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CONTRACTS VS. SALARIES IN MATCHING

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Abstract

Firms and workers may sign complex contracts that govern many aspects of their interactions. I show that when firms regard contracts as substitutes, bargaining over contracts can be understood as bargaining only over wages. Substitutes is the assumption commonly used to guarantee the existence of stable matchings of workers and firms.

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Contracts vs. Salaries in Matching*

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Workers and firms may bargain over general, multi-dimensional contracts; they may negotiate over health benefits, housing, retirement plans, etc. In this note I show that, when firms regard contracts as substitutes, bargaining over contracts can be embedded into a model of bargaining over wages. Substitutes, on the other hand, is the assumption commonly used to guarantee the existence of stable matchings of workers and firms.

The economics of the embedding are straightforward, except for a small twist. When a firm and a worker negotiate over a contract, they may bargain over many dimensions. However, the Pareto frontier of contracts is, in a sense, "one-dimensional:" what is better for the worker is worse for the firm. So Pareto optimal contracts may be viewed as salaries, with the better contracts for the firm meaning lower salaries, and the better contracts for the worker meaning higher salaries. The twist is that the firm's ranking over contracts might be affected by the firm's other hires. For example, health plan A may be better than B for a firm if it has many employees, but B beats A if it has few. When contracts are substitutes, it turns out that the ranking is not affected in this way.

Hatfield and Milgrom (2005) present a model of two-sided worker-firm matching with contracts. A firm will hire a collection of workers, and will negotiate a contract with each one of them. The model is a generalization of Kelso and Crawford (1982), where each firm and each worker negotiate a wage.[1] Kelso and Crawford show that, when firms' demands satisfy gross substitutes, the core of the matching market is nonempty. Hatfield and Milgrom show that, when the firms' preferences over contracts satisfy their notion of substitutes, the core of the market is nonempty.

I show that Hatfield and Milgrom's model can be embedded into the model of Kelso and Crawford. Hatfield and Milgrom's assumption of substitutability enables an embedding where firms' demands for workers satisfy Kelso and Crawford's notion of gross substitutes. As a result, the nonemptiness of the core follows from the argument in Kelso and Crawford, and their salary adjustment algorithm finds a stable matching of workers to firms, and a vector of supporting salaries.

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[1]Kelso and Crawford build on the analysis of Crawford and Knoer (1981); see Roth and Sotomayor (1990) for a description of the models.

Hatfield and Milgrom's paper is an elegant analysis of two-sided matching. It contributes much more than showing the nonemptiness of the core when firms and workers can sign general contracts, and my observation does not diminish their contribution in the least. I believe, however, that there is value in clarifying the relationship between contracts and salaries.

One step in that direction is taken by Hatfield and Kojima (2010), who investigate conditions on preferences over contracts that are weaker than substitutes and still generate stable matchings. My embedding does not work under Hatfield and Kojima's weaker conditions (see 1.3.3 below). Future research should explain the consequences of the added generality of contracts over salaries in different economic environments.

1 Embedding

1.1 Definitions

I shall describe two models. The model of a matching market with contracts with substitutable choices is due to Hatfield and Milgrom (2005). The model of a matching market with salaries and gross-substitutes in demand is due to Kelso and Crawford (1982).

1.1.1 Contracts

A ***matching market with contracts*** is described by:

- (finite, disjoint) sets W of workers, F of firms and X of contracts; each contract $x \in X$ is assigned one worker $x_W \in W$ and one firm $x_F \in F$;
- for each worker $w \in W$, a utility function $u_w : X \cup \{\emptyset\} \to \mathbf{R}$; and for each firm $f \in F$, a utility function $u_f : 2^X \to \mathbf{R}$; all utility functions are one-to-one (preferences are strict).

A firm f's utility function determines a choice rule C_f: for $A \subseteq X$, $C_f(A)$ is the maximal subset of A according to u_f. Note that since u_f is one-to-one, $C_f(A)$ is uniquely defined. The empty set $\emptyset$ represents for f the option of hiring no workers. For notational convenience, I have not restricted the domain of u_f to contracts with $f = x_F$, but of course we want f to sign contracts only in its own name; assume then that $x \in C_f(A)$ implies $f = x_F$. Assume also that $x, x' \in C_f(A)$ implies $x_W \neq x'_W$.

