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TESTABLE IMPLICATIONS OF GROSS SUBSTITUTES IN DEMAND

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Abstract

We present a non-parametric "revealed-preference test" for gross substitutes in demand.

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Testable Implications of Gross Substitutes in Demand *

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1 Introduction

We study the testable implications of the property that a pair of goods are gross substitutes; strictly speaking, for the joint hypotheses of rationality and gross substitutes. We propose a non-parametric test for gross substitutes using expenditure data; the test is in the spirit of the revealed-preference tests first studied by Samuelson (1947) and Afriat (1967).

Two goods are gross substitutes if when the price of one good increases, demand for the other good increases. The usual way of testing for gross substitutes between goods a and b is by estimating the coefficient of the price of good a in a linear regression for the demand for b. We propose instead a non-parametric test; one that does not require assuming a functional form for the demand function or for the underlying preferences.

Our test is for a *pair* of goods: you can use it to test if coffee and tea are substitutes, for example. You cannot use it to test whether wine, beer, and whisky are substitutes. The parametric test we mentioned above is also a two-good exercise.

Consumers, of course, buy more than just two goods. But one can isolate a pair of goods, and test for gross substitutes, under some assumptions about the consumers'

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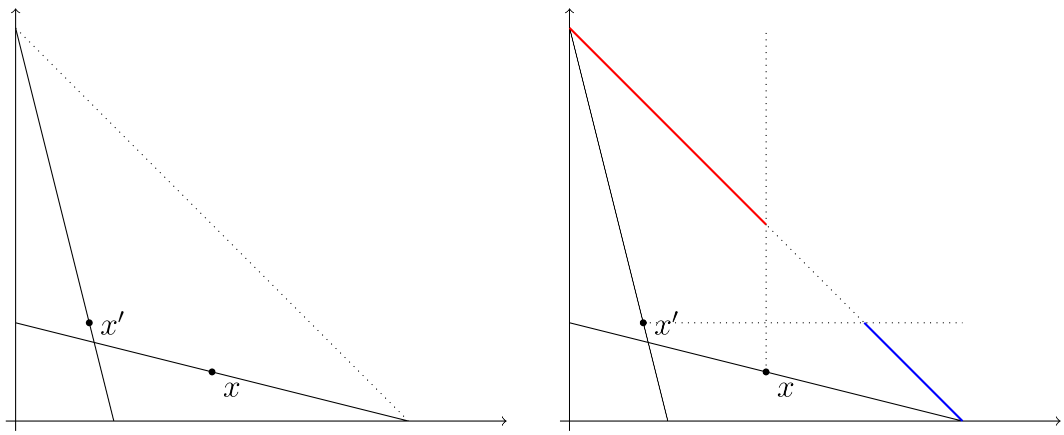
preferences. The assumptions are made routinely in applied studies of demand, see for example Deaton and Muellbauer (1980).[1] Applied researchers use aggregation and separability to study demand for a subset of broad categories of goods. They will, for example, aggregate different types of coffee into a composite coffee good. They assume that agents' preferences are separable, so that what matters for the purchases of coffee and tea is the money spent on other goods; not how many steaks, salads, or pizzas were bought. We stress that assumptions allowing for aggregation and separability are strong but well understood, and seem to be accepted by the community of researchers on applied demand.

Our test is simple. Given is a finite collection of observed demand choices at given prices. We want to reconcile the data with a demand function that satisfies gross substitutes and comes from a rational consumer. That is, we want to know when the data can be rationalized using a rational demand function with the substitutes property.

Consider the example in Figure 1. We have two observations: x is the bundle purchased at prices p, and x' is purchased at prices p'. These purchases do not violate gross substitutes. The observed choices are also consistent with the weak axiom of revealed preferences, so there is an extension of these purchases to a rational demand function that is defined for all prices. There is, however, no extension to a demand function which satisfies gross substitutes: Consider the prices p'' given by the dotted budget line. Gross substitutes and the choice of x at p requires a decrease in the consumption of the good whose price is the same in p and in p'', so demand at p'' should lie in the red segment of the budget line. On the other hand, gross substitutes and x' requires that demand at p'' lies in the blue segment of the budget line. Since the red and blue segments are disjoint, there is no demand function that extends the data and satisfies gross substitutes.

