conditional expected payoff at history h is given by:

$$\mathbb{E}u_i^k(\sigma|h) = \sum_{h' \in I(h)} \pi_i^k(h') \sum_{\{\tau_{-i}: \tau_j < k \forall j \neq i\}} \sum_{z \in Z_h} \nu_i^k(\tau_{-i} | h', \sigma_{-i}^{-k}) \tilde{\rho}_i^k(z | h', \tau_{-i}, \sigma_{-i}^{-k}, \sigma_i^k) u_i(z).$$

⁵Strategic players whose level is $k = 1$ do not update, since they have a degenerate prior belief that all other players are level $k = 0$, who randomize uniformly. Also, note that player i updates their beliefs at every history, not only at histories in H_i .

⁶However, the supports of all levels of all players will always include the type profile τ_{-i}^0 , in which all other players are level-0. That is, $\nu_i^k(\tau_{-i}^0 | h, \sigma_{-i}^{-k}) > 0$ for all i, k, h .

The *dynamic CH solution* of the game is defined as the level-dependent strategy profile, σ^* , such that σ_{ih}^{k*} maximizes $\mathbb{E}u_i^k(\sigma^*|h)$ for all i, k, h .⁷

4 Properties of Dynamic CH in Extensive Form Games

Section 4.1 first establishes the general properties of the belief-updating process. Section 4.2 explores the relationship between the dynamic CH solution and subgame perfect equilibrium. In addition, we point out the possibility that in games of imperfect information, the posterior beliefs could be correlated across histories in Section 4.3. Since the dynamic CH solution is defined with histories, the extensive form game and its corresponding normal form may not have the same CH solution. This representation effect is illustrated in Section 4.4.

4.1 Properties of the Belief-Updating Process

The first result shows that for this important class of games, the updating process satisfies a particular independence property. Specifically, the following proposition establishes that all levels of all players will update their posterior beliefs about other players' levels independently.

Proposition 1. *For any finite extensive form game Γ , any $h \in H \setminus (Z \cup \{h_\emptyset\})$, any $i \in N$, and for any $k \in \mathbb{N}$, level- k player i 's posterior belief about other players' levels at history h is independent across players. That is, $\nu_i^k(\tau_{-i} | h, \sigma_{-i}^{-k}) = \prod_{j \neq i} \nu_{ij}^k(\tau_j | h, \sigma_{-i}^{-k})$.*

Proof: We prove this proposition by induction on the number of predecessors of h , not counting the *initial history* h_\emptyset , which we denote by $|h|$. Let σ be any level-dependent strategy profile and p be any prior distribution over types. First, suppose $|h| = 1$ with $j = P(\alpha(h)) \neq \emptyset$, and consider any $i \neq j$ who is some level $k > 0$. Because player j has made the only move in the game so far, and the prior distribution of types is assumed to be independent across players, we have, for any τ_{-i} such that $\tau_{i'} < k \forall i' \neq i, j$:

$$\begin{aligned} \nu_i^k(\tau_{-i} | h, \sigma_{-i}^{-k}) &= \frac{\sigma_j^{\tau_j}(\alpha(h), h) \mu_{ij}^k(\tau_j)}{\sum_{l=0}^{k-1} \sigma_j^l(\alpha(h), h) \mu_{ij}^k(l)} \prod_{i' \neq i, j} \mu_{ii'}^k(\tau_{i'}) \\ \nu_{ij}^k(\tau_j | h, \sigma_{-i}^{-k}) &= \frac{\sigma_j^{\tau_j}(\alpha(h), h) \mu_{ij}^k(\tau_j)}{\sum_{l=0}^{k-1} \sigma_j^l(\alpha(h), h) \mu_{ij}^k(l)} \\ \nu_{ii'}^k(\tau_{i'} | h, \sigma_{-i}^{-k}) &= \mu_{ii'}^k(\tau_{i'}) \\ &\Rightarrow \\ \nu_i^k(\tau_{-i} | h, \sigma_{-i}^{-k}) &= \prod_{j \neq i} \nu_{ij}^k(\tau_j | h, \sigma_{-i}^{-k}) \end{aligned}$$

⁷We assume (as is typical in level- k models) that players randomize uniformly over optimal actions when indifferent. This assumption is convenient because it ensures a unique dynamic CH solution to every game, so we assume it here. Note that while the dynamic CH solution is defined as a fixed point, it can be solved for recursively, starting with the lowest level and iteratively working up to higher levels.

where, as observed earlier, we know $\sum_{l=0}^{k-1} \sigma_j^l(\alpha(h), h) > 0$ because $\sigma_j^0(\alpha(h), h) = \frac{1}{|A(\alpha(h))|} > 0$. Hence, the result is true for $|h| = 1$. Next, consider any h such that $|h| = t > 1$ and $h \in H \setminus Z$ and suppose that $\nu_i^k(\tau_{-i} | h, \sigma_{-i}^{-k}) = \prod_{j \neq i} \nu_{ij}^k(\tau_j | h, \sigma_{-i}^{-k})$ for all h such that $|h| = 1, 2, \dots, t-1$. Let $j = P(\alpha(h))$ and consider any $i \neq j$ who is some level $k > 0$. Because only player j has moved, going from $\alpha(h)$ to h , we have for any τ_{-i} such that $\tau_{i'} < k \forall i' \neq i, j$:

$$\begin{aligned} \nu_i^k(\tau_{-i} | h, \sigma_{-i}^{-k}) &= \frac{\sigma_j^{\tau_j}(\alpha(h), h) \nu_{ij}^k(\tau_j | \alpha(h), \sigma_{-i}^{-k})}{\sum_{l=0}^{k-1} \sigma_j^l(\alpha(h), h) \nu_{ij}^k(l | \alpha(h), \sigma_{-i}^{-k})} \prod_{i' \neq i, j} \nu_{ii'}^k(\tau_{i'} | \alpha(h), \sigma_{-i}^{-k}) \\ \nu_{ij}^k(\tau_j | h, \sigma_{-i}^{-k}) &= \frac{\sigma_j^{\tau_j}(\alpha(h), h) \nu_{ij}^k(\tau_j | \alpha(h), \sigma_{-i}^{-k})}{\sum_{l=0}^{k-1} \sigma_j^l(\alpha(h), h) \nu_{ij}^k(l | \alpha(h), \sigma_{-i}^{-k})} \\ \nu_{ii'}^k(\tau_{i'} | h, \sigma_{-i}^{-k}) &= \nu_{ii'}^k(\tau_{i'} | \alpha(h), \sigma_{-i}^{-k}) \\ &\text{so} \\ \nu_i^k(\tau_{-i} | h, \sigma_{-i}^{-k}) &= \prod_{j \neq i} \nu_{ij}^k(\tau_j | h, \sigma_{-i}^{-k}) \end{aligned}$$

as desired. ■

What drives this result is that when player j moves, then all players other than j only update their beliefs about the level of player j , and do not update their beliefs about any of the other players. From Proposition 1, we can see that the marginal posterior belief of level- k player i to player j 's belief only depends on player j 's moves along the history. Therefore, we can obtain that $\nu_{ij}^k(\kappa | h, \sigma_{-i}^{-k}) = \nu_{ij}^k(\kappa | h, \sigma_j^{-k})$. Specifically,

$$\nu_{ij}^k(\kappa | h, \sigma_j^{-k}) = \begin{cases} \frac{\mu_{ij}^k(\kappa) f_j(h | \sigma_j^\kappa)}{\sum_{m=0}^{k-1} \mu_{ij}^k(m) f_j(h | \sigma_j^m)} & \text{if } \kappa < k \\ 0 & \text{if } \kappa \geq k, \end{cases}$$

where $f_j(h | \sigma_j^\kappa)$ is the probability that player j moves along the path to reach h given player j is using the strategy σ_j^κ .⁸

The second property of the dynamic cognitive hierarchy model is that in the later histories, the support of the posterior beliefs is (weakly) shrinking. In this sense, the players would have a more precise posterior belief when the history gets longer. For any player $i, j \in N$ such that $i \neq j$, for any $h \in H \setminus (Z \cup \{h_\emptyset\})$, and for any $k \in \mathbb{N}$, we denote $\text{supp}_{ij}^k(h) \equiv \{\tau_j \in \{0, 1, \dots, k-1\} | \nu_{ij}^k(\tau_j | h, \sigma_j^{-k}) > 0\}$. This property is formally stated in the following proposition.

Proposition 2. *In any extensive form game Γ , $\text{supp}_{ij}^k(h) \subseteq \text{supp}_{ij}^k(\alpha(h))$ for all $i, j \in N$, $k \in \mathbb{N}$, and $h \in H \setminus (Z \cup \{h_\emptyset\})$.*

⁸Particularly, the probability $f_j(h | \sigma_j^\kappa)$ can be computed by

$$f_j(h | \sigma_j^\kappa) = \begin{cases} \prod_{h' \in P_j \cap \{\tilde{h}, \tilde{h} \prec h\}} \sigma_{jh'}^\kappa(\{(h', h'') : h' = \alpha(h''), h'' \preceq h\}) & \text{if } P_j \cap \{\tilde{h} : \tilde{h} \prec h\} \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Proof: To prove the statement, it suffices to show that $\kappa \notin \text{supp}_{ij}^k(\alpha(h)) \Rightarrow \kappa \notin \text{supp}_{ij}^k(h)$ for all $\kappa = 0, 1, \dots, k-1$. There are two possibilities. Either $j = P(\alpha(h))$ or $j \neq P(\alpha(h))$. If $j \neq P(\alpha(h))$, then some player other than j moved at $\alpha(h)$, so $\nu_{ij}^k(\tau_j|h, \sigma_j^{-k}) = \nu_{ij}^k(\tau_j|\alpha(h), \sigma_j^{-k})$ for all $\tau_j = 0, 1, \dots, k-1$. Hence $\nu_{ij}^k(\kappa|\alpha(h), \sigma_j^{-k}) = 0 \Rightarrow \nu_{ij}^k(\kappa|h, \sigma_j^{-k}) = 0$, so $\text{supp}_{ij}^k(h) \subseteq \text{supp}_{ij}^k(\alpha(h))$. If $j = P(\alpha(h))$, then j moved at $\alpha(h)$, in which case, by Bayes' rule:

$$\nu_{ij}^k(\tau_j | h, \sigma_j^{-k}) = \frac{\sigma_j^{\tau_j}(\alpha(h), h) \nu_{ij}^k(\tau_j | \alpha(h), \sigma_j^{-k})}{\sum_{l=0}^{k-1} \sigma_j^l(\alpha(h), h) \nu_{ij}^k(l | \alpha(h), \sigma_j^{-k})}$$

for all $\tau_j = 0, 1, \dots, k-1$. Hence, $\nu_{ij}^k(\kappa|\alpha(h), \sigma_j^{-k}) = 0 \Rightarrow \nu_{ij}^k(\kappa|h, \sigma_j^{-k}) = 0$, so $\text{supp}_{ij}^k(h) \subseteq \text{supp}_{ij}^k(\alpha(h))$. ■

There are a few additional remarks about the properties of the updating process in the dynamic CH model that are worth highlighting. First, there is a second source of learning, besides the shrinking support property, which is that after each move by an opponent, each strategic player with $k \geq 2$ updates the probability that the opponent is level 0. This in turn leads to updating of the relative likelihood of the higher strategic types of the opponent, since the probabilities have to sum to 1. Second, as the game unfolds, the beliefs of higher level players about their opponents can be updated in either direction, in the sense of believing an opponent is either more or less sophisticated. Examples in the next section will illustrate this. Third, while players' belief-updating process is adaptive, nonetheless all players are *strategically forward-looking* (rather than myopic) in the sense that players take into account and correctly anticipate how all players in the game will update beliefs at each future history.

Since the players are forward-looking and have truncated rational expectations, it is natural to ask if there is any connection between our model and perfect or sequential equilibrium. We explore this relationship in the next section.

4.2 Dynamic CH and Subgame Perfect Equilibrium

In this section, we study the relationship between the dynamic CH solution and subgame perfect equilibrium through two simple examples. One question we address is whether sufficiently high-level players always behave consistently with rational backward induction. As it turns out, this is *not* generally true. In the following series of simple two-person extensive form games, we demonstrate how high-level players could violate backward induction either on or off the equilibrium path, suggesting the dynamic CH solution is fundamentally different from subgame perfection. For the sake of simplicity, in this section and for the rest of the paper we assume every player's level distribution is identical.

Violating Backward Induction at Some Subgame

Example 1 demonstrates how backward induction could be violated by every level of player at some subgame. The game tree for this two-person game of perfect information is shown in Figure 1. Suppose every player's level is independently drawn from Poisson(1.5), which (Camerer et al., 2004) have suggested is an empirically plausible distribution. Every level of

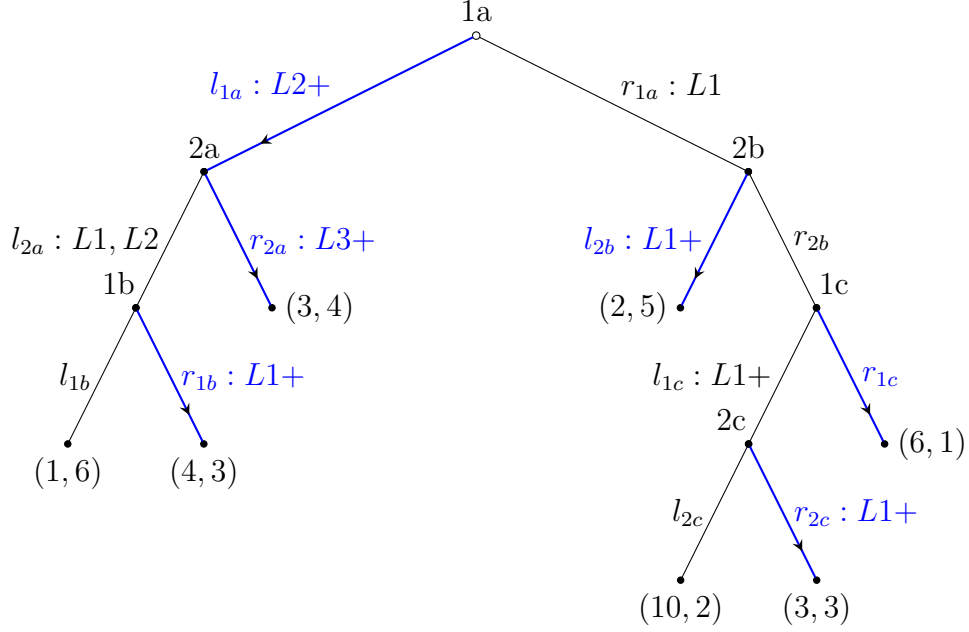


Figure 1: Game Tree of Example 1. A "+" sign indicates a move is chosen by the specified level type and all higher levels. The subgame perfect equilibrium moves are marked with arrows.

players' move choices are labelled in the figure, with a "+" sign indicating a move is chosen by the specified level type and all higher levels. For instance, level-1 player 1 chooses r_{1a} at the beginning while level-2 and above choose l_{1a} . Calculations can be found in Appendix A.

To illustrate the mechanics of the dynamic CH model in this example, it is useful to begin by focusing on subgame 2a. In this subgame levels 2 and higher of player 2 would update from the information that player 1 is not a level-1 player, leading a level-2 player 2 to choose l_{2a} because the updated belief puts all weight on player 1 being level 0. However, a level-3 player 2 places positive posterior probability on player 1 being level-2, and as long as this posterior probability is high enough it is optimal for level-3 player 2 to choose r_{2a} —as if player 2 were engaged in the same backward induction reasoning used to justify the subgame perfect equilibrium. Following a similar logic, all high-level players would behave this way in the left branch of the game, where player 1 chooses l_{1a} at the beginning.

