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APPROXIMATE EXPECTED UTILITY RATIONALIZATION

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Abstract

We propose a new measure of deviations from expected utility, given data on economic choices under risk and uncertainty. In a revealed preference setup, and given a positive number e, we provide a characterization of the datasets whose deviation (in beliefs, utility, or perceived prices) is within e of expected utility theory. The number e can then be used as a distance to the theory.

We apply our methodology to three recent large-scale experiments. Many subjects in those experiments are consistent with utility maximization, but not expected utility maximization. The correlation of our measure with demographics is also interesting, and provides new and intuitive findings on expected utility.

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1 Introduction

Revealed preference theory started out as an investigation into the empirical content of utility maximization, but more recently has turned to the empirical content of specific utility theories. The leading example is expected utility: recent theoretical work seeks to characterize the choice behaviors that are consistent with expected utility maximization. At the same time, a number of empirical papers carry out revealed preference tests on data of choices under risk and uncertainty. We seek to bridge the gap between the theoretical understanding of expected utility theory, and the machinery needed to analyze experimental data on choices under risk and uncertainty.[1]

Imagine an agent making economic decisions, choosing contingent consumption given market prices and income. A long tradition in revealed preference theory studies the consistency of such choices with utility maximization, and a more recent literature has investigated consistency with expected utility theory (EU). [2] Consistency, however, is a black or white question. The choices are either consistent with EU or they are not. Our contribution is to describe the *degree* to which choices are consistent with EU.

Revealed preference theory has developed measures of how far choices are from being compatible with general utility maximization. The most widely used measure is the Critical Cost Efficiency Index (CCEI) proposed by Afriat (1972). Varian (1990) proposes a modification, and Echenique et al. (2011) propose an alternative measure. [3] Such measures are designed to gauge the distance between choices that cannot be rationalized by any utility function, and choices for which there exists some utility function that could explain them. They are not designed to measure consistency with EU.

The CCEI has been widely used to analyze experimental data, including data that involves choice under risk and uncertainty. See, for example, Ahn et al. (2014), Choi et al. (2007), Choi et al. (2014), Carvalho et al. (2016), and Carvalho and Silverman (2017). These studies involve agents making decisions under risk or uncertainty, but the authors have not had tools to investigate consistency with EU, the most commonly used theory

[1]We analyze objective expected utility theory for choice under risk and subjective expected utility theory for choice under uncertainty.

[2] The seminal papers include Samuelson (1938), Afriat (1967) and Varian (1982) (see Chambers and Echenique (2016) for an exposition). The work on EU includes Green and Srivastava (1986), Kubler et al. (2014), and Echenique and Saito (2015).

[3]Dziewulski (2018) provides a foundation for CCEI based on the model in Dziewulski (2016), which seeks to rationalize violations of utility-maximizing behavior with a model of just-noticeable differences.

to explain choices under risk or uncertainty. The purpose of our paper is to provide such a tool.

Of course, there is nothing wrong with studying general utility maximization in environments with risk and uncertainty, but it is surely also of interest to use the same data to look at EU. After a theoretical discussion of our measure (Sections 3 and 5), we carry out an empirical implementation of our proposals to data from the last three of the cited papers (Section 4). [4]

Our empirical application has two purposes. The first is to illustrate how our method can be applied. The second is to a give a new use to existing data. We use data from three large-scale experiments (Choi et al., 2014; Carvalho et al., 2016; Carvalho and Silverman, 2017), each with over 1,000 subjects, that involves choices under risk. Given our methodology, the data can be used to test expected utility theory, not only general utility maximization. The main take aways from our empirical application are as follows. a) The data confirm that CCEI is not a good indication of compliance with EU. Among agents with high CCEI, who seem to be close to consistent with utility maximization, our measure of closeness to EU is very dispersed. b) Correlation between closeness to EU and demographic characteristics yields interesting results. We find that younger subjects, those who have high cognitive abilities, and those who are working, are closer to EU behavior than older, low ability, or passive, subjects. For some of the three experiments, we also find that highly educated, high-income subjects, and males, are closer to EU.