Importantly, the example shows that gross substitutes (and the weak axiom of revealed preference) may be satisfied in the data, but the data may not be rationalizable

[1] Deaton and Muellbauer (1980) explain how most consumption decisions involve in principle an unmanageably large number of goods and, simultaneously, an intertemporal and risk dimensions. They argue that all empirical studies of demand must simplify buy using aggregation and separability



(a) Two observations: (x, p) and (x', p'). (b) Implications of Gross Substitutes.

Figure 1: An example with two observations.

by a demand function satisfying gross substitutes. Our test is based on expenditure shares. We observe that gross substitutes of a demand function is equivalent to the monotonicity of expenditure shares: Demand satisfies gross substitutes if and only if the share of expenditure corresponding to good a increases as the price of good b increases. Our test is simply to verify that the data satisfies the monotonicity of expenditure shares. That is, if the shares in the data are monotonic then they can be extended to a full demand function defined for all prices, and the demand so defined will be smooth, rational, and induce monotonic expenditure shares.

The problem of gross substitutes is thus much simpler than the problem of testing for complements, which we have studied elsewhere (Chambers, Echenique, and Shmaya, 2008). We note that the current paper only deals with the testable implications of substitutes, not the preferences that generate substitutes. The class of preferences generating substitutes is known from the work of Fisher (1972).

The rest of the paper is organized as follows: Section 2 describes the notation and gives our main definitions; Section 3 presents our result; the proof is in Section 4. Finally, in Section 5 we present a discussion of our results.

2 Preliminaries

Let $\mathbb{R}^2_+$ be the domain of consumption bundles, and $\mathbb{R}^2_{++}$ the domain of possible prices. We use standard notational conventions: $x \leq y$ if $x_i \leq y_i$ in $\mathbb{R}$, for $i = 1, 2$; $x < y$ if $x \leq y$ and $x \neq y$; and $x \ll y$ if $x_i < y_i$ in $\mathbb{R}$, for $i = 1, 2$. We write $x \cdot y$ for the inner product $x_1 y_1 + x_2 y_2$.

A function $u : \mathbb{R}^2_+ \to \mathbb{R}$ is *monotone increasing* if $x \leq y$ implies $u(x) \leq u(y)$. It is *monotone decreasing* if $(-u)$ is monotone increasing. Let $A \subseteq \mathbb{R}^2$ be open. A function $u : A \to \mathbb{R}$ is *smooth* if its partial derivatives of all orders exist.

A function $D : \mathbb{R}^2_{++} \times \mathbb{R}_+ \to \mathbb{R}^2_+$ is a *demand function* if it is homogeneous of degree 0 and satisfies $p \cdot D(p, I) = I$, for all $p \in \mathbb{R}^2_{++}$ and $I \in \mathbb{R}_+$.

Say that a demand function satisfies *gross substitutes* if, for fixed p_1 and I, $p_2 \mapsto D_1((p_1, p_2), I)$ is monotone increasing, and for fixed p_2 and I, $p_1 \mapsto D_1((p_1, p_2), I)$ is monotone increasing.

For all $(p, I) \in \mathbb{R}^2_{++} \times \mathbb{R}_+$, define the *budget* $B(p, I)$ by $B(p, I) = \left\{x \in \mathbb{R}^2_+ : p \cdot x \leq I\right\}$. Note that $B(p, I)$ is compact, by the assumption that prices are strictly positive.

A demand function D is *rational* if there is a monotone increasing function $u : \mathbb{R}^2_+ \to \mathbb{R}$ such that

$$D(p, I) = \text{argmax}_{x \in B(p,I)} u(x). \tag{1}$$

In that case, we say that u is a *rationalization of* (or that it *rationalizes*) D. Note that part of the definition of rationalizability is that $D(p, I)$ is the unique maximizer of u in $B(p, I)$.

3 Result

We shall use homogeneity to regard demand as only a function of prices: $D(p, I) = D((1/I)p, 1)$, so we can normalize income to 1. In this case, we regard demand as a function $D : \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ with $p \cdot D(p) = 1$ for all $p \in \mathbb{R}^2_{++}$.

A *partial demand function* is a function $D : P \to \mathbb{R}^2_+$ where $P \subseteq \mathbb{R}^2_{++}$ and $p \cdot D(p) = 1$ for every $p \in P$; P is called *the domain* of D. So a demand function is a partial demand function whose domain is $\mathbb{R}^2_{++}$. The concept of the partial demand function allows us to study finite demand observations. We imagine that we have observed demand at all prices in P (see e.g. Afriat (1967), Diewert and Parkan (1983) or Varian (1982)).