However, this is not the case for the right branch of the game after player 1 chooses r_{1a} at the beginning. At subgame 1c, the move predicted by the subgame perfect equilibrium is *never* chosen by any strategic player 1. Hence, in the dynamic CH solution for this example, high-level player 1 types' behavior is consistent with subgame perfect equilibrium on the left branch but not on the right branch.

Dominated Actions

As we examine this example carefully, we can find the key of this phenomenon is that player 1 knows the subgame $h = 1c$ can be reached only if player 2 chooses a strictly dominated

action⁹ in the previous stage. One can think of player 2's decision at subgame $h = 2b$ as a rationality check in the following sense. Whenever player 2 chooses r_{2b} , the support of strategic player 1's posterior belief will shrink to a singleton—he will believe player 2 is level-0. This extreme posterior belief would lead a strategic player 1 to deviate from subgame perfect strategy.

Generally speaking, if a history contains some player's strictly dominated action, then all other players will immediately believe this player is non-strategic and best respond to such strategy. As a result, it is possible that the strategy profile will not be the subgame perfect equilibrium for every strategic level. This argument holds as long as level-0 player's strategy is the beginning of the hierarchical reasoning process—no matter how small the proportion of level-0 players is. However, since paths with strictly dominated actions can be realized only if some player is level-0, paths containing strictly dominated actions occur with vanishing probability as the proportion of level-0 players converges to 0. Proposition 3 formally shows this conclusion.

Proposition 3. *Consider any finite extensive form game where each player i 's level is independently drawn from the distribution $(p_k)_{k=0}^{\infty}$. If some history h can occur only if some player chooses a strictly dominated action, then the probability for such history being realized converges to 0 as $p_0 \rightarrow 0^+$.*

Proof: Consider any h that can occur only if some player chooses a strictly dominated action. That is, there is $h' \prec h$ with $i = P(h')$ such that there is a strictly dominated action $(h', h'') \in A(h')$ and $h'' \preceq h$. Since this is a strictly dominated action, it can only be chosen by a level-0 player. Therefore, the ex ante probability for player i to choose (h', h'') at h' is

$$\Pr((h', h'') \mid h') = \sum_{j=0}^{\infty} \sigma_i^j(h', h'') p_j = \sigma_i^0(h', h'') p_0 = \frac{1}{|A(h')|} p_0.$$

Finally, the ex ante probability for h to be realized, $\Pr(h)$, is smaller than $\Pr((h', h'') \mid h')$ and hence

$$\lim_{p_0 \rightarrow 0^+} \Pr(h) \leq \lim_{p_0 \rightarrow 0^+} \Pr((h', h'') \mid h') = \lim_{p_0 \rightarrow 0^+} \frac{1}{|A(h')|} p_0 = 0.$$

This completes the proof. ■

The direct implication of Proposition 3 is that if p_0 is sufficiently small, one can effectively “trim” the game by deleting histories containing strictly dominated actions. One sees this principle in play in example 1 where player 1's anomalous behavior only happens when player 2 chooses a strictly dominated action, which is only chosen by level-0. For other parts of the game, if both players are at least level 3 the model predicts the game will follow the subgame perfect equilibrium path.

⁹Formally speaking, at any history $h \in H \setminus Z$ with $i = P(h)$, we say an action $(h, h') \in A(h)$ is strictly dominated if there is an action $(h, h'') \in A(h)$ such that

$$\min_{z \in Z_{h''}} u_i(z) > \max_{z \in Z_{h'}} u_i(z).$$

Since the subgame perfect equilibrium path never contains strictly dominated actions, one might be tempted to conjecture that the equilibrium path is always followed by sufficiently sophisticated players. The next example demonstrates that this is not true. In fact, it is possible that the subgame perfect equilibrium path is *never* chosen by strategic players, so high-level players in our model do not necessarily converge to the subgame perfect equilibrium.

Violating Backward Induction on the Equilibrium Path

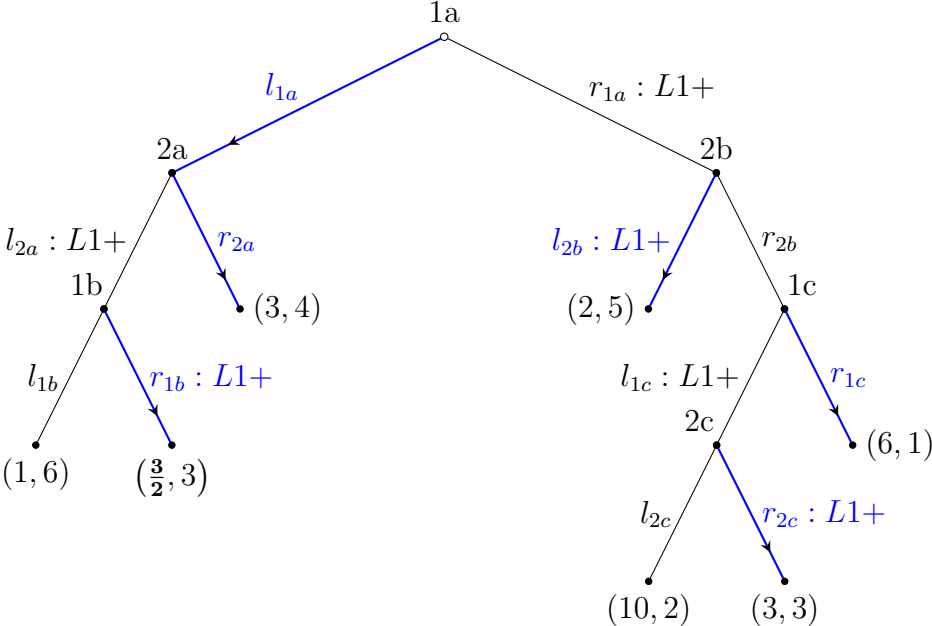


Figure 2: Game Tree of Example 2. A "+" sign indicates a move is chosen by the specified level type and all higher levels. The subgame perfect equilibrium moves are marked with arrows.

Example 2 is modified from the previous example by changing player 1's payoff from 4 to $\frac{3}{2}$ as he chooses r_{1b} at history $1b$. Decreasing the payoff does not affect the subgame perfect equilibrium. However, this change makes low-level players think the subgame perfect equilibrium actions are not profitable, causing a domino effect that high-level players think the equilibrium actions are not optimal as well. Here we consider an arbitrary prior distribution $p = (p_k)_{k=0}^\infty$. The game tree is shown in Figure 2 with every level of players' decisions. The calculations can be found in Appendix A.

Level-1 players will behave the same as in the previous example. However, the change of payoffs makes l_{1a} not profitable for level-2 player 1 at the initial history. Hence, player 2 would believe player 1 is certainly level-0 whenever the game proceeds to the left branch. Moreover, every level of players would behave the same by the same logic. As a result, the subgame perfect equilibrium path is *never* chosen by strategic players. If p_0 is close to 0, the subgame perfect equilibrium outcome will almost never be reached.

Instead, there is an imperfect Nash equilibrium that can be supported by the strategy profile of every strategic level of both players. Loosely speaking, the belief updating process gets “stuck” at this equilibrium, causing all higher-level players behave in the same way.¹⁰

4.3 Correlated Beliefs in Games of Imperfect Information

There is a wide range of applications of extensive form games in economics and political science where players have private information, either due to privately known preferences and beliefs about other players, or from imperfect observability of the histories of play in the game. These applications would include many workhorse models, such as signaling, information transmission, information design, social learning, entry deterrence, reputation building, crisis bargaining, and so forth. Hence the natural next step is to investigate more deeply our approach to dynamic games with incomplete information. In such environments, one complication is that players not only learn about the opponents’ levels of sophistication but also about more basic elements of the game structure, such as the opponents’ private information, payoff types, and prior moves.

One observation is that allowing for incomplete information in the dynamic CH approach does not introduce any problems of off-path beliefs. The reason is that at every information set of the game, all types of all players have posterior beliefs over the opponents’ types that include a positive probability they are facing level-0 players. Hence, there is no issue of specifying off-path beliefs in an ad hoc fashion and therefore we avoid the complications of belief-based refinements.

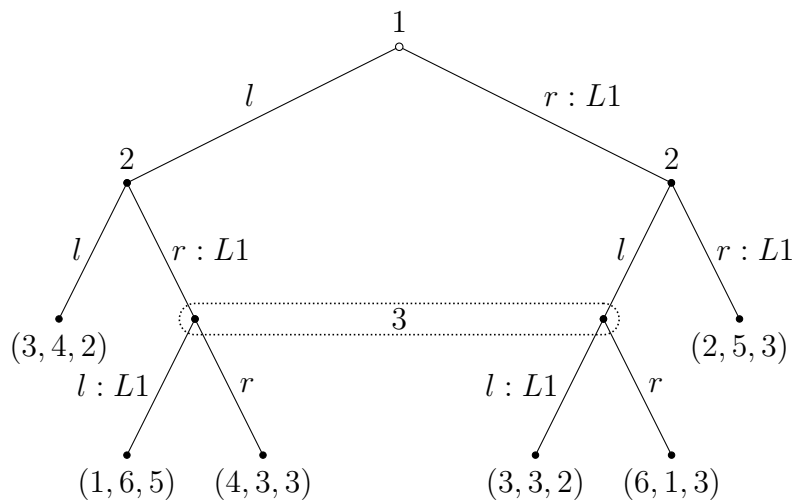


Figure 3: Game Tree of Example 3. Dashed lines are the paths selected by level-1 players.

In games with imperfect information because some information sets contain more than one history, the beliefs about different players’ levels can be correlated across the histories

¹⁰The following strategy profile defines this imperfect equilibrium: player 1 chooses r_{1a} at the beginning, r_{1b} at subgame $h = 1b$, and chooses l_{1c} at subgame $h = 1c$; player 2 chooses l_{2a} at subgame $h = 2a$, l_{2b} at subgame $h = 2b$, and chooses r_{2c} at subgame $h = 2c$. Therefore, $(2, 5)$ is an equilibrium outcome.

in the information set. We illustrate this using the following three-person game where each player moves once. The game tree is shown in Figure 3.

Player 1 chooses first whether to go left or right. After that, player 2 chooses to go left or right. If player 1 and 2 make the same decision, the game ends. Otherwise, player 3 makes the final decision. However, that stage, player 3 only knows that one of the previous players chose l and the other chose r , but does not know which one chose l .

Level-1 players believe all other players are level-0. As we compute the expected payoff of each action, level-1 player 1 will choose r at the initial node. Level-1 player 2 will choose r at subgame $h = l$ and $h = r$. At player 3's information set, since level-1 player 3 thinks both players are level-0, he would believe both histories are equally likely, and hence choose l . Given level-0 and 1 players' strategies, at player 3's information set, level-2 player 3 would think player 1 and player 2 cannot both be level-1 players; otherwise, the game will not reach this information set. Therefore, level-2 player 3's beliefs about player 1 and 2's levels at the two histories in the information set are correlated.

4.4 The Representation Effect and Violations of Strategic Invariance

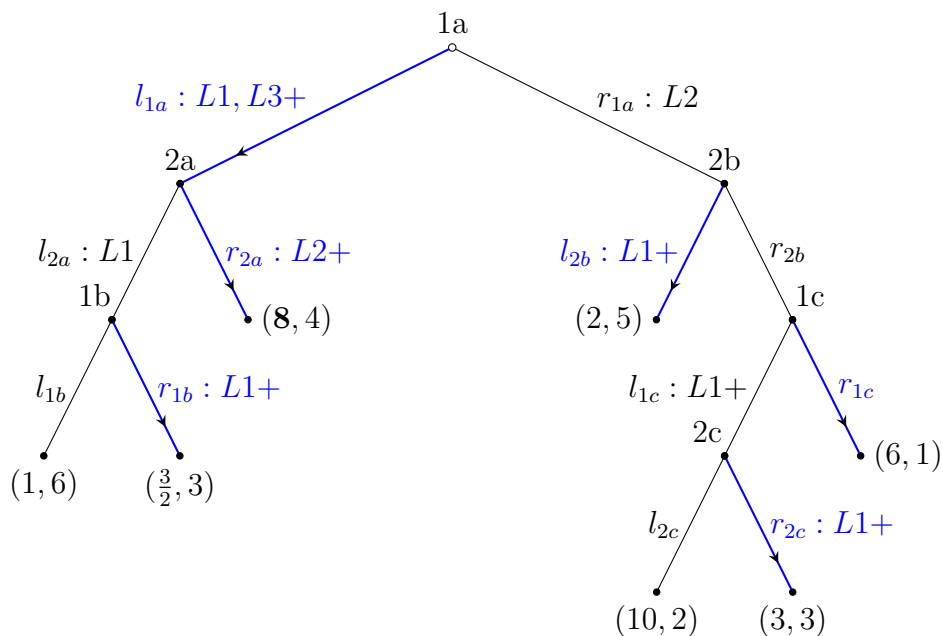


Figure 4: Game Tree of Example 4. A "+" sign indicates a move is chosen by the specified level type and all higher levels. The subgame perfect equilibrium moves are marked with arrows.

An interesting feature of the dynamic CH model is *the representation effect*: strategically equivalent representations of a game have dynamic CH solutions that produce different outcome distributions. We illustrate this with a toy example in this section, and provide a detailed analysis of the representation effect for a class of increasing-sum centipede games in

the subsequent section. The simplest case of this involves a comparison of the dynamic CH solution for a multistage game of perfect information played sequentially to a the dynamic CH solution to a strategically equivalent extensive form representation of the same game where each player chooses a strategy for the whole game without observing the other player's strategy choice.¹¹

Table 1: Reduced Normal Form of Example 4

		Player 2					
Player 1		$l_{2a}l_{2b}$	$r_{2a}l_{2b}$	$l_{2a}r_{2b}l_{2c}$	$l_{2a}r_{2b}r_{2c}$	$r_{2a}r_{2b}l_{2c}$	$r_{2a}r_{2b}r_{2c}$
		$L1+$					
$l_{1a}l_{1b}$		1,6	3,4	1,6	1,6	3,4	3,4
$l_{1a}r_{1b}$		$3/2,3$	3,4	$3/2,3$	$3/2,3$	3,4	3,4
$r_{1a}l_{1c}$	$L1+$	2,5	2,5	10,2	3,3	10,2	3,3
$r_{1a}r_{1c}$		2,5	2,5	6,1	6,1	6,1	6,1

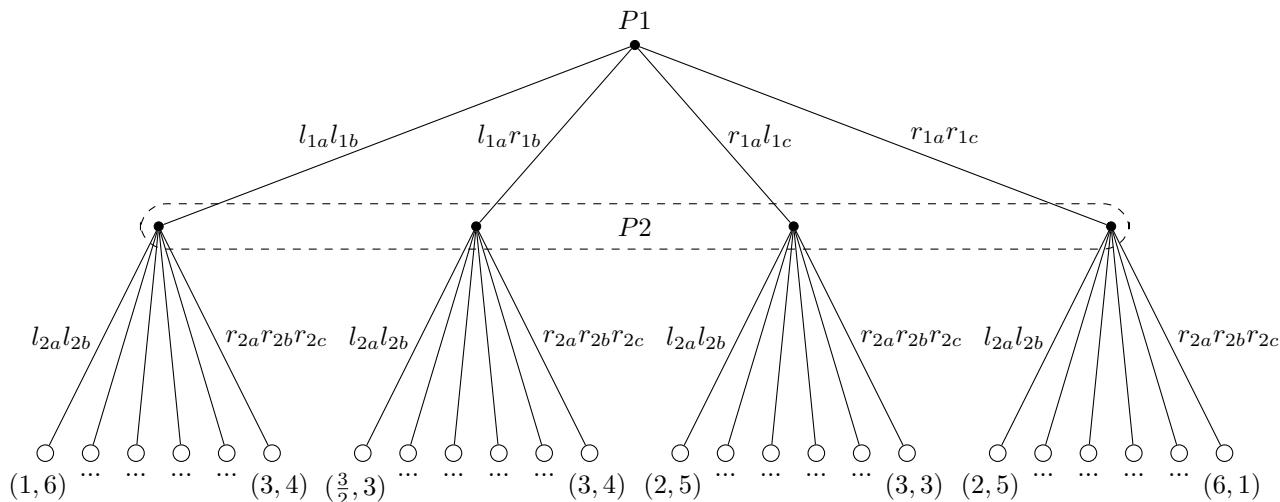


Figure 5: Strategically Equivalent Game of Example 4

We first revisit Example 2 to show how differently the static CH solution is obtained compared with the logic of the dynamic CH solution. Table 1 displays the 4×6 matrix game that is reduced normal form representation of the extensive form game of Example 2. Notice that the reduced normal form can also be represented by the extensive form game of Figure 5. It is easy to see that level 1 and higher of player 1 will choose the strategy $r_{1a}l_{1c}$ and level 1 and higher of player 2 will choose the strategy $l_{2a}l_{2b}$, as indicated in the table. Thus, for this example it turns out that both models lead to a solution where behavior corresponds to an equilibrium outcome that differs from the subgame perfect equilibrium outcome.