In the rest of the introduction, we lay out the argument for why CCEI is inadequate to measure deviations from EU.

The CCEI is meant to test deviations from general utility maximization. If an agent's behavior is not consistent with utility maximization, then it cannot possibly be consistent with expected utility maximization. It stands to reason that if an agent's behavior is far from being rationalizable with a general utility function, as measured by CCEI, then it is also far from being rationalizable with an expected utility function. The problem is, of course, that an agent may be rationalizable with a general utility function but not with an expected utility function.

Broadly speaking, the CCEI proceeds by "amending" inconsistent choices through

[4] These papers involve choices under risk, with given probabilities, and therefore represent a natural unit of analysis.

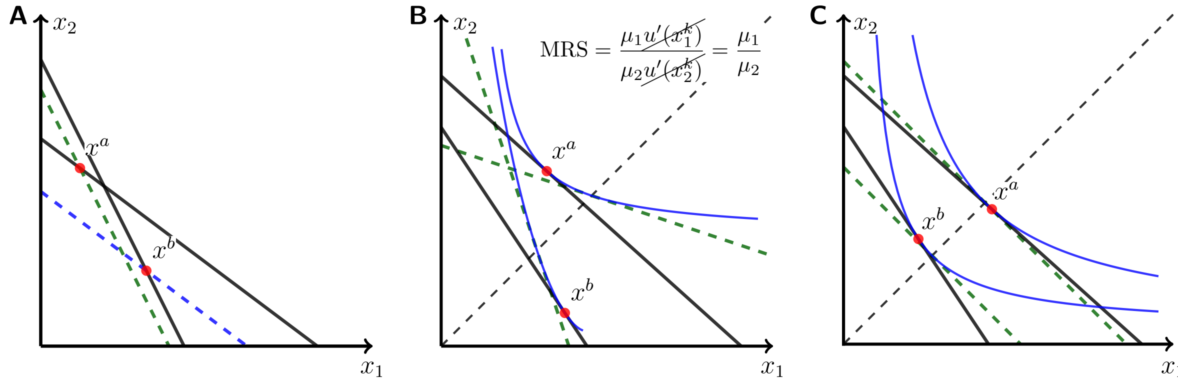

Figure 1: (A) A violation of WARP. (B) A violation of the expected utility theory: $x_2^a > x_1^a$, $x_1^b > x_2^b$, and $p_1^b/p_2^b < p_1^a/p_2^a$. (C) A pattern of choices consistent with EU.

the devise of changing income. This works for general utility maximization, but it is the wrong way to amend choices that are inconsistent with EU: EU is about getting the marginal rates of substitution right, so prices need to be changed, not incomes. The problem is illustrated with a simple example in Figure 1. Suppose that there are two states of the world, labeled 1 and 2. An agent purchases a state-contingent asset $x = (x_1, x_2)$, given Arrow-Debreu prices $p = (p_1, p_2)$ and her income. Prices and incomes define a budget set. In panel A of Figure 1 we are given two choices for the agent, x^a and x^b, for two different budgets. The choices in panel A of Figure 1 are inconsistent with utility maximization: they violate the weak axiom of revealed preference (WARP). When x^b (x^a) was chosen, x^a (x^b, respectively) was strictly inside of the budget set. This violation of WARP can be resolved by shifting down the budget line associated with choice x^b below the dotted green line passing through x^a. Alternatively, the violation can be resolved by shifting down the budget line associated with choice x^a below the dotted blue line passing through x^b. Afriat's CCEI is the smallest of the two shifts that are needed: the smallest proportion of shifting down a budget line to resolve WARP violation. Therefore, the CCEI of this dataset corresponds to the dotted green line passing through x^a. That is, the CCEI is $(p^b \cdot x^a)/(p^b \cdot x^b)$.