Theorem 1. *Let Q be a finite subset of $\mathbb{R}^2_{++}$ and let $D : Q \to \mathbb{R}^2_+$ be a partial demand function. Then D is the restriction to Q of a smooth and rational demand satisfying gross substitutes if and only if $q'_1 D_1(q') \leq q_1 D_1(q)$ for every $q, q' \in Q$ such that $q_1 \leq q'_1$ and $q'_2 \leq q_2$.*

4 Proof of Theorem 1

4.1 Lemmas about demand functions on $\mathbb{R}^2_{++}$

Say that a partial demand function satisfies the *the weak axiom of revealed preference* if $p \cdot D(p') > 1$ whenever $p' \cdot D(p) < 1$. With two goods, the weak axiom is equivalent to the strong axiom of revealed preference, and hence characterizes rational demand.

Throughout this section we fix a demand function $D : \mathbb{R}^2_{++} \to \mathbb{R}^2_+$. We introduce the expenditure share function associated to D: for $i \in \{1, 2\}$, let $\pi_i = \pi_i^{(D)} : \mathbb{R}^2_{++} \to [0, 1]$ be given by $\pi_i(p) = p_i D_i(p)$. Let $\preceq$ be the partial order over $\mathbb{R}^2_{++}$ that is given by

$$p' \preceq p \iff p'_1 \geq p_1 \text{ and } p'_2 \leq p_2.$$

Say that a function $f : \mathbb{R}^2_{++} \to \mathbb{R}$ is $\preceq$*-monotone increasing* if $x \preceq y$ implies $f(x) \leq f(y)$.

Lemma 1. *The following conditions are equivalent*

1. *$p_i \mapsto D_{-i}(p)$ is monotone increasing in p_i for every p_{-i}.*

2. $p_i \mapsto \pi_{-i}(p)$ *is monotone increasing in* p_i *for every* p_{-i}.

3. $p_i \mapsto \pi_i(p)$ *is monotone decreasing in* p_i *for every* p_{-i}.

Proof. $1 \iff 2$ follows from $p_{-i}D_{-i}(p) = \pi_{-i}(p)$ and $2 \iff 3$ follows from $\pi_i(p) = 1 - \pi_{-i}(p)$. ∎

Corollary 1. *D satisfies substitutes if and only if* π_1 *is* $\preceq$*-monotone increasing, i.e.* $\pi_1(p') \geq \pi_1(p)$ *whenever* $p' \preceq p$.

Proof. π_1 is $\preceq$-monotone increasing iff $p_2 \mapsto \pi_1(p_1, p_2)$ is monotone increasing for every p_1 and $p_1 \mapsto \pi_1(p_1, p_2)$ is monotone decreasing for every p_2. By Lemma 1 the first condition is equivalent to $p_2 \mapsto D_1(p)$ being monotone increasing and the second condition is equivalent to $p_1 \mapsto D_2(p)$ being monotone increasing. ∎

Lemma 2. *If D satisfies substitutes then* $p_1 \mapsto D_1(p_1, p_2)$ *is monotone decreasing for fixed* p_2

Proof. Note that $D_1(p_1, p_2) = \pi_1(p_1, p_2)/p_1$. By Lemma 1 the nominator is decreasing in p_1. Also, the denominator is increasing. Therefore the quotient is decreasing. ∎

Lemma 3. *If D satisfies gross substitutes and* $p' \preceq p$ *then* $D(p')_1 \leq D(p)_1$ *and* $D(p')_2 \geq D(p)_2$.

Proof. Let p'' be given by $p''_1 = p'_1$ and $p''_2 = p_2$. Then

$$D(p')_1 \leq D(p'')_1 \leq D(p)_1,$$

where the first inequality follows from the definition of substitutes and the second from Lemma 2. The second assertion follows from symmetry between the products. ∎

The following lemma was shown by Kehoe and Mas-Colell (1984) for *excess demand* functions, and the case of three goods.