¹¹Because the second version of the game is essentially a simultaneous move game, the dynamic CH solution reduces to the standard CH solution for the normal form of the first version of the game. In the remainder of the paper, we will (with minor abuse of terminology) refer to the dynamic CH solution of these strategically equivalent simultaneous-move versions of the alternating move game as the "static" CH solution of the game.

However, it is not true that high-level players in both CH models always lead to the same equilibrium outcome in both representations. Example 4, whose game tree is shown in Figure 4, demonstrates how this can happen. This example is almost exactly the same as Example 2, with the single exception being that player 1’s payoff changes from 3 to 8 after choosing r_{2a} at subgame $2a$. This change does not affect the subgame perfect equilibrium, but makes choosing l_{1a} profitable again for high-level player 1. (Here we again assume the prior distribution follows Poisson(1.5).) Consequently, higher levels of dynamic CH players in this game will choose actions that lead to the subgame perfect equilibrium outcome, (8, 4). This switch to the subgame perfect outcome is a direct result of the belief-updating process in the dynamic model. Although the payoff 10 is really attractive to player 1, player 1 in the dynamic model will realize he can get it only if player 2 is level-0. Therefore, if there is a high enough probability of higher levels of player 2, player 1 will realize he is likely to get the lower payoff of 3 at node $2c$. Hence, a high-level player 1 will choose l_{1a} at the beginning (as if conducting backward induction). As long as there are enough strategic types of player 1 choosing l_{1a} , higher levels of player 2 will update accordingly and choose the subgame perfect equilibrium action r_{2a} . The calculations can be found in Appendix A.

In contrast, the static CH model makes exactly the same prediction as in Example 2. That is, even though player 1’s subgame perfect equilibrium payoff has increased from 3 to 8, *all* strategic types of players 1 and 2 in the reduced normal form will still choose $r_{1a}l_{1c}$ and $l_{2a}l_{2b}$, respectively, again producing the outcome (2, 5) that is not subgame perfect.

One property of the static model identified by Camerer et al. (2004) is that if a k -level type plays a (pure) equilibrium strategy, then all higher levels of that player will play that strategy too. One may wonder if an analogous property holds in the dynamic CH model. That is, if some level type of a player chooses on the equilibrium path, do all higher-levels of that player choose that action too? Example 4 provides a counterexample for this conjecture. At the initial history, level-1 player 1 chooses the equilibrium path l_{1a} . However, level-2 player 1 switches to r_{1a} , and level-3 (and above) player 1 switches back to l_{1a} .

The underlying reason is that even if a level- k player chooses the equilibrium path, a higher-level player could still deviate from the equilibrium path if other players do not move along the equilibrium path in later subgames. In this example, level-1 player 1 chooses l_{1a} at the beginning to best respond to level-0 player 2. Yet, level-1 player 2 does not choose the equilibrium path at the subgame $h = 2a$, causing level-2 player 1 to choose r_{1a} at the beginning. Level-2 (and above) player 2 switches to the equilibrium path at the subgame $h = 2a$, and this information can only be updated by level-3 (and above) player 1. Finally, as long as there are enough level-2 (and above) players, high-level player 1 would switch back to the equilibrium path, creating a non-monotonicity.

5 An Application: Centipede Games

In this section, we demonstrate the representation effect on the class of “linear centipede games,” which is illustrated in Figure 6. The games in this class are described in the following way. Player 1, the first-mover, and player 2, the second-mover, alternate over a sequence of moves. At each move, the player whose turn it is can either end the game (“take”)

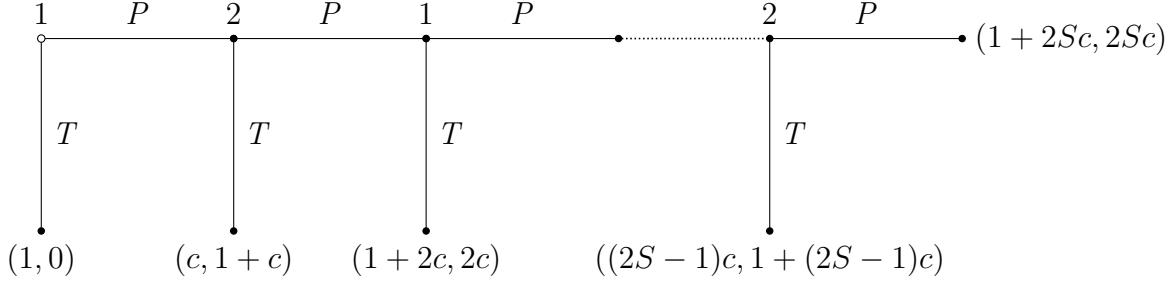


Figure 6: $2S$ -node Centipede Game

and receive the larger of two payoffs or allow the game to continue (“pass”), in which case both the large and the small payoff are incremented by an amount $c > 0$. The difference between the large and the small payoff equals 1 and does not change. The game continues for at most $2S$ decision nodes (stages) where $S \geq 2$, and we label the decision nodes by $\{1, 2, \dots, 2S\}$. Player 1 moves at odd nodes and player 2 moves at even nodes. If the game is ended by a player at stage $j \leq 2S$, the payoffs are $(1 + (j - 1)c, (j - 1)c)$ if j is odd and $((j - 1)c, 1 + (j - 1)c)$ if j is even. If no player ever takes, the payoffs are $(1 + 2Sc, 2Sc)$. Thus, a linear centipede game has two parameters: (S, c) . To avoid trivial cases, we assume $\frac{1}{3} < c < 1$.¹²

Specifically, we will compare each level of players’ behavior in two different representations of the game given the same prior distribution. Theoretically, the direct-response method and the strategy method correspond to the extensive form and the reduced normal form,¹³ respectively. The key difference between the extensive form and the reduced normal form is that players can observe the other player’s previous actions in the extensive form. As the game continues, the only information that can be observed is *how many times the opponent has passed*.¹⁴ This history seems uninformative at the first glance. However, players can still update their beliefs about the other player’s level from this history, and hence behave differently.

5.1 CH Solution for the Extensive Form Centipede Games

In the extensive form centipede game, since each player can move at S stages, then a (behavioral) strategy for player i is an S -tuple where each element is the probability to take at the corresponding decision node. That is, $\sigma_1 = (\sigma_{1,1}, \dots, \sigma_{1,S})$ and $\sigma_2 = (\sigma_{2,1}, \dots, \sigma_{2,S})$ are player 1 and 2’s strategies, respectively. For every $1 \leq j \leq S$, $\sigma_{1,j}$ is the probability that player 1 would take at stage $2j - 1$ and $\sigma_{2,j}$ is the probability that player 2 would take at stage $2j$.

Following the notation introduced earlier, we use σ_1^k and σ_2^k to denote level- k player’s

¹²If $c > 1$, then the unique equilibrium is for every player to pass at every node. If $c < \frac{1}{3}$, then all players with level $k > 0$ will always take, so CH behavior is the same as subgame perfect Nash equilibrium behavior.

¹³In the reduced normal form, a player’s strategy is the node at which they will stop the game (or never). Therefore, each player has $S + 1$ available strategies.

¹⁴Players can also update their beliefs about the opponent’s level if the other player surprisingly takes at some stage. However, the game is over at that point.

strategy. Level-0 players uniformly randomize at each stage. That is, $\sigma_1^0 = \sigma_2^0 = (\frac{1}{2}, \dots, \frac{1}{2})$. Finally, to simplify the notation, for every stage $1 \leq j \leq 2S$, we let $\nu_j^k(\cdot) : \mathbb{N}_0 \rightarrow \Delta(\mathbb{N}_0)$ be level- k stage j -mover's belief about the opponent's level at stage j where $\nu_j^k(\tau_{-P(j)}) \equiv \nu_{P(j), -P(j)}^k(\tau_{-P(j)} \mid j, \sigma_{-P(j)}^{-k})$.

To fully characterize every level of players' strategies, we need to compute every level of players' best responses at every subgame. In principle, we have to solve the behavior of each level recursively. However, since each level of players' strategy is *monotonic*—when the player decides to take at some stage, he will take in all of his later subgames—we can alternatively characterize the solution by identifying the lowest level of player to take at every subgame.

In Lemma 1, we characterize level-1 players' behavior and establish the monotonicity result. These results are straightforward and follow from the assumption that $\frac{1}{3} < c < 1$.

Lemma 1. *In the extensive form linear centipede game, as $\frac{1}{3} < c < 1$,*

1. $\sigma_{2,S}^k = 1$ for all $k \geq 1$.
2. $\sigma_1^1 = (0, \dots, 0)$ and $\sigma_2^1 = (0, \dots, 0, 1)$.
3. For every $k \geq 2$ and every $1 \leq j \leq S - 1$,
 - (i) $\sigma_{1,j}^k = 0$ if $\sigma_{2,j}^m = 0$ for every $1 \leq m \leq k - 1$;
 - (ii) $\sigma_{2,j}^k = 0$ if $\sigma_{1,j+1}^m = 0$ for every $1 \leq m \leq k - 1$.

Proof: See Appendix B. ■

Lemma 1 has three parts: (1) every strategic player 2 takes at the last stage; (2) completely characterizes level-1 strategies—player 1 passes at every stage and player 2 passes at every stage except for the last stage; (3) provides necessary conditions for higher levels to take at some stage. For any level $k \geq 2$ and any stage $1 \leq j \leq 2S - 1$, a level- k player would take at stage j only if there is some lower level player that would take at the next stage. Otherwise, it is optimal for level- k player to pass at stage j .

The general characterization of level- k optimal strategies is in terms of the following **cutoffs**, specifying, for each stage, the lowest level type to take at that stage.

Definition 1. *For every stage j where $1 \leq j \leq 2S$, define the **cutoff**, K_j^* be the lowest level of player that would take at this stage. In other words,*

$$K_j^* = \begin{cases} \arg \min_k \left\{ \sigma_{1, \frac{j+1}{2}}^k = 1 \right\}, & \text{if } j \text{ is odd} \\ \arg \min_k \left\{ \sigma_{2, \frac{j}{2}}^k = 1 \right\}, & \text{if } j \text{ is even} \\ \infty, & \text{if } \nexists k \text{ s.t. } \sigma_{1, \frac{j+1}{2}}^k = 1 \text{ or } \sigma_{2, \frac{j}{2}}^k = 1. \end{cases}$$

Based on Definition 1, the monotonicity obtained in part (3) of Lemma 1 implies the following two results about cutoffs and strategies. Together they show that for any stage, a player's strategy will be to take at that stage if and only if his level is greater or equal to the cutoff.

Proposition 4. For every $1 \leq j \leq 2S - 1$,

1. $K_j^* \geq K_{j+1}^* + 1$ if $K_{j+1}^* < \infty$;
2. $K_j^* = \infty$ if $K_{j+1}^* = \infty$.

Proof: See Appendix B. ■

Proposition 5. For every $1 \leq j \leq 2S - 1$,

1. $\sigma_{1, \frac{j+1}{2}}^k = 1$ for all $k \geq K_j^*$ if j is odd and $K_j^* < \infty$;
2. $\sigma_{2, \frac{j}{2}}^k = 1$ for all $k \geq K_j^*$ if j is even and $K_j^* < \infty$.

Proof: See Appendix B. ■

Hence, cutoffs characterize optimal strategies of each level of each player, with a cutoff defining the lowest level that would take at each stage and all higher levels of that player would also take at that stage. The next two propositions establish recursive necessary and sufficient conditions for the existence of some level of some player to take at each stage. The proofs of these propositions provide a recipe for computing cutoffs.

Proposition 6. $K_{2S-1}^* < \infty \iff p_0 < \frac{2^S}{2^S + \left(\frac{3c-1}{1-c}\right)}$

First, we note that the proofs are simplified somewhat by observing the following identity:

$$p_0 < \frac{2^S}{2^S + \left(\frac{3c-1}{1-c}\right)} \iff \frac{p_0 \left(\frac{1}{2}\right)^S}{p_0 \left(\frac{1}{2}\right)^{S-1} + (1-p_0)} < \frac{1-c}{1+c}.$$

Proof: Only if: Suppose $K_{2S-1}^* < \infty$. By Proposition 4, $K_j^* \geq K_{2S-1}^*$ for all $j < 2S - 1$. Hence, the belief of level K_{2S-1}^* of player 1 that player 2 is level-0 at stage $2S - 1$ equals to

$$\nu_{2S-1}^{K_{2S-1}^*}(0) = \frac{p_0 \left(\frac{1}{2}\right)^{S-1}}{p_0 \left(\frac{1}{2}\right)^{S-1} + \sum_{l=1}^{K_{2S-1}^*-1} p_l},$$

since it is optimal for $K_{2S-1}^* < \infty$ to take at $2S - 1$. This implies $\frac{p_0 \left(\frac{1}{2}\right)^S}{p_0 \left(\frac{1}{2}\right)^{S-1} + \sum_{l=1}^{K_{2S-1}^*-1} p_l} < \frac{1-c}{1+c}$

and hence:

$$\begin{aligned} \frac{p_0 \left(\frac{1}{2}\right)^S}{p_0 \left(\frac{1}{2}\right)^{S-1} + (1-p_0)} &< \frac{1-c}{1+c} \\ \iff & \\ p_0 &< \frac{2^S}{2^S + \left(\frac{3c-1}{1-c}\right)}. \end{aligned}$$

If: Suppose $K_{2S-1}^* = \infty$. Then from Proposition 4, $K_j^* = \infty$ for all $j < 2S - 1$. That is, all levels of both players pass at every stage up to and including $2S - 1$. Hence, the belief of level $k \geq 1$ of player 1 that player 2 is level-0 at stage $2S - 1$ equals to

$$\nu_{2S-1}^k(0) = \frac{p_0 \left(\frac{1}{2}\right)^{S-1}}{p_0 \left(\frac{1}{2}\right)^{S-1} + \sum_{l=1}^{k-1} p_l} > \frac{p_0 \left(\frac{1}{2}\right)^{S-1}}{p_0 \left(\frac{1}{2}\right)^{S-1} + (1 - p_0)}.$$

Since $K_{2S-1}^* = \infty$, it is optimal to pass at $2S - 1$ for all levels $k \geq 1$ of player 1, which implies

$$\frac{p_0 \left(\frac{1}{2}\right)^S}{p_0 \left(\frac{1}{2}\right)^{S-1} + (1 - p_0)} \geq \frac{1 - c}{1 + c}.$$

This completes the proof. ■

Thus, p_0 must be sufficiently small, and the condition is easier to satisfy the smaller c is (the potential gains to passing) and the larger is S (the horizon). If this condition holds, there exists some strategic player 1 that takes at stage $2S - 1$. The proof also provides an insight for how the cutoffs can be computed. Specifically, the K_{2S-1}^* cutoff is computed as:

$$K_{2S-1}^* = \arg \min_k \left\{ \frac{p_0 \left(\frac{1}{2}\right)^S}{p_0 \left(\frac{1}{2}\right)^{S-1} + \sum_{l=1}^{k-1} p_l} < \frac{1 - c}{1 + c} \right\}.$$

Cutoffs for earlier stages can be derived recursively as the following proposition establishes.