Now consider the example in panel B of Figure 1. There are again two choices made by a subject, x^a and x^b, for two different budgets. These choices do not violate WARP, and CCEI indicates perfect compliance with the theory of utility maximization. The choices in the panel are *not*, however, compatible with EU. To see why, assume that the dataset were rationalized by an expected utility: $\mu_1 u(x_1^k) + \mu_2 u(x_2^k)$, where (μ_1, μ_2) are the probabilities of the two states, and u is a (smooth) concave utility function over money. Note that the slope of a tangent line to the indifference curve at a point x^k is equal to the marginal rate of substitution (MRS): $\mu_1 u'(x_1^k)/\mu_2 u'(x_2^k)$. Moreover, at the 45-degree line (i.e., when $x_1^k = x_2^k$), the slope must be equal to $\mu_1 \cancel{u'(x_1^k)}/\mu_2 \cancel{u'(x_2^k)} = \mu_1/\mu_2$. This is a contradiction because in Figure 1 panel B, the two tangent lines (green dotted lines) associated with x^a and x^b cross each other. In contrast with panel B, the figure in panel C shows choices that are consistent with EU. Tangent lines at the 45-degree line are parallel in this case.

Importantly, the violation in panel B cannot be resolved by shifting budget lines up or down, or more generally by adjusting agents' expenditures. The reason is that *the empirical content of expected utility is captured by the relation between prices and marginal rates of substitution. The slope, not the level, of the budget line is what matters.*

Our contribution is to propose a measure of how close a dataset is to being consistent with expected utility maximization. Our measure is based on the idea that marginal rates of substitution have to conform to expected utility maximization. If one "perturbs" marginal utility enough, then a dataset is always consistent with expected utility. Our measure is simply a measure of how large of a perturbation is needed to rationalize a dataset. Perturbations of marginal utility can be interpreted in three different, but equivalent, ways: as measurement error on prices, as random shocks to marginal utility in the spirit of random utility theory, or as perturbations to agents' beliefs. For example, if the data in panel B of Figure 1 is e away from being consistent with expected utility, then one can find beliefs μ^a and μ^b, one for each observation, so that expected utility is maximized for these observation-specific beliefs, and such that the data is consistent with such perturbed beliefs.

Our measure can be applied in settings where probabilities are known and objective, for which we develop a theory in Section 3, and an application to experimental data in Section 4. It can also be applied to settings where probabilities are not known, and therefore subjective (see Section 5).

Finally, we propose a statistical methodology for testing the null hypothesis of consistency with EU. Our test relies on a set of auxiliary assumptions: the methodology is developed in Section 4.3. The test indicates moderate levels of rejection of the EU hypothesis.

2 Model

Let S be a finite set of *states*. We occasionally use S to denote the number $|S|$ of states. Let $\Delta_{++}(S) = \{\mu \in \mathbf{R}^S_{++} \mid \sum_{s=1}^S \mu_s = 1\}$ denote the set of strictly positive probability measures on S. In our model, the objects of choice are state-contingent monetary payoffs, or *monetary acts*. A monetary act is a vector in $\mathbf{R}^S_+$.

Definition 1. *A* dataset *is a finite collection of pairs* $(x, p) \in \mathbf{R}^S_+ \times \mathbf{R}^S_{++}$.

The interpretation of a dataset $(x^k, p^k)_{k=1}^K$ is that it describes K purchases of a state-contingent payoff x^k at some given vector of prices p^k, and income $p^k \cdot x^k$.

For any prices $p \in \mathbf{R}^S_{++}$ and positive number $I > 0$, the set

$$B(p, I) = \{y \in \mathbf{R}^S_+ \mid p \cdot y \leq I\}$$

is the *budget set* defined by p and I.

Expected utility theory requires a decision maker to solve the problem

$$\max_{x \in B(p,I)} \sum_{s \in S} \mu_s u(x_s) \tag{1}$$

when faced with prices $p \in \mathbf{R}^S_{++}$ and income $I > 0$, where $\mu \in \Delta_{++}(S)$ is a belief and u is a concave utility function over money. We are interested in concave u; an assumption that corresponds to risk aversion.