For a worker w, $\emptyset$ represents an outside option: a contract that is always available to her if she chooses to reject the contract some firm offers her. Suppose that if $x_W \neq w$ then $u_w(x) < u_w(\emptyset)$.

A set of contracts A is ***feasible*** if, for all workers w, there is at most one $x \in A$ with $w = x_W$.

A firm f's utility satisfies ***substitutability*** if, for any set of contracts A, and any two contracts x and x', $x \notin C_f(A \cup \{x\})$ implies $x \notin C_f(A \cup \{x, x'\})$.

The tuple $(F, W, X, (u_f), (u_w))$ describes a a matching market with contracts.

A set of contracts $A \subseteq X$ is ***individually rational*** if, for all $x \in A$, $u_{x_W}(x) \geq u_{x_W}(\emptyset)$; and for all firms f, $C_f(A) = \{x \in A : f = x_F\}$.

A set of contracts $A \subseteq X$ is ***stable*** if it is individually rational and if for any firm f and set of contracts $A' \neq A$ with $A' = C_f(A \cup A')$, there is one contract $x' \in A'$ such that either $u_{x'_W}(x') < u_{x'_W}(\emptyset)$ or there is $x \in A$ with $x_W = x'_W$ and $u_{x'_W}(x') < u_{x'_W}(x)$.

1.1.2 Salaries

A ***matching market with salaries*** is described by:

- (finite, disjoint) sets W of workers, F of firms and $S \subseteq \mathbf{R}_+$ of salaries;
- for each worker $w \in W$, a utility function $v_w : F \cup \{\emptyset\} \times S \to \mathbf{R}$; and for each firm $f \in F$, a utility function $v_f : \cup_{A \subseteq W} A \times S \to \mathbf{R}$; all utility functions are one-to-one (preferences are strict).

We can suppose that S is the set $\{0, 1, \ldots L\}$ of the first $L+1$ non-negative integers, for some L.

For a firm f, u_f defines a demand function $D_f : S^W \to 2^W$ by

$$D_f(\mathbf{s}) = \text{argmax}_{A \subseteq W} v_f(\{(w, s_w) : w \in A\}).$$

Say that D satisfies ***gross substitutes*** if, for any two vectors of salaries, $\mathbf{s}$ and $\mathbf{s}'$, if $\mathbf{s} \leq \mathbf{s}'$ and $s_w = s'_w$ then $w \in D(\mathbf{s})$ implies that $w \in D(\mathbf{s}')$.

A ***matching*** is a function $\mu : W \to F \times S$. A matching assigns to each worker a firm and a salary; I use the $\mu(w) = (\emptyset, 0)$ notation for when w is unmatched (unemployed). A matching specifies, for each firm f a collection of workers with their corresponding salaries: $\mu^0(f) = \{(w, s) : (f, s) = \mu(w)\}$. The set $\mu^0(f)$ is thus the set of workers employed by f, and their salaries, in the matching μ.

The tuple $(F, W, S, (v_f), (v_w))$ describes an matching market with salaries.

A matching μ is ***individually rational*** if, for every f and w, $v_f(\mu^0(f)) \geq v_f(B)$ for all $B \subseteq \mu^0(f)$ and $u_w(\mu(w)) \geq u_w(\emptyset, 0)$.

A matching μ is ***stable*** if it is individually rational and if, for any firm f and $A \subseteq W$, if there is a vector of salaries $(s_w)_{w \in A}$ with $v_f(\{(w, s_w) : w \in A\}) > v_f(\mu^0(f))$ then there is $w \in A$ with $u_w(\mu(w)) > u_w(f, s_w)$.

1.2 Embedding

Let $(F, W, X, (u_f), (u_w))$ be a matching market with contracts, and $(F, W, S, (v_f), (v_w))$ a matching market with salaries. An ***embedding*** of $(F, W, X, (u_f), (u_w))$ into $(F, W, S, (v_f), (v_w))$ is a one-to-one function g which maps each $x \in X$ into a triple $(x_F, x_W, s) \in F \times W \times S$.