Proposition 7. *For every $1 \leq j \leq 2S - 2$,*

$$K_j^* < \infty \iff \frac{p_0 \left(\frac{1}{2}\right)^{\lfloor \frac{j}{2} \rfloor + 1} + \sum_{l=1}^{K_{j+1}^* - 1} p_l}{p_0 \left(\frac{1}{2}\right)^{\lfloor \frac{j}{2} \rfloor} + (1 - p_0)} < \frac{1 - c}{1 + c}. \quad (2)$$

Proof: The logic of the proof is similar to Proposition 6. See Appendix B for details. ■

A simple economic interpretation of the conditions obtained in Proposition 6 and 7 is as follows. At any stage s , if the other player will take at the next stage, the net gain to taking at s is $[1 + (s - 1)c] - [sc] = 1 - c$. On the other hand, if the other player passes at the next stage, the net gain to taking at stage $s + 2$ is $[1 + (s + 1)c] - [sc] = 1 + c$. Hence, the right-hand side is simply the ratio of payoffs to the current player depending on the opponent taking or passing at the next stage, assuming the current player will take in the subsequent stage. Thus, a player will take in the current stage if and only if the posterior probability the opponent will take in the next stage is less than this ratio.

The information contained in the history is that *if the game proceeds to later stages, the opponent is less likely to be a level-0 player*. If the game reaches stage j , the player would know the opponent has passed $\lfloor \frac{j}{2} \rfloor$ times, which would occur with probability (conditional on the opponent being level-0) $1/2^{\lfloor \frac{j}{2} \rfloor}$ which rapidly approaches 0.

5.2 CH Solution for the Reduced Normal Form Centipede Game

In order to demonstrate the effect of CH learning on behavior in centipede games, we compare the results from the dynamic CH model with the static CH model where the game is played in its reduced normal form. The reduced normal form of the $2S$ -leg centipede game is simply a simultaneous move game where $A_1 = A_2 = \{1, \dots, S+1\}$ is the set of actions for each player. Action $s \leq S$ represents a plan to pass at the first $s-1$ opportunities and take at the s -th opportunity. Strategy $S+1$ is the plan to always pass. Denote by a_1 and a_2 player 1 and 2's strategies, respectively. If $a_1 \leq a_2$, then the payoffs are $(1 + (2a_1 - 2)c, (2a_1 - 2)c)$; if $a_1 > a_2$, then the payoffs are $((2a_2 - 1)c, 1 + (2a_2 - 1)c)$.

Let a_i^k denote level- k of player i 's strategy. A level-0 player uniformly randomizes across all available strategies. With a minor abuse of notation, denote $a_i^0 = \frac{1}{S+1}$ for $i \in \{1, 2\}$. Lemma 2 establishes level-1 players' behavior and the monotonicity, similarly to Lemma 1.

Lemma 2. *In the reduced normal form linear centipede game, as $\frac{1}{3} < c < 1$,*

1. $a_1^1 = S+1$ and $a_2^1 = S$.
2. For every $k \geq 2$,
 - (i) $a_1^k \geq \min\{a_2^m : 1 \leq m \leq k-1\}$;
 - (ii) $a_2^k \geq \min\{a_1^m : 1 \leq m \leq k-1\} - 1$.
3. $a_i^{k+1} \leq a_i^k$ for all $k \geq 1$ and for all $i \in \{1, 2\}$.

Proof: See Appendix B. ■

Lemma 2 has essentially the same three parts as Lemma 1, but stated in terms of the stopping point strategies rather than behavioral strategies. Therefore, as in the extensive form centipede game, optimal strategies are given by **cutoffs**, defined analogously to Definition 1.

Definition 2. *For every stage s where $1 \leq j \leq 2S$, define the **cutoff** \tilde{K}_j^* to be the lowest level of player that would take no later than this stage. In other words,*

$$\tilde{K}_j^* = \begin{cases} \arg \min_k \{a_1^k \leq \frac{j+1}{2}\}, & \text{if } j \text{ is odd} \\ \arg \min_k \{a_2^k \leq \frac{j}{2}\}, & \text{if } j \text{ is even} \\ \infty, & \text{if } \nexists k \text{ s.t. } a_1^k \leq \frac{j+1}{2} \text{ or } a_2^k \leq \frac{j}{2}. \end{cases}$$

By Lemma 2, we know $a_2^1 = S$. Therefore, $\tilde{K}_{2S}^* = 1$. Proposition 8 and Proposition 9 are parallel to Proposition 6 and 7, providing necessary and sufficient conditions for the existence of some strategic players to take before a particular stage.

Proposition 8. $\tilde{K}_{2S-1}^* < \infty \iff p_0 < \frac{S+1}{(S+1) + (\frac{3c-1}{1-c})}$.

Proof: See Appendix B. ■

From Proposition 6 and Proposition 8, we can find a class of prior distributions such that the dynamic and static CH models would generate different predictions. In this class of prior distributions, the dynamic model predicts there is some level of player 1 would take at some stage while the static model predicts every level of player 1 would pass at every stage.

Corollary 1. *If $\frac{S+1}{(S+1)+(\frac{3c-1}{1-c})} \leq p_0 < \frac{2^S}{2^S+(\frac{3c-1}{1-c})}$, then $K_{2S-1}^* < \infty$ and $\tilde{K}_{2S-1}^* = \infty$.*

Proof: Since $2^S > S + 1$ for all $S \geq 2$, the corollary follows directly from Propositions 6 and 8. ■

Proposition 9. *For every $1 \leq j \leq 2S - 2$,*

$$\tilde{K}_j^* < \infty \iff p_0 \left(\frac{S}{S+1} - \frac{2\lfloor \frac{j}{2} \rfloor c}{(S+1)(1+c)} \right) + \sum_{k=1}^{\tilde{K}_{j+1}^*-1} p_k < \frac{1-c}{1+c}. \quad (3)$$

Proof: The logic of the proof is similar to Proposition 8. See Appendix B for details. ■

An immediate implication of propositions 7 and 9 is that if p_0 is small, then the difference in behavior under the static and the dynamic versions of the game will also be small, since the left hand side of inequalities (2) and (3) both converge to $\sum_{k=1}^{\tilde{K}_{j+1}^*-1} p_k$. This result is intuitive. If p_0 is very small, then there is essentially no learning in the dynamic game so behavior will be almost the same as the static game.

However, regardless how small p_0 is (as long as it is positive), the dynamic and static models lead to systematically different behavioral predictions. These differences lead to the main result of this section, Theorem 1, which establishes that players are more likely to take at every stage in the extensive form.

Theorem 1. *For every stage $1 \leq j \leq 2S$,*

$$K_j^* \leq \tilde{K}_j^*.$$

Proof: See Appendix B. ■

This result provides a testable prediction that these centipede games will end earlier if played in the extensive form rather than in the reduced normal form. Moreover, this result is robust to the prior distributions of levels. The intuition is that as long as there exists some level-0 players, players in the dynamic model exhaust all available information while players in the static model only use the prior distribution to form their beliefs about the other player's level. Therefore, players in the dynamic model exhibit more sophisticated behavior since the information from the history is that the opponent is less likely to be a level-0 player.

5.3 Results for the Poisson CH Model

In previous applications of the CH model, it has been useful to assume the distribution of levels is given by a Poisson distribution (Camerer et al., 2004). We obtain some additional

results here for this one-parameter family of distributions that allow us to further pin down the differences between the extensive form and the reduced normal form of the centipede game. The Poisson CH model assumes:

$$p_k \equiv \frac{e^{-\lambda} \lambda^k}{k!}, \text{ for all } k = 0, 1, 2, \dots$$

where $\lambda > 0$ is the mean of the Poisson distribution.

Finally, we write the cutoffs as functions of λ . In the dynamic model, the cutoff function for stage j is $K_j^*(\lambda)$. In the static model, the cutoff function for stage j is $\tilde{K}_j^*(\lambda)$.

As we have discussed before, in the dynamic model, players become more sure that the opponent is not level 0 when the game moves to later stages. Therefore, player 1 has the best information in stage $2S - 1$. Proposition 10 quantitatively demonstrates the difference between two models at stage $2S - 1$.

Proposition 10. *As the prior distribution follows $\text{Poisson}(\lambda)$, then*

- (i) $K_{2S-1}^*(\lambda) < \infty \iff \lambda > \ln \left[1 + \left(\frac{1}{2}\right)^S \left(\frac{3c-1}{1-c}\right) \right];$
- (ii) $\tilde{K}_{2S-1}^*(\lambda) < \infty \iff \lambda > \ln \left[1 + \left(\frac{1}{S+1}\right) \left(\frac{3c-1}{1-c}\right) \right].$

Proof: The result is obtained by substituting $p_0 = e^{-\lambda}$ in the formulas given by Propositions 6 and 8, and with some algebra. See Appendix B for details. ■

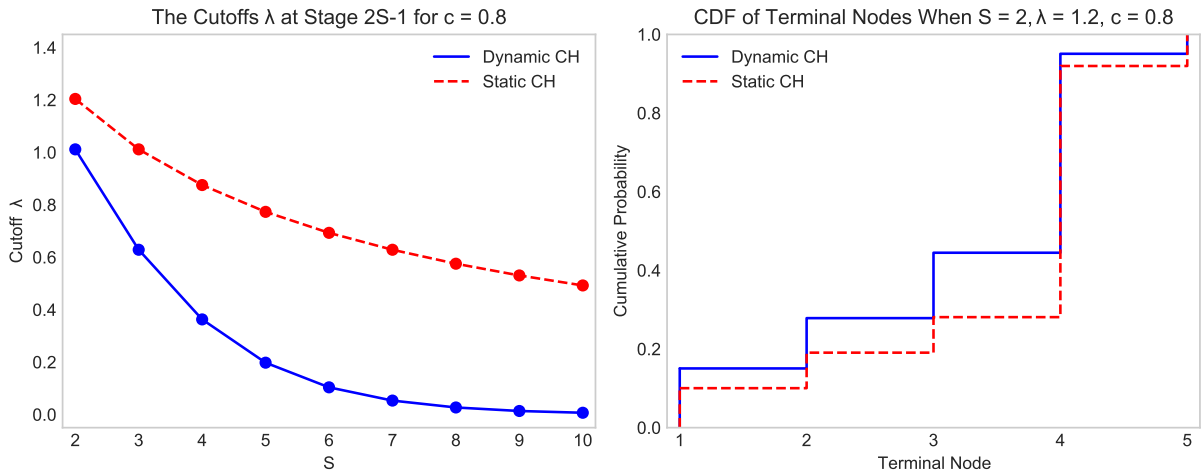


Figure 7: (Left) The minimum value of λ to support taking at stage $2S - 1$ for both the dynamic CH model (solid) and the static CH model (dash) when $c = 0.8$ with S on the horizontal axis and λ on the vertical axis. (Right) The CDF of terminal nodes in four-node centipede games predicted by the dynamic CH and static CH models.

Proposition 10 provides a closed form solution for the minimum λ to support some level of player 1 to take at stage $2S - 1$ in both the dynamic and static models. The left panel of Figure 7 plots the lowest λ . From the figure, we can notice that at stage $2S - 1$, the minimum

value of λ to start unraveling is much smaller for the dynamic model than the static model. Moreover, the minimum λ converges to 0 much faster in the dynamic model than the static model as S gets higher, which is derived from the belief updating in the dynamic CH.

On the other hand, in the right panel of Figure 7, we focus on the four-node centipede game ($S = 2$) and plot the CDF of terminal nodes predicted by both models. First of all, we can observe the distribution of terminal nodes of the static CH model first order stochastically dominates the dynamic CH model. In fact, the FOSD relationship holds for any S, c, λ . This leads to a second interpretation of Theorem 1—since the cutoffs of the dynamic CH model are uniformly smaller than the cutoffs of the static CH model, there are more levels of players that would take at every stage, generating the FOSD relationship.

When λ gets larger, the distribution of levels will shift to the right and players tend to be more sophisticated at the aggregate level. Proposition 11 shows that for sufficiently large λ , highly sophisticated players would take at every stage in both the extensive form and reduced normal form of the centipede game.

Proposition 11. *For both the dynamic and static models, there exists sufficiently high λ such that unraveling occurs. That is, for each S :*

- (i) $\exists \lambda^* < \infty$ such that $K_1^*(\lambda) < \infty$ for all $\lambda > \lambda^*$;
- (ii) $\exists \tilde{\lambda}^* < \infty$ such that $\tilde{K}_1^*(\lambda) < \infty$ for all $\lambda > \tilde{\lambda}^*$.

Proof: See Appendix B. ■

This result shows that unravelling occurs if λ is sufficiently high, in both models. However, it leaves open questions about how this unravelling differs between the static and dynamic models. To this end, Proposition 12 provides some insight on this issue, in particular that the static model requires strictly more “density shift” (higher λ) in order to completely unravel for high-level players.

Proposition 12. *For any j where $1 \leq j \leq 2S - 1$, let λ_{2S-j}^{**} be the lowest λ such that $K_{2S-j}^*(\lambda) = j + 1$ for all $\lambda > \lambda_{2S-j}^{**}$, and let $\tilde{\lambda}_{2S-j}^{**}$ be the lowest λ such that $\tilde{K}_{2S-j}^*(\lambda) = j + 1$ for all $\lambda > \tilde{\lambda}_{2S-j}^{**}$. Then $\lambda_{2S-j}^{**} < \tilde{\lambda}_{2S-j}^{**}$ for all $1 \leq j \leq 2S - 1$.*

Proof: See Appendix B. ■

In other words, we can view the difference of density shifts between two models (so that every level of players completely unravels) as a measure of the effect of belief updating. As shown in Proposition 12, we can always find a non-trivial set of λ such that players have already unravelled in the extensive form but not in the reduced normal form.

Finally, in the Poisson family, we can obtain an unambiguous comparative static result on the change of λ . Proposition 13 shows that when λ increases, the cutoff level of each stage is weakly decreasing. That is, when the average sophistication of the players increases, play is closer to the fully rational model—i.e., there is more taking.

Proposition 13. *For every stage $1 \leq j \leq 2S$, $K_j^*(\lambda)$ and $\tilde{K}_j^*(\lambda)$ are weakly decreasing in $\lambda > 0$.*

Proof: See Appendix B. ■

5.4 Non-linear centipede games

The results of this section about the exact characterization of behavior in dynamic and static centipede games only consider games with a linearly increasing pie. A natural robustness question is whether the qualitative findings apply more generally to other families of centipede games. The key assumption in our analysis is that the increment of pie is not too fast or too slow. If the increment is too fast (i.e., $c > 1$), then it is optimal to pass everywhere. On the other hand, if the increment is too slow (i.e., $c < \frac{1}{3}$), even the lowest level of players would take at the first stage. In all cases within this range, learning occurs in the dynamic version of the game, but not in the static version. This would seem to be a general property of increasing-pie centipede games. That is, unless the pie sizes grow so fast that all positive levels of players will always pass, or so slowly that positive levels will always take, then there will be some opportunity for learning, which will lead to different behavior in the two versions of the game. Moreover, the main effect of learning in the dynamic version will be to update the prior probability of level-0 players in a downward direction, which in turn will lead to earlier taking.