The belief μ will have two interpretations in our model. First, in Section 3, we shall focus on decisions taken under *risk*. The belief μ will be a known "objective" probability measure $\mu^* \in \Delta_{++}(S)$. Then, in Section 5, we study choice under *uncertainty*. Consequently, The belief μ will be a subjective beliefs, which is unobservable to us as outside observers.

When imposed on a dataset, expected utility maximization (1) may be too demanding. We are interested in situations where the model in (1) holds approximately. As a result, we shall relax (1) by "perturbing" some elements of the model. The exercise will be to see if a dataset is consistent with the model in which some elements have been perturbed. Specifically, we shall perturb beliefs, utilities or prices.

First, consider a perturbation of utility u. We allow u to depend on the choice problem k and the realization of the state s. We suppose that the utility of consumption x_s in state s is given by $\varepsilon_s^k u(x_s)$, with ε_s^k being a (multiplicative) perturbation in utility. To sum up, given price p and income I, a decision maker solves the problem

$$\max_{x \in B(p,I)} \sum_{s \in S} \mu_s \varepsilon_s^k u(x_s) \tag{2}$$

when faced with prices $p \in \mathbf{R}^S_{++}$ and income $I > 0$. Here $\{\varepsilon_s^k\}$ is a set of perturbations, and u is, as before, a concave utility function over money.

In the second place, consider a perturbation of beliefs. We allow μ to be different for each choice problem k. That is, given price p and income I, a decision maker solves the problem

$$\max_{x \in B(p,I)} \sum_{s \in S} \mu_s^k u(x_s) \tag{3}$$

when faced with prices $p \in \mathbf{R}^S_{++}$ and income $I > 0$, where $\{\mu^k\} \subset \Delta_{++}(S)$ is a set of beliefs and u is a concave utility function over money.

Finally, consider a perturbation of prices. Our consumer faces perturbed prices $\tilde{p}_s^k = \varepsilon_s^k p_s^k$, with a perturbation ε_s^k that depends on the choice problem k and the state s. Given price p and income I, a decision maker solves the problem

$$\max_{x \in B(\tilde{p},I)} \sum_{s \in S} \mu_s u(x_s), \tag{4}$$

when faced with income $I > 0$ and the perturbed prices $\tilde{p}_s^k = \varepsilon_s^k p_s^k$ for each $k \in K$ and $s \in S$.

Observe that our three sources of perturbations have different interpretations. Perturbed prices can be thought of a prices subject to measurement error. Perturbed utility is an instance of random utility models. Finally, perturbations of beliefs can be thought of as a kind of random utility, or as an inability to exactly use probabilities.

3 Perturbed Objective Expected Utility

In this section we treat the problem under risk: there exists a known "objective" belief $\mu^* \in \Delta_{++}(S)$ that determines the realization of states.

As mentioned above, we go through each of the sources of perturbation: beliefs, utility and prices. We seek to understand how large a perturbation has to be in order to rationalize a dataset. It turns out that, for this purpose, all sources of perturbations are equivalent.

3.1 Belief Perturbation

We allow the decision maker to have a belief μ^k for each choice k. We seek to understand how much the belief μ^k deviates from the objective belief μ^* by evaluating how far the ratio,

$$\frac{\mu_s^k/\mu_t^k}{\mu_s^*/\mu_t^*},$$

where $s \neq t$, differs from 1. If the ratio is larger (smaller) than one, then it means that in choice k, the decision maker believes the relative likelihood of state s with respect to state t is larger (smaller, respectively) than what he should believe, given the objective belief μ^*.