Let g be such an embedding and $A \subseteq X$. Say that $g(A)$ ***defines a matching*** if for any w there is at most one s and f with $(f, w, s) \in g(A)$. The matching defined by A under g is the function $\mu : W \to F \times S$ defined by $\mu(w) = (f, s)$ if $g^{-1}(f, w, s) \in A$ and $\mu(w) = (\emptyset, 0)$ otherwise.

Let $(F, W, X, (u_f), (u_w))$ be a matching market with contracts.

Theorem 1. *If firms' choices satisfy substitutability, then there is a matching market with salaries $(F, W, S, (v_f), (v_w))$, and an embedding g of $(F, W, X, (u_f), (u_w))$ into $(F, W, S, (v_f), (v_w))$ such that*

1. *firms' demand functions in $(F, W, S, (v_f), (v_w))$ satisfy gross substitutes;*
2. *$A \subseteq X$ is a set of stable contracts if and only if $g(A)$ defines a stable matching.*

Proof. Say that a contract x is ***dominated*** for x_F and x_W if there is a contract x' with $x_F = x'_F$, $x_W = x'_W$, $u_{x_F}(x') > u_{x_F}(x)$ and $u_{x_W}(x') > u_{x_W}(x)$. Let X_{fw} be the set of all contracts x with $x_F = f$ and $x_W = w$ that are not dominated for f and w. Note that X_{fw} can be ordered by u_w; that is I can enumerate the elements of X_{fw} as $x_1, \ldots, x_{|X_{fw}|}$ with $u_w(x_i) < u_w(x_{i+1})$. Then I can write $X_{fw} = \{(w, s) : s = 1, \ldots, |X_{fw}|\}$ with the understanding that (w, s) correspond to offering worker w the contract x_s in X_{fw}. Note that $s < s'$ if and only if $u_f(\{w, s\}) > u_f(\{w, s'\})$: by definition, if $s < s'$ then $u_w(f, s) < u_w(f, s')$ so $u_f(\{w, s\}) < u_f(\{w, s'\})$ would imply that (w, s) is dominated.

Let $K = \max\{|X_{fw}| : f \in F, w \in W\}$ and $S = \{1, 2, \ldots, K + 1\}$. For convenience, we augment the contracts in X_{fw} to include (w, s) with $|X_{fw}| < s \leq K + 1$ and assume that $u_w(w, s) < u_w(\emptyset)$ and $(w, s) \notin C_f(A)$ for any A if $s > |X_{fw}|$. The embedding g is the mapping that takes $x \in X$ into (f, w, s) with (w, s) being the representation of x in X_{fw} if x is not dominated, and into $(x_F, x_F, K + 1)$ if it is.[2]

Define firms' and workers' utilities as follows. Let $v_w(f, s) = u_w(x)$, where $x \in X_{fw}$ corresponds to s. Let $v_f((w_1, s_1), \ldots, (w_n, s_n)) = u_f((w_1, s_1), \ldots, (w_n, s_n))$. For a vector of wages $\mathbf{s} = (s_w)_{w \in W} \in S^W$, let $X_f^{\mathbf{s}} \subseteq X_f$ be the set of contracts (w, s_w). Define a demand function D_f for firm f by $D_f(\mathbf{s}) = C_f(X_f^{\mathbf{s}})$ (this is consistent with our definition of utility for f).

I shall now prove that D satisfies the GS property. Let $X_f^{\mathbf{s}+} = \{(w, s) \in W \times S : s \geq s_w\}$, where s_w denotes w's salary in $\mathbf{s}$. I first prove that $C_f(X_f^{\mathbf{s}}) = C_f(X_f^{\mathbf{s}+})$. Let $(w, s) \in$

[2]Strictly speaking, for dominated x we would need to set $g(x) = (x_F, x_F, K + l)$, choosing $l \geq 1$ so that g remains one-to-one.