For example, the analysis can be extended to the class of centipede games with an exponentially increasing pie, as studied in the [McKelvey and Palfrey \(1992\)](#) experiment. Similar to the previous analysis, two players alternate over a sequence of moves in an exponential centipede game with $2S$ legs. At each move, if a player passes, both the large and small (positive) payoffs would be *multiplied* by $c > 1$. In addition, the ratio between the large and the small payoff is equal to $\pi > 1$ and does not change as the game progresses. Therefore, an exponential centipede game is parameterized by (S, π, c) : if the game is terminated by a player at stage $j \leq 2S$, the payoffs are $(c^{j-1}\pi, c^{j-1})$ if j is odd and $(c^{j-1}, c^{j-1}\pi)$ if j is even. If no one ever takes, then the payoffs will be $(c^{2S}\pi, c^{2S})$. In this class of centipede games, the multiplier c governs the growth rate of pie, and the logic of the proofs of propositions for the linear games is similar for exponential games as long as:

$$\frac{-1 + \sqrt{1 + 8\pi^2}}{2\pi} < c < \pi,$$

which rules out trivial cases, in the same way as the assumption of $\frac{1}{3} < c < 1$ rules out trivial cases in linear centipede games.

All of the qualitative results for linearly increasing centipede games also hold for exponential centipede games, with the only difference being the analytical expression of the cutoffs. In particular, [Theorem 1](#), the representation effect continues to hold.

6 Conclusions

We conclude by emphasizing the key motivation for this paper: to provide a theoretical framework that characterizes hierarchical reasoning in sequential games. As documented in the literature, sequential equilibrium based on backward induction is not only mathematically fragile but also empirically implausible to hold. To narrow the gap between the theory and empirical patterns in sequential games, it is natural to extend the level- k approach to

such games, as it has already demonstrated considerable success in narrowing the gap for games played simultaneously. However, the conundrum for directly applying the standard level- k approach is that players may observe actions that are incompatible with their beliefs, which leads to the widely known problem of specifying off-path beliefs. The dynamic CH model avoids this issue with a simple structure that allows players with heterogeneous levels of sophistication to update their beliefs everywhere as history unfolds, using Bayes' rule.

We characterize properties of the belief-updating process and explore how it can affect players' strategic behavior. The key of our framework is that the history of play contains substantial information about other players' levels of sophistication, and therefore as play unfolds, players learn about their opponents' strategic sophistication and update their beliefs about the continuation play in the game accordingly. In this way our dynamic CH model departs from the standard level- k approach and generates new insights, including experimentally testable implications.

We obtain two main results that apply generally to all finite extensive form games. Proposition 1 establishes that a player's updating process is independent across the other players. That is, for every player and every non-terminal history, the joint distribution of the beliefs of the levels of the other players is the product of the individual posterior distribution of the levels of each of those other players. In games of imperfect information, the information sets are non-singleton and the beliefs could be correlated across the histories at some information set.

In addition, Proposition 2 establishes that the updating process filters out possible level types of opponents as the game proceeds, and it is irreversible. That is, over the course of play, it is possible that a player eliminates some levels of another player from the support of his beliefs, and as the game continues, these levels can never be added back to the support. Hence, in addition to updating posterior beliefs over the support of level types, the support also shrinks over time. However, the level-0 players always remain in the support of beliefs, and hence every player believes every future information set can be reached with positive probability.

The second half of the paper provides a rigorous analysis of a class of increasing-pie centipede games and generates testable predictions about how play depends on whether the game is played sequentially or as a simultaneous move game in its reduced normal form. One direct implication is the representation effect given by Theorem 1. The theorem states that playing a centipede game in its extensive form representation, i.e., as a sequential move game, would lead to more taking than the reduced normal form representation, where the two players simultaneously announce the stage at which they will take.

This result provides a prediction that may be useful for experimental testing, since the claim is independent of the length of the centipede and the increment of the pie. Moreover, the statement is true for any prior belief about the strategic levels. A natural next step would be to conduct an experiment to test this strong prediction of our model. [García-Pola et al. \(2020a\)](#) report the results of an experiment that compares behavior in sequential and simultaneous move centipede games. They find that games end significantly earlier in the sequential version for two of four games they study, with no significant effect in the other two games. This provides some support for the representation effect we identify, but unfortunately that experiment does not apply directly to Theorem 1. Their experiment does

not study linearly increasing centipede games, instead exploring behavior in constant sum, decreasing sum, exponentially increasing, and irregularly increasing variations of centipede games.

Another direction worth pursuing would be to incorporate some salient features of alternative behavioral models of learning in extensive form games into our approach. In the approach taken here, the learning process is “extreme” in the sense that players will completely rule out some levels from their beliefs whenever they observe incompatible actions. For example, players will believe the opponent is level-0 with certainty if a strictly dominated action is taken. Yet, it is possible that the player is strategic and the action is taken by mistake. In this sense, one could incorporate some elements of the extensive form QRE, where players choose actions at each information set stochastically, and the choice probabilities are increasing in the continuation values. In fact, this approach has been used with some success in simultaneous move games (Crawford and Iriberri, 2007a). As shown in Proposition 2, in the present model of dynamic CH, there is no way to *expand* the support of a player’s belief about the other players’ types. However, if players choose stochastically, then no level type is ever ruled out from the support, which smoothes out the updating process. Because players’ beliefs maintain full support on lower types throughout the game, a natural conjecture is that arbitrarily high-level players will approach backward induction when the error is sufficiently small.

As a final remark, while the main point of the present paper is to develop a general theoretical foundation for applying CH to extensive form games, the ultimate hope is that this framework can be usefully applied to gain insight into specific economic models. There are a number of possible such applications one might imagine, where some or all agents in the model have opportunities to learn about the strategic sophistication of the other agents in ways that could significantly affect their choices in the game. For example, Chamley and Gale (1994) consider a dynamic investment game with social learning, where investments are valuable only if enough other agents are able to invest, and learning occurs as investment decisions are observed over time. The dynamic CH model, which combines learning and updating, but without common knowledge of rationality or fully rational expectations, might be a useful alternative approach to this problem. Models of sequential voting on agendas (McKelvey and Niemi (1978) and Banks (1985)), limit pricing and entry deterrence (Selten (1978) and Milgrom and Roberts (1982)), and dynamic public good provision (Marx and Matthews (2000), Duffy et al. (2007), and Choi et al. (2008)) are some additional areas of applied interest where the dynamic CH approach could be useful.

Appendix A: Proofs of Results in Section 4

Let τ_i be player i 's level. Following previous notations, we use $\sigma_i(h)$ to denote player i 's (pure) action at h . In addition, $\mu_i^k(\tau_{-i})$ is level- k player i 's prior belief about the opponent's level, and $\nu_i^k(\tau_{-i} | h)$ is level- k player i 's posterior belief about the opponent's level at history h . Finally, level-0 players would uniformly randomize at every node. The analysis of the examples is summarized in the following claims.

Example 1

Claim 1. *In Example 1, each level of players' strategies are:*

1. for any $k \in \mathbb{N}$, $\sigma_1^k(1b) = r_{1b}$, $\sigma_1^k(1c) = l_{1c}$, $\sigma_2^k(2b) = l_{2b}$, and $\sigma_2^k(2c) = r_{2c}$;
2. $\sigma_1^1(1a) = r_{1a}$ and $\sigma_1^k(1a) = l_{2a}$ for $k \geq 2$; $\sigma_2^1(2a) = \sigma_2^2(2a) = l_{2a}$ and $\sigma_2^k(2a) = r_{2a}$ for $k \geq 3$.

Proof: **1.** First, all strategic levels of players would choose the action with a higher payoff at the last node. Hence, $\sigma_1^k(1b) = r_{1b}$ and $\sigma_2^k(2c) = r_{2c}$ for all $k \geq 1$. Player 2 has a dominant action at history $h = 2b$, so $\sigma_2^k(2b) = l_{2b}$ for all $k \geq 1$. Notice that whenever a dominant action is not chosen, players would believe the opponent is level-0 with certainty. At history $h = 1c$, every level of player 1 thinks player 2 is level-0 and hence for all $k \geq 1$, $\sigma_1^k(1c) = l_{1c}$ since the expected payoff is $13/2 > 6$.

2. Level-1 players believe the other player would randomize at every node. On the one hand, $\sigma_1^1(1a) = r_{1a}$ and $\sigma_2^1(2a) = l_{2a}$ so that they can maximize the expected payoff. On the other hand, level-2 players' initial beliefs are $\mu_i^2(0) = e^{-1.5}/(e^{-1.5} + 1.5e^{-1.5}) = 2/5$ and $\mu_i^2(1) = 3/5$. Thus, $\sigma_1^2(1a) = l_{1a}$ since the expected payoff for l_{1a} is $19/5 > 29/10$. On the other hand, when history $h = 2a$ is realized, level-2 player 2 would believe the opponent is definitely level-0 and hence $\sigma_2^2(2a) = \sigma_2^1(2a) = l_{2a}$.

The behavior of higher-level players can be solved by induction. Level-3 players' prior beliefs are $\mu_i^3(0) = 8/29$, $\mu_i^3(1) = 12/29$, and $\mu_i^3(2) = 9/29$. In this case, $\sigma_1^3(1a) = l_{1a}$ since the expected payoff for l_{1a} is $112/29 > 76/29$. In addition, when history $h = 2a$ is realized, level-3 player 2's posterior belief becomes $\nu_2^3(0 | 2a) = 0.5e^{-1.5}/(0.5e^{-1.5} + 1.125e^{-1.5}) = 4/13$ and $\nu_2^3(2 | 2a) = 9/13$, and hence $\sigma_2^3(2a) = r_{2a}$ since $4 > 45/13$. Suppose for some $k > 3$, $\sigma_1^\kappa(1a) = l_{1a}$ for all $2 \leq \kappa \leq k$ and $\sigma_2^\kappa(2a) = r_{2a}$ for all $3 \leq \kappa \leq k$. We want to show that $\sigma_1^{k+1}(1a) = l_{1a}$ and $\sigma_2^{k+1}(2a) = r_{2a}$. Level- $(k+1)$ players' prior beliefs are $\mu_i^{k+1}(\kappa) = p_\kappa / (\sum_{\kappa=0}^k p_\kappa)$ for $0 \leq \kappa \leq k$. By the induction hypothesis, $\sigma_1^{k+1}(1a) = l_{1a}$ if and only if

$$\frac{7}{2} \left(\frac{p_0}{\sum_{\kappa=0}^k p_\kappa} \right) + 4 \left(\frac{p_1 + p_2}{\sum_{\kappa=0}^k p_\kappa} \right) + 3 \left(\frac{\sum_{\kappa=3}^k p_\kappa}{\sum_{\kappa=0}^k p_\kappa} \right) > \frac{17}{4} \left(\frac{p_0}{\sum_{\kappa=0}^k p_\kappa} \right) + 2 \left(1 - \frac{p_0}{\sum_{\kappa=0}^k p_\kappa} \right),$$

which is equivalent to $(7/4)p_0 - p_1 - p_2 < \sum_{\kappa=0}^k p_\kappa$. This holds because $(7/4)p_0 - p_1 - p_2 = -(7/8)e^{-1.5} < 0$. Finally, by the induction hypothesis, level- $(k+1)$ player 2's posterior belief

at $h = 2a$ is $\nu_2^{k+1}(0 | 2a) = 0.5p_0 / (0.5p_0 + \sum_{\kappa=2}^k p_\kappa)$ and $\nu_2^{k+1}(j | 2a) = p_j / (0.5p_0 + \sum_{\kappa=2}^k p_\kappa)$ where $2 \leq j \leq k$. Thus, $\sigma_2^{k+1}(2a) = r_{2a}$ if and only if

$$\frac{9}{2}\nu_2^{k+1}(0 | 2a) + 3(1 - \nu_2^{k+1}(0 | 2a)) < 4 \iff \nu_2^{k+1}(0 | 2a) < \frac{2}{3}.$$

Moreover, the induction hypothesis suggests that

$$\nu_2^{k+1}(0 | 2a) = \frac{\frac{1}{2}p_0}{\frac{1}{2}p_0 + \sum_{\kappa=2}^k p_\kappa} < \frac{\frac{1}{2}p_0}{\frac{1}{2}p_0 + \sum_{\kappa=2}^{k-1} p_\kappa} = \nu_2^k(0 | 2a) < \frac{2}{3},$$

implying the optimal choice for level- $(k+1)$ player 2 is r_{2a} . This completes the proof. ■

Example 2

Claim 2. Suppose τ_i 's are independently drawn from $p = (p_k)_{k=0}^\infty$, then in Example 2,

1. for any $k \in \mathbb{N}$, $\sigma_1^k(1a) = r_{1a}$, $\sigma_1^k(1b) = r_{1b}$, $\sigma_1^k(1c) = l_{1c}$, $\sigma_2^k(2a) = l_{2a}$, $\sigma_2^k(2b) = l_{2b}$, and $\sigma_2^k(2c) = r_{2c}$;
2. the ex ante probability of the subgame perfect equilibrium path being realized converges to 0 as $p_0 \rightarrow 0^+$.

Proof: **1.** By the analysis of Example 1, we only need to check player 1's action at the initial node and player 2's action at history $h = 2a$. We can prove the statement by induction on k . For $k = 1$, players would think the opponent is level-0. In this case, $\sigma_1^1(1a) = r_{1a}$ since the expected payoff is $17/4 > 9/4$ and $\sigma_2^1(2a) = l_{2a}$ with the expected payoff being $9/2 > 4$. Suppose there is some K such that $\sigma_1^k(1a) = r_{1a}$ and $\sigma_2^k(2a) = l_{2a}$ for all $1 \leq k \leq K$. For level- $(K+1)$ player 1, the prior belief is $\mu_1^{K+1}(0) = p_0 / (\sum_{\kappa=0}^K p_\kappa)$ and $\sigma_1^{K+1}(1a) = r_{1a}$ if and only if

$$\frac{17}{4}\mu_1^{K+1}(0) + 2(1 - \mu_1^{K+1}(0)) > \frac{9}{4}\mu_1^{K+1}(0) + \frac{3}{2}(1 - \mu_1^{K+1}(0)),$$

which holds as $\mu_1^{K+1}(0) > 0$. On the other hand, by the induction hypothesis, player 2 would believe player 1 is level-0 with certainty when history $h = 2a$ is realized, so $\sigma_2^{K+1}(2a) = \sigma_2^1(2a) = l_{2a}$.

2. Statement 1 implies the probability of the subgame perfect equilibrium path r_{2a} being realized is

$$\Pr(r_{2a}) = \Pr((1a, 2a) | 1a) \Pr(r_{2a} | 2a) = [\sigma_1^0(1a, 2a)p_0] [\sigma_{2,2a}^0(r_{2a})p_0] = \frac{1}{4}p_0^2.$$

Therefore, we can find the limit of the probability is

$$\lim_{p_0 \rightarrow 0^+} \Pr(r_{2a}) = \lim_{p_0 \rightarrow 0^+} \frac{1}{4}p_0^2 = 0.$$

This completes the proof. ■

Example 4

Claim 3. In Example 4, each level of players' strategies are:

1. for any $k \in \mathbb{N}$, $\sigma_1^k(1b) = r_{1b}$, $\sigma_1^k(1c) = l_{1c}$, $\sigma_2^k(2b) = l_{2b}$, and $\sigma_2^k(2c) = r_{2c}$;
2. $\sigma_1^k(1a) = l_{1a}$ for all $k \neq 2$, and $\sigma_1^2(1a) = r_{1a}$; $\sigma_2^1(2a) = l_{2a}$, and $\sigma_2^k(2a) = r_{2a}$ for all $k \geq 2$.