Given a nonnegative number e, we say that a dataset is e-belief-perturbed objective expected utility (OEU) rational, if it can be rationalized using expected utility with perturbed beliefs for which the relative likelihood ratios do not differ by more than e from their objective equivalents. Formally:

Definition 2. *Let $e \in \mathbf{R}_+$. A dataset $(x^k, p^k)_{k=1}^K$ is* e-belief-perturbed OEU rational *if there exist $\mu^k \in \Delta_{++}$ for each $k \in K$, and a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$, such that, for all k,*

$$y \in B(p^k, p^k \cdot x^k) \implies \sum_{s \in S} \mu_s^k u(y_s) \leq \sum_{s \in S} \mu_s^k u(x_s^k). \tag{5}$$

and for each $k \in K$ and $s, t \in S$,

$$\frac{1}{1+e} \leq \frac{\mu_s^k/\mu_t^k}{\mu_s^*/\mu_t^*} \leq 1 + e. \tag{6}$$

When $e = 0$, e-belief-perturbed OEU rationality requires that $\mu_s^k = \mu_s^*$, so the case of exact consistency with expected utility is obtained with a zero bound of belief perturbations. Moreover, it is easy to see that by taking e to be large enough, any data set can be e-belief-perturbed rationalized.

We should note that e bounds belief perturbations for all states and observations. As such, it is sensitive to extreme observations and outliers (the CCEI is also subject to this critique: see Echenique et al. (2011)). In our empirical results, we carry our a robustness analysis to account for such sensitivity: see Appendix D.2.

Finally, we mention a potential relationship with models of nonexpected utility. One could think of rank-dependent utility, for example, as a way of allowing agent's beliefs to adapt to his observed choices. However, unlike e-belief-perturbed OEU, the nonexpected utility theory requires some consistencies on the dependency. For example, for the case of rank dependent utility, the agent's belief over the sates is affected by the ranking of the outcomes across states.

3.2 Price Perturbation

We now turn to perturbed prices: think of them as prices measured with error. The perturbation is a multiplicative noise term ε_s^k to the Arrow-Debreu state price p_s^k. Thus, perturbed state price are $\varepsilon_s^k p_s^k$. Note that if $\varepsilon_s^k = \varepsilon_t^k$ for all s, t, then introducing the noise does not affect anything because it only changes the scale of prices. In other words, what matters is how perturbations affect relative prices, that is $\varepsilon_s^k / \varepsilon_t^k$.

We can measure how much the noise ε^k perturbs relative prices by evaluating how much the ratio,

$$\frac{\varepsilon_s^k}{\varepsilon_t^k},$$

where $s \neq t$, differs from 1.

Definition 3. *Let $e \in \mathbf{R}_+$. A dataset $(x^k, p^k)_{k=1}^K$ is* e-price-perturbed OEU rational *if there exists a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$, and $\varepsilon^k \in \mathbf{R}_+^S$ for each $k \in K$ such that, for all k,*

$$y \in B(\tilde{p}^k, \tilde{p}^k \cdot x^k) \implies \sum_{s \in S} \mu_s^* u(y_s) \leq \sum_{s \in S} \mu_s^* u(x_s^k), \tag{7}$$

where for each $k \in K$ and $s \in S$

$$\tilde{p}_s^k = p_s^k \varepsilon_s^k \text{[5]} \tag{8}$$

and for each $k \in K$ and $s, t \in S$

$$\frac{1}{1+e} \leq \frac{\varepsilon_s^k}{\varepsilon_t^k} \leq 1 + e. \tag{9}$$

The idea is illustrated in Figure 2 (panels A-D). The figure shows how the perturbations to relative prices affect budget lines, under the assumption that $|S| = 2$. For each value of $e \in \{0.1, 0.25, 0.5, 1\}$ and $k \in K$, the blue area is the set $\{x \in \mathbf{R}_+^S \mid x \cdot \tilde{p}^k = x^k \cdot \tilde{p}^k \text{ and } (9)\}$ of perturbed budget lines. The dataset in the figure is the same as in panel B of Figure 1, which is not rationalizable with any expected utility function.

[5] It is without loss of generality to add an additional restriction that $\tilde{p}_k \cdot x_k = p_k \cdot x_k$ for each $k \in K$ because what matters are the relative prices.