$X_f^{\mathbf{s}+} \setminus X_f^{\mathbf{s}}$. There must exist some s', with $s' < s$ and $(w, s') \in X_f^{\mathbf{s}+}$. Note that $s' < s$ implies $(w, s) \notin C_f(\{(w, s), (w, s')\})$; so $\{(w, s), (w, s')\} \subseteq X_f^{\mathbf{s}+}$ implies, by the property of substitutability, that $(w, s) \notin C_f(X_f^{\mathbf{s}+})$. Thus I have shown that $C_f(X_f^{\mathbf{s}+}) \subseteq C_f(X_f^{\mathbf{s}})$. Since $X_f^{\mathbf{s}} \subseteq X_f^{\mathbf{s}+}$, the definition of C_f implies that $C_f(X_f^{\mathbf{s}+}) = C_f(X_f^{\mathbf{s}})$.

Now, let $\mathbf{s} = (s_w)$ and $\mathbf{s}' = (s'_w)$ be vectors with $\mathbf{s} \leq \mathbf{s}'$ while for $w_0 \in W$, $s_{w_0} = s'_{w_0}$ and $w_0 \in D(\mathbf{s})$. Suppose by way of contradiction that $w_0 \notin D(\mathbf{s}')$. Then $(w_0, s'_{w_0}) \notin C_f(X_f^{\mathbf{s}'}) = C_f(X_f^{\mathbf{s}'+})$. But then $X_f^{\mathbf{s}'+} \subseteq X_f^{\mathbf{s}+}$ so substitutability implies that $(w_0, s'_{w_0}) \notin C_f(X_f^{\mathbf{s}+})$. Now, $C_f(X_f^{\mathbf{s}+}) = C_f(X_f^{\mathbf{s}})$ and $(w_0, s_{w_0}) = (w_0, s'_{w_0})$ implies that $(w_0, s_{w_0}) \notin C_f(X_f^{\mathbf{s}})$, a contradiction.

The proof that A is stable in $(F, W, X, (u_f), (u_w))$ if and only if $g(A)$ is stable in $(F, W, S, (v_f), (v_w))$ is straightforward. $\square$

1.3 Discussion

1.3.1 Antecedents

The identification of contracts and salaries is not new. Kelso and Crawford (1982) mention how salaries can be interpreted as contracts. Roth (1984), who presents an early model of matching with contracts, also identifies contracts with salaries. The observation that substitutability is important for this identification to hold seems to be new. It is not mentioned in the literature that follows Hatfield and Milgrom (2005).

1.3.2 Quasilinearity

In Kelso and Crawford, firms' profits are quasilinear, but their existence proof is more general and does not depend on quasilinearity. One detail is that they require a salary that is high enough so no worker would be hired at that salary (see their Lemma 2). In the embedding in the theorem, we do have such a salary.

1.3.3 Bilateral substitutes

In a model of matching with contracts, Hatfield and Kojima (2010) present a generalization of substitutes, called bilateral substitutes. They show stable matching with contracts exists under bilateral substitutes. Here I show that the model of Hatfield-Kojima cannot be embedded into the Kelso-Crawford model.

The following example is from Hatfield and Kojima (2010). Let the set of firms be $\{f, f'\}$, the set of workers $\{w, w'\}$ and the set of contracts be $\{x, \tilde{x}, z, \tilde{z}, z'\}$. Let x and $\tilde{x}$ involve firm f and worker w, while z and $\tilde{z}$ involve worker w' and firm f. Contract z' is between w' and f'. Suppose that agents' utilities are such that their preferences are:

$$\begin{array}{rrrr} \succeq_f & \succeq_{f'} & \succeq_w & \succeq_{w'} \\ \hline \{x,z\} & \{z'\} & \tilde{x} & z \\ \{\tilde{z}\} & \emptyset & x & z' \\ \{\tilde{x}\} & & \emptyset & \tilde{z} \\ x & & & \emptyset \\ z & & & \\ \emptyset, & & & \end{array}$$

I have omitted the alternatives that are worse than $\emptyset$. Suppose that there is a embedding, where x maps to the salary s_x and $\tilde{x}$ to the salary $s_{\tilde{x}}$. Then $x \notin C_f(\{x, \tilde{x}\})$ means that $s_x > s_{\tilde{x}}$. This would imply that

$$u_f(x_W, z_W, s_x, s_z) < u_f(\tilde{x}_W, z_W, s_{\tilde{x}}, s_z),$$

as $x_W = \tilde{x}_W$, which is incompatible with the preferences above.

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