Proof: **1.** The proof is the same as the proof of Claim 1.

2. First, level-1 players believe the other player randomizes everywhere, so $\sigma_1^1(1a) = l_{1a}$ and $\sigma_2^1(2a) = l_{2a}$ in order to maximize their expected payoffs. Level-2 players' prior beliefs are $\mu_i^2(0) = 2/5$ and $\mu_i^2(1) = 3/5$. Therefore, $\sigma_1^2(1a) = r_{1a}$ since the expected payoff is $29/10 > 28/10$. Level-2 player 2's posterior belief at history $h = 2a$ is $\nu_2^2(0 | 2a) = 0.5e^{-1.5}/(0.5e^{-1.5} + 1.5e^{-1.5}) = 1/4$ and $\nu_2^2(1 | 2a) = 3/4$. In this case, $\sigma_2^2(2a) = r_{2a}$ because $4 > 27/8$.

Finally, we can solve higher-level players' behavior by induction. Level-3 players' prior beliefs are $\mu_i^3(0) = 8/29$, $\mu_i^3(1) = 12/29$, and $\mu_i^3(2) = 9/29$, and hence $\sigma_1^3(1a) = l_{1a}$ since the expected payoff is $128/29 > 76/29$. At history $h = 2a$, level-3 player 2's posterior belief is the same as level-2, and so $\sigma_2^3(2a) = \sigma_2^2(2a) = r_{2a}$. Suppose there is some $K > 3$ such that $\sigma_1^k(1a) = l_{1a}$ for all $3 \leq k \leq K$ and $\sigma_2^k(2a) = r_{2a}$ for all $2 \leq k \leq K$. Level- $(K+1)$ players' prior beliefs are $\mu_i^{K+1}(j) = p_j / \sum_{i=0}^K p_i$ for $0 \leq j \leq K$. By the induction hypothesis, $\sigma_1^{K+1}(1a) = l_{1a}$ if and only if

$$\frac{19}{4} \left(\frac{p_0}{\sum_{i=0}^K p_i} \right) + \frac{3}{2} \left(\frac{p_1}{\sum_{i=0}^K p_i} \right) + 8 \left(\frac{\sum_{i=2}^K p_i}{\sum_{i=0}^K p_i} \right) > \frac{17}{4} \left(\frac{p_0}{\sum_{i=0}^K p_i} \right) + 2 \left(1 - \frac{p_0}{\sum_{i=0}^K p_i} \right),$$

which is equivalent to $5.5p_0 + 6.5p_1 < 6 \sum_{i=0}^K p_i$. This holds when the distribution of levels follows Poisson(1.5). On the other hand, by the induction hypothesis, level- $(K+1)$ player 2's posterior belief at history $h = 2a$ is $\nu_2^{K+1}(0 | 2a) = 0.5p_0 / (0.5p_0 + p_1 + \sum_{i=3}^K p_i)$ and $\nu_2^{K+1}(j | 2a) = p_j / (0.5p_0 + p_1 + \sum_{i=3}^K p_i)$ where $j \neq 0$ or 2 , and hence $\sigma_2^{K+1}(2a) = r_{2a}$ if and only if

$$\frac{9}{2} \nu_2^{K+1}(0 | 2a) + 3(1 - \nu_2^{K+1}(0 | 2a)) < 4 \iff \nu_2^{K+1}(0 | l) < \frac{2}{3}.$$

Moreover, the induction hypothesis implies:

$$\nu_2^{K+1}(0 | 2a) = \frac{\frac{1}{2}p_0}{\frac{1}{2}p_0 + p_1 + \sum_{i=3}^K p_i} < \frac{\frac{1}{2}p_0}{\frac{1}{2}p_0 + p_1 + \sum_{i=3}^{K-1} p_i} = \nu_2^K(0 | 2a) < \frac{2}{3},$$

as desired. ■

Appendix B: Proofs of Results in Section 5

Proof of Lemma 1

1. Since stage $2S$ is the last stage of the game, for any $k \geq 1$, player 2 would take at this stage if and only if

$$1 + (2S - 1)c > 2Sc \iff 1 > c,$$

which holds by assumption. Therefore, $\sigma_{2,S}^k = 1$ for all $k \geq 1$.

2. Consider a level-1 type of player 1 and any of player 1's decision nodes $j \in \{1, \dots, S\}$. The payoff from Take is $1 + (2j - 2)c$ and the expected payoff from Pass is greater than or equal to $\frac{1}{2}(2j - 1)c + \frac{1}{2}(1 + (2j)c)$. Thus, $\sigma_{1,j}^k = 0$ is strictly optimal if and only if:

$$\begin{aligned} 1 + (2j - 2)c &< \frac{1}{2}(2j - 1)c + \frac{1}{2}(1 + (2j)c) \\ &\iff \frac{1}{3} < c. \end{aligned}$$

Hence, $\sigma_1^1 = (0, \dots, 0)$. A similar argument shows that $\sigma_2^1 = (0, \dots, 0, 1)$.

3. The argument is similar to the proof of the first statement. Consider a level- k type of player 1 and any of player 1's decision nodes $j \in \{1, \dots, S - 1\}$, and suppose $\sigma_{2,j}^m = 0$ for every $1 \leq m \leq k - 1$. Then the payoff from Take is $1 + (2j - 2)c$ and the expected payoff from Pass is greater than or equal to $\frac{1}{2}\nu_{2j-1}^k(0)(2j - 1)c + (1 - \frac{1}{2}\nu_{2j-1}^k(0))(1 + (2j)c)$, which in turn is greater than or equal to $\frac{1}{2}(2j - 1)c + \frac{1}{2}(1 + (2j)c)$ because $\nu_{2j-1}^k(0) \leq 1$. Thus $\sigma_{1,j}^k = 0$ is strictly optimal if and only if:

$$\begin{aligned} 1 + (2j - 2)c &< \frac{1}{2}(2j - 1)c + \frac{1}{2}(1 + (2j)c) \\ &\iff \frac{1}{3} < c. \end{aligned}$$

Hence, $\sigma_{1,j}^k = 0$. A similar argument shows that $\sigma_{2,j}^k = 0$ if $\sigma_{1,j+1}^m = 0$ for every $1 \leq m \leq k - 1$. This completes the proof. ■

Proof of Proposition 4

1. The statement can be proved by induction. Consider stage $2S - 1$. By Lemma 1, we know $K_{2S}^* = 1$ and $\sigma_{1,S}^1 = 0$, suggesting $K_{2S-1}^* \geq 2 = K_{2S}^* + 1$. Now, fix any $2 \leq m \leq 2S - 1$ and suppose the statement holds for all stages $m \leq j \leq 2S - 1$. Without loss of generality, we consider an even m . We want to show that if $K_m^* < \infty$, then $K_{m-1}^* \geq K_m^* + 1$. By construction, we know $\sigma_{2,\frac{m}{2}}^k = 0$ for all $1 \leq k \leq K_m^* - 1$. Therefore, Lemma 1 implies $\sigma_{1,\frac{m}{2}}^m = 0$ for all $1 \leq k \leq K_m^*$, and $K_{m-1}^* \geq K_m^* + 1$.

2. Consider any j such that $K_{j+1}^* = \infty$. Without loss of generality, we consider an odd j . Hence, $\sigma_{2,\frac{j+1}{2}}^k = 0$ for all $k \geq 1$ and we want to show $\sigma_{1,\frac{j+1}{2}}^k = 0$ for all $k \geq 1$ by induction. Lemma 1 implies $\sigma_{1,\frac{j+1}{2}}^1 = 0$. Suppose there is $\bar{K} \geq 2$ such that $\sigma_{1,\frac{j+1}{2}}^k = 0$ for all $1 \leq k \leq \bar{K}$. Since $\sigma_{2,\frac{j+1}{2}}^k = 0$ for all $k \geq 1$, Lemma 1 implies $\sigma_{1,\frac{j+1}{2}}^{\bar{K}+1} = 0$, as desired. ■

Proof of Proposition 5

We prove this by induction. Consider stage $2S - 1$. Level K_{2S-1}^* player 1 believes that only level-0 player 2 will pass at stage $2S$, so:

$$\begin{aligned} 1 + (2S - 2)c &> \left(1 - \frac{1}{2}\nu_{2S-1}^{K_{2S-1}^*}(0)\right) [(2S - 1)c] + \frac{1}{2}\nu_{2S-1}^{K_{2S-1}^*}(0)[1 + 2Sc] \\ &> \left(1 - \frac{1}{2}\nu_{2S-1}^k(0)\right) [(2S - 1)c] + \frac{1}{2}\nu_{2S-1}^k(0)[1 + 2Sc] \text{ for all } k > K_{2S-1}^* \end{aligned}$$

since $\frac{1}{2}\nu_{2S-1}^k(0) < \frac{1}{2}\nu_{2S-1}^{K_{2S-1}^*}(0)$ and therefore $\sigma_{1,S}^k = 1$ for all $k \geq K_{2S-1}^*$.

Next, suppose for any m where $2 \leq m \leq 2S - 1$, the statement holds for all j such that $m \leq j \leq 2S - 1$. Suppose m is odd and $K_{m-1}^* < \infty$. (A similar argument applies if m is even.) By construction, $\sigma_{2, \frac{m-1}{2}}^k = 0$ for all $1 \leq k \leq K_{m-1}^* - 1$. By Lemma 1, we have $\sigma_{1,s}^k = 0$ for all $1 \leq s \leq \frac{m-1}{2}$ and for all $1 \leq k \leq K_{m-1}^*$. Level K_{m-1}^* player 2's belief at stage $m - 1$ that the other player would pass at stage m is

$$\begin{aligned} \frac{1}{2}\nu_{m-1}^{K_{m-1}^*}(0) + \sum_{\kappa=1}^{K_{m-1}^*} \nu_{m-1}^{K_{m-1}^*}(\kappa) &= \frac{1}{2} \frac{p_0 \left(\frac{1}{2}\right)^{\frac{m-1}{2}}}{p_0 \left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa}} + \frac{\sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa}}{p_0 \left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa}} \\ &= \frac{p_0 \left(\frac{1}{2}\right)^{\frac{m+1}{2}} + \sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa}}{p_0 \left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa}}. \end{aligned}$$

Since $\sigma_{1,s}^{K_{m-1}^*} = 0$ for all $1 \leq s \leq \frac{m-1}{2}$, then for any $k > K_{m-1}^*$, at stage $m - 1$ level- k player 2's belief about the probability that the other player would pass at stage m is

$$\begin{aligned} \frac{1}{2}\nu_{m-1}^k(0) + \sum_{\kappa=1}^{K_{m-1}^*} \nu_{m-1}^k(\kappa) &\leq \frac{1}{2}\nu_{m-1}^{K_{m-1}^*+1}(0) + \sum_{\kappa=1}^{K_{m-1}^*} \nu_{m-1}^{K_{m-1}^*+1}(\kappa) \\ &= \frac{1}{2} \frac{p_0 \left(\frac{1}{2}\right)^{\frac{m-1}{2}}}{p_0 \left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa}} + \frac{\sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa}}{p_0 \left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa}} \\ &< \frac{p_0 \left(\frac{1}{2}\right)^{\frac{m+1}{2}} + \sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa}}{p_0 \left(\frac{1}{2}\right)^{\frac{m-1}{2}} + \sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa}}. \end{aligned}$$

since $\sum_{\kappa=1}^{K_{m-1}^*} p_{\kappa} > \sum_{\kappa=1}^{K_{m-1}^*-1} p_{\kappa}$. This implies that for any level $k > K_{m-1}^*$, higher level of player 2 at stage $m - 1$ would think the other player is less likely to pass at stage m . Since it is already profitable for level K_{m-1}^* player 2 to take at stage $m - 1$, we can conclude that $\sigma_{2, \frac{m-1}{2}}^k = 1$ for all $k \geq K_{m-1}^*$. ■

Proof of Proposition 7

Without loss of generality, we can consider an even j , so $\lfloor \frac{j}{2} \rfloor = \frac{j}{2}$ and it is player 2's turn at stage j .

Only if: Suppose $K_j^* < \infty$. Then from Proposition 4, $K_{j'}^* \geq K_j^* + 1$ for all $j' < j$. Hence, the belief of level K_j^* of player 2 that player 1 is level-0 at stage j equals to

$$\nu_j^{K_j^*}(0) = \frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}}}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_j^*-1} p_\kappa} > \frac{p_0 \left(\frac{1}{2}\right)^{S-1}}{p_0 \left(\frac{1}{2}\right)^{S-1} + (1-p_0)}.$$

Level K_j^* player 2's belief at stage j that the player 1 would pass at stage $j+1$ is

$$\begin{aligned} \frac{1}{2} \nu_j^{K_j^*}(0) + \sum_{\kappa=1}^{K_{j+1}^*-1} \nu_j^{K_j^*}(\kappa) &= \frac{1}{2} \frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}}}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_j^*-1} p_\kappa} + \frac{\sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_j^*-1} p_\kappa} \\ &= \frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}+1} + \sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_j^*-1} p_\kappa} \end{aligned}$$

where we know that $K_{j+1}^* \leq K_j^* - 1$. Since it is optimal for level $K_j^* < \infty$ to take at j this implies $\frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}+1} + \sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_j^*-1} p_\kappa} < \frac{1-c}{1+c}$ and therefore

$$\frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}+1} + \sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + (1-p_0)} < \frac{1-c}{1+c}.$$

If: Suppose $K_j^* = \infty$. Then from Proposition 4, $K_{j'}^* = \infty$ for all $j' < j$. That is, all levels of both players pass at every stage up to and including j . Hence the belief of level $k \geq 1$ of player 2 that player 1 is level-0 at stage j equals to

$$\nu_j^k(0) = \frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}}}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{k-1} p_\kappa} > \frac{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}}}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + (1-p_0)}.$$

Since $K_j^* = \infty$ it is optimal to pass at j for all levels $k \geq 1$ of player 2, which implies

$$\frac{\frac{1}{2} p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{k-1} p_\kappa} \geq \frac{1-c}{1+c}, \text{ for all } k, \text{ where possibly } K_{j+1}^* = \infty, \text{ so:}$$

$$\frac{\frac{1}{2} p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + \sum_{\kappa=1}^{K_{j+1}^*-1} p_\kappa}{p_0 \left(\frac{1}{2}\right)^{\frac{j}{2}} + (1-p_0)} \geq \frac{1-c}{1+c},$$

as desired. ■

Proof of Lemma 2

1. To prove the statement, we can discuss player 1 and 2 separately.

Player 2:

- (i) $a_2^1 = S$ strictly dominates $a_2^1 = S + 1$: $\mathbb{E}[u_2(a_1^0, S)] - \mathbb{E}[u_2(a_1^0, S + 1)] = \frac{1-c}{S+1} > 0$ since $c < 1$.
- (ii) $a_2^1 = j + 1$ strictly dominates $a_2^1 = j$ for all $1 \leq j \leq S - 1$: For $1 \leq j \leq S - 1$, since $c > \frac{1}{3}$,

$$\mathbb{E}[u_2(a_1^0, j + 1)] - \mathbb{E}[u_2(a_1^0, j)] = \frac{1}{S+1} [-1 + (2S - 2j + 1)c] \geq \frac{1}{S+1} (-1 + 3c) > 0.$$

Hence, we can obtain that $a_2^1 = S$.

Player 1: By the same logic as (ii) above, $a_1^1 = j + 1$ strictly dominates $a_1^1 = j$ for all $1 \leq j \leq S$: For $1 \leq j \leq S$, since $c > \frac{1}{3}$,

$$\mathbb{E}[u_1(j + 1, a_2^0)] - \mathbb{E}[u_1(j, a_2^0)] = \frac{1}{S+1} [-1 + (2S - 2j + 3)c] \geq \frac{1}{S+1} (-1 + 3c) > 0.$$

Hence, we can obtain that $a_1^1 = S + 1$.