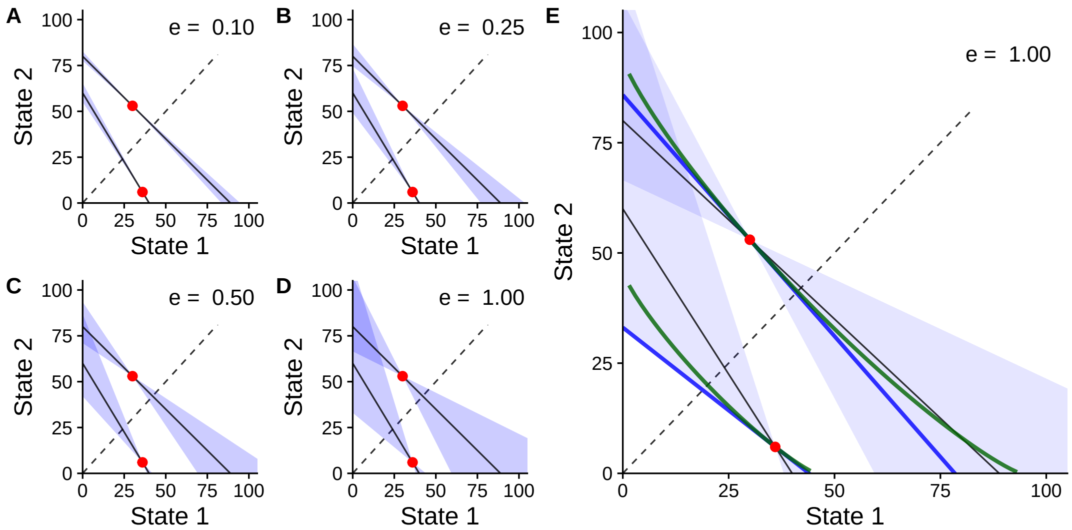

Figure 2: (A-D) Illustration of perturbed budget sets with $e \in \{0.1, 0.25, 0.5, 1\}$. (E) Example of price-perturbed expected utility rationalization.

Figure 2, panel E illustrates how we rationalize the dataset in panel B of Figure 1. The blue bold lines are perturbed budget lines and the green bold curves are (fixed) indifference curves passing through each of the x^k in the data. Note that the indifference curves have the same slope at the 45-degree line. The blue shaded areas are the sets of perturbed budget lines bounded by $e = 1$. Perturbed budget lines needed to rationalize the choices are indicated with blue bold lines. Since these are inside the shaded areas, the dataset is price-perturbed OEU rational with $e = 1$.

3.3 Utility Perturbation

Finally, we turn to perturbed utility. As explained above, perturbations are multiplicative and take the form $\varepsilon_s^k u(x_s^k)$. It is easy to see that this method is equivalent to belief perturbation. As for price perturbations, we seek to measure how much the ε^k perturbs utilities at choice problem k by evaluating how much the ratio,

$$\frac{\varepsilon_s^k}{\varepsilon_t^k},$$

where $s \neq t$, differs from 1.

Definition 4. *Let $e \in \mathbf{R}_+$. A dataset $(x^k, p^k)_{k=1}^K$ is* e-utility-perturbed OEU rational *if there exists a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$ and $\varepsilon^k \in \mathbf{R}_+^S$ for each*

$k \in K$ such that, for all k,

$$y \in B(p^k, p^k \cdot x^k) \implies \sum_{s \in S} \mu_s^* \varepsilon_s^k u(y_s) \leq \sum_{s \in S} \mu_s^* \varepsilon_s^k u(x_s^k), \tag{10}$$

and for each $k \in K$ and $s, t \in S$

$$\frac{1}{1+e} \leq \frac{\varepsilon_s^k}{\varepsilon_t^k} \leq 1 + e. \tag{11}$$

3.4 Equivalence of the Three Measures

The first observation we make is that the three sources of perturbations are equivalent, in the sense that for any e a data set is e-perturbed rationalizable according to one of the sources if and only if it is also rationalizable according to any of the other sources. By virtue of this result, we can interpret our measure deviations from OEU in any of the ways we have introduced.