Lemma 4. *If D satisfies gross substitutes then D satisfies the weak axiom of revealed preference.*

Proof. Assume that $p, p' \in \mathbb{R}^2_{++}$ and let $x = D(p)$ and $x' = D(p')$. Assume that $p' \cdot x < 1$. We claim that $p \cdot x' > 1$. Indeed, if $p' \geq p$ then $p' \cdot x \geq p \cdot x = 1$, a contradiction. Assume therefore without loss of generality that $p'_2 < p_2$. If $p'_1 < p_1$ then $p' \ll p$ and therefore

$$p \cdot x' > p' \cdot x' = 1,$$

as desired. Assume therefore that $p'_1 \geq p_1$, so that $p' \preceq p$. Then, it follows from Lemma 3 that $x'_1 \leq x_1$ and $x'_2 \geq x_2$. Therefore

$$p \cdot x' = p_1 \cdot x'_1 + p_2 \cdot x'_2 = \\ p \cdot x + p' \cdot x' - p' \cdot x + (p_1 - p'_1) \cdot (x'_1 - x_1) + (p'_2 - p_2) \cdot (x_2 - x'_2) > 1,$$

since $p \cdot x = p' \cdot x' = 1$. ∎

4.2 Partial Demand

The condition in the theorem is equivalent to $q'_1 D_1(q') \leq q_1 D_1(q)$ for every $q, q' \in Q$ such that $q' \preceq q$. Necessity follows immediately from Corollary 1. For sufficiency, fix $\epsilon > 0$ such that

$$\text{if } q'_i > q_i - 2\epsilon \text{ then } q'_i \geq q_i, \tag{2}$$

for every $q, q' \in Q$ and every $i \in \{1, 2\}$. Let $\pi : \mathbb{R}^2 \to [0, 1]$ be given by

$$\pi(p) = \max\{q_1 \cdot D_1(q) | q \in S(p)\} \tag{3}$$

where $S(p) = \{q \in Q | q_1 > p_1 - \epsilon \text{ and } q_2 < p_2 + \epsilon\}$ and the maximum of the empty set is by definition 0. If $p' \preceq p$ then $S(p') \subseteq S(p)$. Therefore π is $\preceq$-monotone.

Claim 5. *If $\|p - q\|_\infty < \epsilon$ for some $p \in \mathbb{R}^2_{++}$ and $q \in Q$ then $\pi(p) = q_1 D_1(q)$.*

Proof. Since $q \in S(p)$ it follows from (3) that $\pi(p) \geq q_1 D_1(q)$. On the other hand, let $q' \in S(p)$. Then $q'_1 > p_1 - \epsilon > q_1 - 2\epsilon$ and therefore $q'_1 \geq q_1$ by (2) and $q'_2 < p_2 + \epsilon < q_2 + 2\epsilon$

and therefore $q'_2 \leq q_2$ by (2). Thus $q' \preceq q$ and therefore $q'_1 D_1(q') \leq q_1 D_1(q)$. Since this is true for every $q' \in S(p)$ it follows from (3) that $\pi(p) \leq q_1 D_1(q)$. ∎

Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth function such that $\psi(\tau) \geq 0$ for every $\tau \in \mathbb{R}^2$, $\psi(\tau) = 0$ whenever $\|\tau\|_\infty \geq \epsilon$ and

$$\int_{\mathbb{R}^2} \psi = 1. \tag{4}$$

For example, we can choose

$$\psi(x, y) = \begin{cases} \frac{1}{C} e^{-1/(1-(x/\epsilon)^2)-1/(1-(y/\epsilon)^2)}, & \text{if } |x| < \epsilon \text{ and } |y| < \epsilon \\ 0, & \text{otherwise,} \end{cases}$$

for a suitable normalizing factor C.

Let $\tilde{\pi} : \mathbb{R}^2_{++} \to [0, 1]$ be given by $\tilde{\pi} = \psi * \pi$, i.e.

$$\tilde{\pi}(p) = \int_{\mathbb{R}^2} \psi(\tau)\pi(p - \tau)d\tau.$$

Then $\tilde{\pi}$ is smooth (as a convolution of a smooth function with a bounded function), $\preceq$-monotone (as a convolution of a nonnegative function with a $\preceq$-monotone function) and $\tilde{\pi}(q) = q_1 D_1(q)$ for every $q \in Q$ by Claim 5 and the properties of ψ. Finally, Let $\tilde{D} : \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ be the demand function given by $\tilde{D}_1(p) = \tilde{\pi}(p)/p_1$ and $\tilde{D}_2(p) = (1-\tilde{\pi}(p))/p_2$. Then $\tilde{D}$ is a smooth demand function that satisfies gross substitutes by Corollary 1. By Lemma 4 $\tilde{D}$ satisfies the weak axiom, and hence it is rational.