2. (i) Notice that for any a_2 , $u_1(a_1, a_2)$ is maximized at $a_1 = a_2$. Fix level $k \geq 2$. If level- k player 1 chooses s , then the expected payoff is:

$$\begin{aligned} V_1^k(s) &\equiv \sum_{\kappa=0}^{k-1} \tilde{p}_\kappa^k \mathbb{E}[u_1(s, a_2^\kappa)] \\ &= \tilde{p}_0^k \mathbb{E}[u_1(s, a_2^0)] + \sum_{\kappa=1}^{k-1} \tilde{p}_\kappa^k u_1(s, a_2^\kappa). \end{aligned}$$

Suppose $\min\{a_2^m : 1 \leq m \leq k-1\} = 1$, then (i) holds trivially. If $\min\{a_2^m : 1 \leq m \leq k-1\} \geq 2$, then we can prove the statement by contradiction. Suppose $a_1^k < \min\{a_2^m : 1 \leq m \leq k-1\}$, then

$$\begin{aligned} V_1^k(a_1^k) &= \tilde{p}_0^k \underbrace{\mathbb{E}[u_1(a_1^k, a_2^0)]}_{< \mathbb{E}[u_1(a_1^k+1, a_2^0)]} + \sum_{\kappa=1}^{k-1} \tilde{p}_\kappa^k \underbrace{u_1(a_1^k, a_2^\kappa)}_{\leq u_1(a_1^k+1, a_2^\kappa)} \\ &< V_1^k(a_1^k + 1). \end{aligned}$$

$\mathbb{E}[u_1(a_1^k, a_2^0)] < \mathbb{E}[u_1(a_1^k + 1, a_2^0)]$ follows from the first statement. Furthermore, $a_1^k < \min\{a_2^m : 1 \leq m \leq k-1\}$ implies $u_1(a_1^k, a_2^\kappa) \leq u_1(a_1^k + 1, a_2^\kappa)$ for all $1 \leq \kappa \leq k-1$. Hence, $a_1^k < \min\{a_2^m : 1 \leq m \leq k-1\}$ is not optimal for level- k player 1, a contradiction.

(ii) The logic is similar for player 2.

3. We prove this statement by induction on k . First, it holds for $k = 1$, by the first statement. Next, we suppose it holds for any k where $1 \leq k \leq K - 1$ and prove it holds for $k = K$. For level $K + 1$ player 1, the expected payoff for choosing s is:

$$\begin{aligned} V_1^{K+1}(s) &= \tilde{p}_0^{K+1} \mathbb{E}[u_1(s, a_2^0)] + \sum_{\kappa=1}^K \tilde{p}_\kappa^{K+1} u_1(s, a_2^\kappa) \\ &= \left(\frac{\sum_{\kappa=0}^{K-1} \tilde{p}_\kappa^{K+1}}{\sum_{\kappa=0}^K \tilde{p}_\kappa^{K+1}} \right) V_1^K(s) + \tilde{p}_K^{K+1} u_1(s, a_2^K). \end{aligned}$$

Suppose, by way of contradiction, that $a_1^{K+1} > a_1^K$. Then $V_1^K(a_1^{K+1}) < V_1^K(a_1^K)$. From the induction hypothesis, $a_2^K \leq a_2^{K-1}$, and from the second statement, $a_1^K \geq a_2^{K-1}$ and hence $a_1^{K+1} > a_1^K \geq a_2^{K-1} \geq a_1^K$. This implies $u_1(a_1^{K+1}, a_2^K) \leq u_1(a_1^K, a_2^K)$, so $V_1^{K+1}(a_1^{K+1}) < V_1^{K+1}(a_1^K)$, which contradicts that a_1^{K+1} is the optimal strategy for level $K+1$ player 1. Hence $a_1^{K+1} \leq a_1^K$, so the result is proved for $i = 1$. A similar argument proves the result for $i = 2$. ■

Proof of Proposition 8

With slight abuse of notation, denote a level- k player's prior belief that the opponent is level- κ by $\mu_\kappa^k \equiv \frac{p_\kappa}{\sum_{j=0}^{k-1} p_j}$, $\kappa = 1, \dots, k-1$.

Only if: Suppose $p_0 \geq \frac{S+1}{(S+1) + (\frac{3c-1}{1-c})}$, then we want to show that $a_1^k = S+1$ for all $k \geq 1$.

We can prove this statement by induction on k . By Lemma 2, we know $a_1^1 = S+1$. Now, suppose this statement holds for all $1 \leq k \leq K$ for some $K \in \mathbb{N}$, then we want to show this holds for level $K+1$ player 1. First, by Lemma 2, we have $a_2^k = S$ for all $1 \leq k \leq K$. Level $K+1$ player 1 would choose S if and only if

$$\begin{aligned} & \mu_0^{K+1} \left[\frac{1}{S+1} \left[1 + 2Sc + \sum_{i=2}^{S+1} (2i-3)c \right] \right] + (1 - \mu_0^{K+1}) (2S-1)c \\ & < \mu_0^{K+1} \left[\frac{1}{S+1} \left[2(1 + (2S-2)c) + \sum_{i=2}^S (2i-3)c \right] \right] + (1 - \mu_0^{K+1}) (1 + (2S-2)c) \\ & \iff \mu_0^{K+1} \left[\frac{1}{S+1} (1-3c) \right] + (1 - \mu_0^{K+1}) (1-c) > 0 \\ & \iff \mu_0^{K+1} < \frac{S+1}{(S+1) + (\frac{3c-1}{1-c})}. \end{aligned}$$

However, we know $\mu_0^K > p_0$ and we have assumed $p_0 \geq \frac{S+1}{(S+1) + (\frac{-1+3c}{1-c})}$, so $\mu_0^{K+1} > p_0 \geq \frac{S+1}{(S+1) + (\frac{3c-1}{1-c})}$, implying that $a_1^{K+1} = S+1$.

If: Suppose $p_0 < \frac{S+1}{(S+1) + (\frac{3c-1}{1-c})}$, then there exists $N^* < \infty$ such that $\mu_0^{N^*} < \frac{S+1}{(S+1) + (\frac{3c-1}{1-c})}$. Therefore, by a previous calculation we have that

$$\tilde{K}_{2S-1}^* = \arg \min_{N^*} \left\{ \mu_0^{N^*} < \frac{S+1}{(S+1) + (\frac{3c-1}{1-c})} \right\} < \infty,$$

which is the lowest level of player 1 who would take at no later than stage $2S-1$. ■

Proof of Proposition 9

First, an immediate implication of Lemma 2 is that for all level $k \geq 1$, the optimal choice for level- $(k+1)$ is either the same as level- k or to take at one stage earlier. Given this observation, the logic of the proof is similar to Proposition 7.

Only if: For any $1 \leq j \leq 2S - 2$, suppose $p_0 \left(\frac{S}{S+1} - \frac{2\lfloor \frac{j}{2} \rfloor c}{(S+1)(1+c)} \right) + \sum_{\kappa=1}^{\tilde{K}_{j+1}^*-1} p_\kappa \geq \frac{1-c}{1+c}$, then we want to show $\tilde{K}_j^* = \infty$. Without loss of generality, we consider an odd j . If $\tilde{K}_{j+1}^* = \infty$, then the statement holds immediately. Otherwise, we can prove $a_1^k > \frac{j+1}{2}$ for all $k \geq 1$ by induction. By construction, we know $a_2^m > \frac{j+1}{2}$ for all $1 \leq m \leq \tilde{K}_{j+1}^* - 1$ and $a_1^k > \frac{j+1}{2}$ for all $1 \leq k \leq \tilde{K}_{j+1}^*$ by Lemma 2. Suppose there is some $K \geq \tilde{K}_{j+1}^* + 1$ such that $a_1^k > \frac{j+1}{2}$ for all $1 \leq k \leq K$. We want to show this holds for level $K+1$ player 1. Level $K+1$ player 1 would choose $\frac{j+1}{2} + 1$ if and only if

$$p_0 \left[\frac{1}{S+1} (1 - (2S - j + 2)c) \right] + \left(\sum_{\kappa=1}^{\tilde{K}_{j+1}^*-1} p_\kappa \right) (-2c) + \left(\sum_{\kappa=\tilde{K}_{j+1}^*}^K p_\kappa \right) (1-c) \leq 0.$$

Moreover, we can observe that this condition is implied by:

$$\begin{aligned} & p_0 \left[\frac{1}{S+1} (1 - (2S - j + 2)c) \right] + \left(\sum_{\kappa=1}^{\tilde{K}_{j+1}^*-1} p_\kappa \right) (-2c) + \left(1 - p_0 - \sum_{\kappa=1}^{\tilde{K}_{j+1}^*-1} p_\kappa \right) (1-c) \leq 0 \\ \iff & p_0 \left[\frac{S}{S+1} - \frac{(j-1)c}{(S+1)(1+c)} \right] + \sum_{\kappa=1}^{\tilde{K}_{j+1}^*-1} p_\kappa \geq \frac{1-c}{1+c}. \end{aligned}$$

By our assumption, we can conclude that the optimal choice for level $(K+1)$ player 1 is $\frac{j+1}{2} + 1$,¹⁵ which completes the only if part of the proof.

If: For any $1 \leq j \leq 2S - 2$, suppose

$$p_0 \left(\frac{S}{S+1} - \frac{2\lfloor \frac{j}{2} \rfloor c}{(S+1)(1+c)} \right) + \sum_{\kappa=1}^{\tilde{K}_{j+1}^*-1} p_\kappa < \frac{1-c}{1+c},$$

then there exists N^* where $\tilde{K}_{j+1}^* + 1 \leq N^* < \infty$ such that

$$\mu_0^{N^*} \left(\frac{S}{S+1} - \frac{2\lfloor \frac{j}{2} \rfloor c}{(S+1)(1+c)} \right) + \frac{\sum_{\kappa=1}^{\tilde{K}_{j+1}^*-1} p_\kappa}{\sum_{\kappa=0}^{N^*-1} p_\kappa} < \frac{1-c}{1+c}.$$

Therefore, by previous calculation and the existence of such $N^* < \infty$, we can obtain that

$$\tilde{K}_j^* = \arg \min_{N^*} \left\{ \mu_0^{N^*} \left(\frac{S}{S+1} - \frac{2\lfloor \frac{j}{2} \rfloor c}{(S+1)(1+c)} \right) + \frac{\sum_{\kappa=1}^{\tilde{K}_{j+1}^*-1} p_\kappa}{\sum_{\kappa=0}^{N^*-1} p_\kappa} < \frac{1-c}{1+c} \right\} < \infty,$$

which is the lowest level of player who would take at no later than stage j . ■

¹⁵If j is even, then by the same argument, we can obtain level $(K+1)$ player 2 would choose $\frac{j}{2} + 1$ as

$$p_0 \left(\frac{S}{S+1} - \frac{jc}{(S+1)(1+c)} \right) + \sum_{l=1}^{\tilde{K}_{j+1}^*-1} p_l \geq \frac{1-c}{1+c}.$$

Proof of Theorem 1

Step 1: By Lemma 1 and Lemma 2, we can obtain that $1 = K_{2S}^* \leq \tilde{K}_{2S}^* = 1$, suggesting that the inequality holds at stage $2S$.

Step 2: By Proposition 6 and 8, we know K_{2S-1}^* and \tilde{K}_{2S-1}^* are the lowest levels such that

$$\begin{aligned} \frac{p_0}{\sum_{\kappa=0}^{K_{2S-1}^*-1} p_\kappa} &< \frac{2^S}{2^S + \left(\frac{-1+3c}{1-c}\right)}, \text{ and} \\ \frac{p_0}{\sum_{\kappa=0}^{\tilde{K}_{2S-1}^*-1} p_\kappa} &< \frac{S+1}{(S+1) + \left(\frac{-1+3c}{1-c}\right)}, \text{ respectively.} \end{aligned}$$

We can observe that $\frac{S+1}{(S+1) + \left(\frac{-1+3c}{1-c}\right)} < \frac{2^S}{2^S + \left(\frac{-1+3c}{1-c}\right)}$, suggesting the inequality for the dynamic model is less stringent. Hence, we can obtain that $K_{2S-1}^* \leq \tilde{K}_{2S-1}^*$.

Step 3: We can finish the proof by induction on the stages. At stage $2S-2$, as we rearrange the condition from Proposition 7, we can obtain K_{2S-2}^* is the lowest level such that

$$\sum_{\kappa=1}^{K_{2S-2}^*-1} p_\kappa > p_0 \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right) + \left(\sum_{\kappa=1}^{K_{2S-1}^*-1} p_\kappa\right) \left(\frac{1+c}{1-c}\right). \quad (4)$$

Similarly, as we rearrange the necessary and sufficient condition from Proposition 9, we can find that \tilde{K}_{2S-2}^* is the lowest level such that

$$\sum_{\kappa=1}^{\tilde{K}_{2S-2}^*-1} p_\kappa > p_0 \left(\frac{1}{S+1}\right) \left(\frac{-1+3c}{1-c}\right) + \left(\sum_{\kappa=1}^{\tilde{K}_{2S-1}^*-1} p_\kappa\right) \left(\frac{1+c}{1-c}\right). \quad (5)$$

It suffices to prove $K_{2S-2}^* \leq \tilde{K}_{2S-2}^*$ by showing the right-hand side of Condition (4) is smaller than the right-hand side of (5). This holds because $\left(\frac{1}{2}\right)^S < \frac{1}{S+1}$ for all $S \geq 2$ and $K_{2S-1}^* \leq \tilde{K}_{2S-1}^*$ as we have shown in *step 2*.

Step 4: Consider any j where $3 \leq j \leq 2S-1$ and suppose $K_{2S-i}^* \leq \tilde{K}_{2S-i}^*$ for all $0 \leq i \leq j-1$. We want to show $K_{2S-j}^* \leq \tilde{K}_{2S-j}^*$. Without loss of generality, we consider an odd j . That is, player 1 owns stage $2S-j$. By Proposition 7, we know K_{2S-j}^* is the lowest level such that

$$\sum_{\kappa=1}^{K_{2S-j}^*-1} p_\kappa > p_0 \left(\frac{1}{2}\right)^{S-\frac{j+1}{2}+1} \left(\frac{-1+3c}{1-c}\right) + \left(\sum_{\kappa=1}^{K_{2S-j+1}^*-1} p_\kappa\right) \left(\frac{1+c}{1-c}\right). \quad (6)$$

Similarly, as we rearrange the necessary and sufficient condition from Proposition 9, we can obtain that \tilde{K}_{2S-j}^* is the lowest level such that

$$\sum_{\kappa=1}^{\tilde{K}_{2S-j}^*-1} p_\kappa > p_0 \left(\frac{1}{S+1}\right) \left[\frac{-1+(j+2)c}{1-c}\right] + \left(\sum_{\kappa=1}^{\tilde{K}_{2S-j+1}^*-1} p_\kappa\right) \left(\frac{1+c}{1-c}\right). \quad (7)$$

Similar to the previous step, we can finish the proof by showing the right-hand side of Condition (6) is smaller than the right-hand side of (7). The induction hypothesis implies the second term of (7) is larger than the second term of (6). Hence, the only thing left to show is

$$\left(\frac{1}{2}\right)^{S-\frac{j+1}{2}+1} \left(\frac{-1+3c}{1-c}\right) < \left(\frac{1}{S+1}\right) \left[\frac{-1+(j+2)c}{1-c}\right].$$

Or equivalently,

$$(S+1)(-1+3c) < 2^{S-\frac{j+1}{2}+1}(-1+(j+2)c). \quad (8)$$

Since $3 \leq j \leq 2S-1$, there is nothing to show if $S < \frac{j+1}{2}$. When $S \geq \frac{j+1}{2}$, we know (8) would hold in the following three different cases.