Theorem 1. *Let $e \in \mathbf{R}_+$, and D be a dataset. The following are equivalent:*

- *D is e-belief-perturbed OEU rational;*
- *D is e-price-perturbed OEU rational;*
- *D is e-utility-perturbed OEU rational.*

In light of Theorem 1 we shall simply say that a data set is *e-perturbed OEU rational* if it is e-belief-perturbed OEU rational, and this will be equivalent to being e-price-perturbed OEU rational, and e-utility-perturbed OEU rational.

3.5 Characterizations

We proceed to give a characterization of the dataset that are e-perturbed OEU rational. Specifically, given $e \in \mathbf{R}_+$, we propose a revealed preference axiom and prove that a dataset satisfies the axiom if and only if it is e-perturbed OEU rational.

Before we state the axiom, we need to introduce some additional notation. In the current model, where μ^* is known and objective, what matters to an expected utility maximizer is not the state price itself, but instead the *risk-neutral* price:

Definition 5. *For any dataset* $(p^k, x^k)_{k=1}^K$*, the* risk neutral price $\rho_s^k \in \mathbf{R}_{++}^S$ *in choice problem* k *at state* s *is defined by*

$$\rho_s^k = \frac{p_s^k}{\mu_s^*}.$$

As in Echenique and Saito (2015), the axiom we propose involves a sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ of pairs satisfying certain conditions.

Definition 6. *A sequence of pairs* $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n \equiv \sigma$ *is called a* test sequence *if*

(1) $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ *for all* i*;*

(2) each k *appears as* k_i *(on the left of the pair) the same number of times it appears as* k_i' *(on the right).*

Echenique and Saito (2015) provide an axiom, termed the Strong Axiom for Revealed Objective Expected Utility (SAROEU), which states for any test sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$, we have

$$\prod_{i=1}^n \frac{\rho_{s_i}^{k_i}}{\rho_{s_i'}^{k_i'}} \le 1. \tag{12}$$

SAROEU is equivalent to the axiom provided by Kubler et al. (2014).

It is easy to see why SAROEU is necessary. Assuming (for simplicity of exposition) that u is differentiable, the first order condition of the maximization problem (1) for choice problem k

$$\lambda^k p_s^k = \mu_s^* u'(x_s^k), \text{ or equivalently, } \rho_s^k = \frac{u'(x_s^k)}{\lambda^k},$$

where $\lambda^k > 0$ is a Lagrange multiplier.

By substituting this equation on the left hand side of (12), we have

$$\prod_{i=1}^n \frac{\rho_{s_i}^{k_i}}{\rho_{s_i'}^{k_i'}} = \prod_{i=1}^n \frac{\lambda^{k_i'}}{\lambda^{k_i}} \cdot \prod_{i=1}^n \frac{u'(x_{s_i}^{k_i})}{u'(x_{s_i'}^{k_i'})} \le 1.$$

To see that this term is smaller than 1, note that the first term of the product of the λ-ratios is equal to one because of the condition (2) of the test sequence: all λ^k must cancel out. The second term of the product of u'-ratio is less than one because of the concavity of u, and the condition (1) of the test sequence (i.e., $u'(x_{s_i}^{k_i})/u'(x_{s_i'}^{k_i'}) \le 1$). Thus SAROEU is implied. It is more complicated to show that SAROEU is sufficient (see Echenique and Saito (2015) for details).