5 Conclusion and remarks

The property of gross substitutes is of obvious interest to economists. One indication of this fact is that all the "principles of economics" courses that we are aware of discuss gross substitutes. We believe that the simple test we have developed is of interest as well.

We have determined the testable implications of gross substitutes of a pair of goods. The implications are very simple: the data itself must satisfy the definition in the form of the monotonicity of expenditure shares (Corollary 1). One might want to study more than two goods, for example whether beer, wine and whisky are substitutes. We note that such a study would still require the type of assumptions we implicitly made: one needs to aggregate different types of beer, for example, into a composite beer good. One would also need to separate the three goods from the steaks, fishes and salads the consumer is also deciding on.

In any case, with more than two goods there are additional implications than just the data satisfying the monotonicity of expenditure shares. We finalize with a four-good example making this point.

Example 1. We suppose that we have data $\{(x^k, p^k), (x^l, p^l)\}$ given, meaning that we have a partial demand function defined on two observations, k and l, where $x^k = D(p^k)$ and $x^l = D(p^l)$. Thus, $P = \{p^k, p^l\}$. All vectors lie in $\mathbb{R}^4$; the observations are in the table below.

It is easy to verify that the two observations satisfy the weak axiom of revealed preference, and hence the strong axiom (as there are only two observations). Note that $p^k \cdot x^l = 3/4 + 9/32 = 33/32 > 1$, so x^k is not revealed preferred to x^l. Hence $\{(x^k, p^k), (x^l, p^l)\}$ satisfy the weak axiom of revealed preference. The observations also satisfy monotonicity of expenditure shares with four goods because one price increases and one decreases.

Now, we claim that there is no x one can choose at prices $p = (1/2, 3, 3, 2)$ such that both the weak axiom and gross substitutes are satisfied. Concretely, the only x chosen at p compatible with substitutes violates the weak axiom with respect to $\{(x^l, p^l), (x, p)\}$. Comparing p and p^k, gross substitutes demands that $x_1 \geq x_1^k$: The table shows the case where x is such that $x_1 = x_1^k = 1$ and the rest is spent on x_4. We show that this is the most favorable case.

We write R and S for the weak and strict revealed preference relations, respectively. So $y \, S \, z$ if y is demanded at prices p_y and $p_y \cdot z < 1$, and $y \, R \, z$ if y is demanded at prices p_y and $p_y \cdot z \leq 1$.

	p_1	p_2	p_3	p_4	x_1	x_2	x_3	x_4
(p^k, x^k)	1	1	1	1	1	0	0	0
(p^l, x^l)	1/3	1/3	8/3	8/3	3/4	0	0	9/32
(p, x)	1/2	3	3	2	1	0	0	1/4

First, note that $x \; S \; x^l$, as $p \cdot x^l = (1/2)(3/4) + 2(9/32) = 30/32 < 1$, and $x^l \; R \; x$: $p^l \cdot x = 1/3 + (8/3)(1/4) = 1$. So $\{(x^l, p^l), (x, p)\}$ with x chosen as in the table violate the weak axiom.

Second, we argue that this is the case for any x that satisfies gross substitutes with respect to (p^k, x^k). We have $x \, S \, x^l$ independently of the choice of x, so we need to see that $p^l \cdot x \leq 1$ for all x with $x_1 \geq 1$. Given a choice for x_1, consider the x that maximize $p^l \cdot x$ subject to $p \cdot x = 1$. That is: maximize $x_2/3 + 8x_3/3 + 8x_4/3$ s.t. $3(x_2 + x_3) + 2x_4 = 1 - x_1/2$. The solution is to set $(x_2, x_3, x_4) = (0, 0, (1/2)(1 - x_1/2))$. Now,

$$\begin{aligned} p^l \cdot x \leq (1/3)x_1 + (1/2 - x_1/4)(8/3) &= 8/6 + (1/3 - 2/3)x_1 \\ &= 8/6 - (1/3)x_1 \leq 4/3 - 1/3 = 1, \end{aligned}$$

as $x_1 \geq 1$. Thus $x^l \; R \; x$, which together with $x \; S \; x^l$ is a violation of the weak axiom of revealed preference.

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