- *Case 1:* If $S+1 = 2^{S-\frac{j+1}{2}+1}$, then (8) becomes $-1+3c < -1+(j+2)c \iff j > 1$.
- *Case 2:* If $S+1 < 2^{S-\frac{j+1}{2}+1}$, then (8) is equivalent to

$$2^{S-\frac{j+1}{2}+1} - (S+1) < \left[(j+2)2^{S-\frac{j+1}{2}+1} - 3(S+1)\right]c \iff 1 < \left[3 + \underbrace{\frac{(j-1)2^{S-\frac{j+1}{2}+1}}{2^{S-\frac{j+1}{2}+1} - (S+1)}}_{>0 \text{ as } j \geq 3}\right]c,$$

which holds under our assumption $c > \frac{1}{3}$.

- *Case 3:* If $S+1 > 2^{S-\frac{j+1}{2}+1}$, then (8) can be rearranged as

$$(S+1) - 2^{S-\frac{j+1}{2}+1} > \left[3(S+1) - (j+2)2^{S-\frac{j+1}{2}+1}\right]c \iff 1 > \left[3 - \frac{j-1}{\left(\frac{S+1}{2^{S-\frac{j+1}{2}+1}}\right) - 1}\right]c.$$

The right-hand side of the inequality is negative since

$$3 - \frac{j-1}{\left(\frac{S+1}{2^{S-\frac{j+1}{2}+1}}\right) - 1} \leq 3 - \frac{j-1}{\left(\frac{\frac{j+1}{2}+1}{2}\right) - 1} = -1.$$

This completes the proof. ■

Proof of Proposition 10

By Proposition 6, we know

$$K_{2S-1}^* < \infty \iff p_0 < \frac{2^S}{2^S \left(\frac{-1+3c}{1-c}\right)}.$$

As the prior distribution follows Poisson(λ), the condition becomes

$$\begin{aligned} K_{2S-1}^*(\lambda) < \infty &\iff e^{-\lambda} < \frac{2^S}{2^S \left(\frac{-1+3c}{1-c}\right)} \\ &\iff \lambda > \ln \left[1 + \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right) \right]. \end{aligned}$$

Similarly, by Proposition 8, we know

$$\tilde{K}_{2S-1}^* < \infty \iff p_0 < \frac{S+1}{(S+1) \left(\frac{-1+3c}{1-c}\right)},$$

which can be rearranged to the following expression when the prior distribution follows Poisson(λ):

$$\begin{aligned} \tilde{K}_{2S-1}^*(\lambda) < \infty &\iff e^{-\lambda} < \frac{S+1}{(S+1) \left(\frac{-1+3c}{1-c}\right)} \\ &\iff \lambda > \ln \left[1 + \left(\frac{1}{S+1}\right) \left(\frac{-1+3c}{1-c}\right) \right]. \end{aligned}$$

This completes the proof. ■

Proof of Proposition 11

Here we show the existence of λ^* . The existence of $\tilde{\lambda}^*$ can be proven by the same argument.

Step 1: By Proposition 1, we know for all $\lambda > 0$, $K_{2S}^*(\lambda) = 1$.

Step 2: By Proposition 10, we know

$$K_{2S-1}^*(\lambda) < \infty \iff \lambda > \ln \left[1 + \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right) \right] \equiv \lambda_{2S-1}^*.$$

Since $\lambda_{2S-1}^* < \infty$, we know $K_{2S-1}^*(\lambda) < \infty \iff \lambda > \lambda_{2S-1}^*$.

Step 3: By Proposition 7, we know

$$\begin{aligned} K_{2S-2}^*(\lambda) < \infty &\iff \frac{e^{-\lambda} \left(\frac{1}{2}\right)^S + e^{-\lambda} \sum_{l=1}^{K_{2S-1}^*(\lambda)-1} \frac{\lambda^l}{l!}}{e^{-\lambda} \left(\frac{1}{2}\right)^{S-1} + (1-e^{-\lambda})} < \frac{1-c}{1+c} \\ &\iff 1 - e^{-\lambda} \left[1 + \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right) \right] - e^{-\lambda} \sum_{\kappa=1}^{K_{2S-1}^*(\lambda)-1} \frac{\lambda^\kappa}{\kappa!} \left(\frac{1+c}{1-c}\right) > 0. \end{aligned}$$

Notice that by *step 2*, we know there exists some $M < \infty$, such that for all $\lambda > \lambda_{2S-1}^*$, $K_{2S-1}^*(\lambda) < M$. Moreover, by Proposition 4, we know $K_{2S-1}^*(\lambda) \geq 2$. Hence,

$$0 = \lim_{\lambda \rightarrow \infty} \frac{\lambda}{e^\lambda} \leq \lim_{\lambda \rightarrow \infty} e^{-\lambda} \sum_{\kappa=1}^{K_{2S-1}^*(\lambda)-1} \frac{\lambda^\kappa}{\kappa!} \leq \lim_{\lambda \rightarrow \infty} e^{-\lambda} \sum_{\kappa=1}^{M-1} \frac{\lambda^\kappa}{\kappa!} = 0.$$

Coupled with the fact that $\lim_{\lambda \rightarrow \infty} e^{-\lambda} = 0$, we can conclude that there exists λ_{2S-2}^* such that $\lambda_{2S-1}^* < \lambda_{2S-2}^* < \infty$ and $K_{2S-2}^*(\lambda) < \infty \iff \lambda > \lambda_{2S-2}^*$.

Step 4: Now we can prove this statement by induction on each stage. Consider any j where $3 \leq j \leq 2S - 1$ and suppose there exists $\lambda_{2S-j+1}^* < \infty$ such that $K_{2S-j+1}^*(\lambda) < \infty$ for all $\lambda > \lambda_{2S-j+1}^*$. By Proposition 7, we know

$$\begin{aligned} K_{2S-j}^*(\lambda) < \infty &\iff \frac{e^{-\lambda} \left(\frac{1}{2}\right)^{\lfloor \frac{2S-j}{2} \rfloor + 1} + e^{-\lambda} \sum_{\kappa=1}^{K_{2S-j+1}^*(\lambda)-1} \frac{\lambda^\kappa}{\kappa!}}{e^{-\lambda} \left(\frac{1}{2}\right)^{\lfloor \frac{2S-j}{2} \rfloor} + (1 - e^{-\lambda})} < \frac{1-c}{1+c} \\ &\iff 1 - e^{-\lambda} \left[1 + \left(\frac{1}{2}\right)^{\lfloor \frac{2S-j}{2} \rfloor + 1} \left(\frac{-1+3c}{1-c}\right) \right] - e^{-\lambda} \sum_{\kappa=1}^{K_{2S-j+1}^*(\lambda)-1} \frac{\lambda^\kappa}{\kappa!} \left(\frac{1+c}{1-c}\right) > 0. \end{aligned}$$

By the induction hypothesis, we know there exists some $L < \infty$ such that for all $\lambda > \lambda_{2S-j+1}^*$, $K_{2S-j+1}^*(\lambda) < L$. Proposition 4 gives us $K_{2S-j+1}^*(\lambda) \geq j$, and hence,

$$0 = \lim_{\lambda \rightarrow \infty} e^{-\lambda} \left(\sum_{\kappa=1}^{j-1} \frac{\lambda^\kappa}{\kappa!} \right) \leq \lim_{\lambda \rightarrow \infty} e^{-\lambda} \sum_{\kappa=1}^{K_{2S-j+1}^*(\lambda)-1} \frac{\lambda^\kappa}{\kappa!} \leq \lim_{\lambda \rightarrow \infty} e^{-\lambda} \sum_{\kappa=1}^{L-1} \frac{\lambda^\kappa}{\kappa!} = 0.$$

Combined with the fact that $\lim_{\lambda \rightarrow \infty} e^{-\lambda} = 0$, we have proved that there exists λ_{2S-j}^* such that $\lambda_{2S-j+1}^* < \lambda_{2S-j}^* < \infty$ and $K_{2S-j}^*(\lambda) < \infty \iff \lambda > \lambda_{2S-j}^*$. Thus, λ_1^* is the desired λ^* . ■

Proof of Proposition 12

The proofs of Propositions 6 through 9 provide a recipe to derive the necessary and sufficient conditions for complete unraveling at each stage. That is, when the prior distribution follows Poisson distribution, we can compute the minimum λ for both models such that the predictions coincide with the standard level- k model. In the dynamic model, we can obtain from Proposition 6 and 7 that for any stage $2S - j$ where $1 \leq j \leq 2S - 1$,

$$\begin{aligned} K_{2S-1}^*(\lambda) = 2 &\iff \frac{e^{-\lambda}}{e^{-\lambda} + \lambda e^{-\lambda}} < \frac{2^S}{2^S + \left(\frac{-1+3c}{1-c}\right)} \iff \lambda > \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right) \equiv \lambda_{2S-1}^{**}, \text{ and} \\ K_{2S-j}^*(\lambda) = j + 1 &\iff \sum_{\kappa=1}^j \frac{\lambda^\kappa e^{-\lambda}}{\kappa!} > e^{-\lambda} \left(\frac{1}{2}\right)^{S - \lfloor \frac{j+1}{2} \rfloor + 1} \left(\frac{-1+3c}{1-c}\right) + \left(\sum_{\kappa=1}^{j-1} \frac{\lambda^\kappa e^{-\lambda}}{\kappa!}\right) \left(\frac{1+c}{1-c}\right) \\ &\iff \frac{1}{j!} \lambda^j - \left(\frac{2c}{1-c}\right) \left(\sum_{\kappa=1}^{j-1} \frac{\lambda^\kappa}{\kappa!}\right) > \left(\frac{1}{2}\right)^{S - \lfloor \frac{j+1}{2} \rfloor + 1} \left(\frac{-1+3c}{1-c}\right) \equiv M_{2S-j}^{**}. \end{aligned}$$

Similarly, we know from Proposition 8 and Proposition 9 that for any stage $2S - j$ where

$$1 \leq j \leq 2S - 1,$$

$$\begin{aligned} \tilde{K}_{2S-1}^*(\lambda) = 2 &\iff \frac{e^{-\lambda}}{e^{-\lambda} + \lambda e^{-\lambda}} < \frac{S+1}{(S+1) + \left(\frac{-1+3c}{1-c}\right)} \iff \lambda > \frac{-1+3c}{(S+1)(1-c)} \equiv \tilde{\lambda}_{2S-1}^{**}, \text{ and} \\ \tilde{K}_{2S-j}^*(\lambda) = j+1 &\iff \sum_{\kappa=1}^j \frac{\lambda^\kappa e^{-\lambda}}{\kappa!} > e^{-\lambda} \left(\frac{-1 + (2\lfloor \frac{j+1}{2} \rfloor + 1)c}{(S+1)(1-c)} \right) + \left(\sum_{\kappa=1}^{j-1} \frac{\lambda^\kappa e^{-\lambda}}{\kappa!} \right) \left(\frac{1+c}{1-c} \right) \\ &\iff \frac{1}{j!} \lambda^j - \left(\frac{2c}{1-c} \right) \left(\sum_{\kappa=1}^{j-1} \frac{\lambda^\kappa}{\kappa!} \right) > \left(\frac{-1 + (2\lfloor \frac{j+1}{2} \rfloor + 1)c}{(S+1)(1-c)} \right) \equiv \tilde{M}_{2S-j}^{**}. \end{aligned}$$

First, we can find $\lambda_{2S-1}^{**} < \tilde{\lambda}_{2S-1}^{**}$ since $(\frac{1}{2})^S < \frac{1}{S+1}$. Moreover, because the LHS of each inequality is a degree of j polynomial of λ , it has only one positive root by Descartes' rule of signs. Hence, it suffices to prove $M_{2S-j}^{**} < \tilde{M}_{2S-j}^{**}$, or equivalently, $\frac{\tilde{M}_{2S-j}^{**}}{M_{2S-j}^{**}} > 1$, for all $2 \leq j \leq 2S - 1$. Due to the property of floor functions, we can focus on odd j without loss of generality. Also, we can observe that this ratio is decreasing in j since for any odd j where $3 \leq j \leq 2S - 3$,

$$\begin{aligned} \frac{\tilde{M}_{2S-(j+2)}^{**}}{M_{2S-(j+2)}^{**}} &= \left(\frac{2^{S-\frac{j+1}{2}}}{S+1} \right) \left(\frac{-1 + (j+4)c}{-1 + 3c} \right) \\ &= \frac{1}{2} \left(\frac{\tilde{M}_{2S-j}^{**}}{M_{2S-j}^{**}} \right) + \frac{1}{2} \left[\left(\frac{2^{S-\frac{j-1}{2}}}{S+1} \right) \left(\frac{2c}{-1 + 3c} \right) \right] < \frac{\tilde{M}_{2S-j}^{**}}{M_{2S-j}^{**}} \\ &\iff 2c < -1 + (j+2)c \iff 1 < jc, \end{aligned}$$

which holds because of the assumption $c > \frac{1}{3}$. The monotonicity implies that the ratio is minimized when $j = 2S - 1$, and we can obtain the conclusion by showing $\frac{\tilde{M}_1^{**}}{M_1^{**}} > 1$:

$$\frac{\tilde{M}_1^{**}}{M_1^{**}} = \left(\frac{2}{S+1} \right) \left(\frac{-1 + (2S+1)c}{-1 + 3c} \right) > 1 \iff (S-1)(1+c) > 0,$$

as desired. ■

Proof of Proposition 13

Here we only provide the proof for the dynamic model. A very similar argument can be applied to the static model. First of all, by Proposition 1, we know $K_{2S}^*(\lambda) = 1$ for all $\lambda > 0$. Therefore, it is weakly decreasing in λ .

To show the monotonicity of $K_{2S-1}^*(\lambda)$, we need to introduce the function $F_k(\lambda) : \mathbb{R}_{++} \rightarrow \mathbb{R}$ where $k \in \mathbb{N}$ and $F_k(\lambda) = \sum_{\kappa=1}^k \frac{\lambda^\kappa}{\kappa!}$. Notice that $F_{k+1}(\lambda) > F_k(\lambda)$ for all $\lambda > 0$, and $F_k(\lambda)$ is strictly increasing since $F_k'(\lambda) = \sum_{\kappa=0}^{k-1} \frac{\lambda^\kappa}{\kappa!} > 0$ for all $\lambda > 0$. We prove the monotonicity toward contradiction. By Proposition 6, we know $K_{2S-1}^*(\lambda)$ is the lowest level such that

$$F_{K_{2S-1}^*(\lambda)-1}(\lambda) > \left(\frac{1}{2} \right)^S \left(\frac{-1 + 3c}{1-c} \right).$$

If $K_{2S-1}^*(\lambda)$ is not weakly decreasing in λ , then there exists $\lambda' > \lambda$ such that $K_{2S-1}^*(\lambda') > K_{2S-1}^*(\lambda)$. By the construction and the monotonicity of $F_k(\lambda)$, we can find that

$$F_{K_{2S-1}^*(\lambda)-1}(\lambda') > F_{K_{2S-1}^*(\lambda)-1}(\lambda) > \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right).$$

Also, $K_{2S-1}^*(\lambda')$ is the lowest level such that

$$F_{K_{2S-1}^*(\lambda')-1}(\lambda') > \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right),$$

implying that

$$F_{K_{2S-1}^*(\lambda)-1}(\lambda') > F_{K_{2S-1}^*(\lambda')-1}(\lambda') > \left(\frac{1}{2}\right)^S \left(\frac{-1+3c}{1-c}\right).$$

This contradicts the assumption that $K_{2S-1}^*(\lambda') > K_{2S-1}^*(\lambda)$. ■

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