Now, e-perturbed OEU rationality allows the decision maker to use different beliefs $\mu^k \in \Delta_{++}(S)$ for each choice problem k. Consequently, SAROEU is not necessary for e-perturbed OEU rationality. To see that SAROEU can be violated, note that the first order condition of the maximization (3) for choice k is as follows: there exists a positive number (Lagrange multiplier) λ^k such that for each $s \in S$,

$$\lambda^k p^k_s = \mu^k_s u'(x^k_s), \text{ or equivalently, } \rho^k_s = \frac{\mu^k_s}{\mu^*_s}\frac{u'(x^k_s)}{\lambda^k}.$$

Suppose that $x^k_s > x^k_t$. Then (x^k_s, x^k_t) is a test sequence (of length one). We have

$$\frac{\rho^k_s}{\rho^k_t} = \left(\frac{\mu^k_s}{\mu^*_s}\frac{u'(x^k_s)}{\lambda^k}\right) \Big/ \left(\frac{\mu^k_t}{\mu^*_t}\frac{u'(x^k_t)}{\lambda^k}\right) = \frac{u'(x^k_s)}{u'(x^k_t)}\frac{\mu^k_s/\mu^k_t}{\mu^*_s/\mu^*_t}.$$

Even though $x^k_s > x^k_t$ implies the first term of the ratio of u' is less than one, the second term can be strictly larger than one. When x^k_s is close enough to x^k_t, the first term is almost one; the second term is strictly larger than one. Consequently, SAROEU can be violated.

However, by (6), we know that the second term is bounded by $1+e$. So we must have

$$\frac{\rho^k_s}{\rho^k_t} \leq 1 + e.$$

In general, for a sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ of pairs, one may suspect that the bound is calculated as $(1+e)^n$. This is not true because if x^k_s appears as both $x^{k_i}_{s_i}$ for some i and as $x^{k'_j}_{s'_j}$ for some j, then all μ^k_s can be canceled out. What matters is the number of times x^k_s appears without being canceled out. The number can be defined as follows.

Definition 7. *Consider any sequence* $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ *of pairs. Let* $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1} \equiv \sigma$. *For any* $k \in K$ *ands* $\in S$,

$$d(\sigma, k, s) = \#\{i \mid x^k_s = x^{k_i}_{s_i}\} - \#\{i \mid x^k_s = x^{k'_i}_{s'_i}\}.$$

and

$$m(\sigma) = \sum_{s \in S} \sum_{k \in K: d(\sigma,k,s)>0} d(\sigma, k, s).$$

Note that, if $d(\sigma, k, s)$ is positive, then $d(\sigma, k, s)$ is the number of times μ^k_s appears as a numerator without being canceled out. If it is negative, then $d(\sigma, k, s)$ is the number of times μ^k_s appears as a denominator without being canceled out. So $m(\sigma)$ is the "net" number of terms such as μ_s^k/μ_t^k that are present in the numerator. Thus the relevant bound is $(1+e)^{m(\sigma)}$.

Given the discussion above, it is easy to see that the following axiom is necessary for e-perturbed OEU rationality.

Axiom 1 (e-Perturbed Strong Axiom for Revealed Objective Expected Utility (e-PSAROEU))**.** *For any test sequence of pairs* $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n \equiv \sigma$*, we have*

$$\prod_{i=1}^{n} \frac{\rho_{s_i}^{k_i}}{\rho_{s_i'}^{k_i'}} \leq (1+e)^{m(\sigma)}.$$

The main result of this section is to show that the axiom is also sufficient.

Theorem 2. *Given* $e \in \mathbf{R}_+$*, and* D *be a dataset. The following are equivalent:*

- *D is e-belief-perturbed OEU rational.*
- *D satisfies e-PSAROEU.*

Axioms like e-PSAROEU can be interpreted as a statement about downward sloping demand (see Echenique et al., 2016). For example $(x_s^k, x_{s'}^k)$ with $x_s^k > x_{s'}^k$ is a test sequence. If risk neutral prices satisfy $\rho_s^k > \rho_{s'}^k$, then the data violate downward sloping demand. Now e-PSAROEU measures the extent of the violation by controlling the size of $\rho_s^k/\rho_{s'}^k$.

In its connection to downward sloping demand, Theorem 2 formalizes the idea of testing OEU through the correlation of risk-neutral prices and quantities: see Friedman et al. (2018) and our discussion in Section 4.2. Theorem 2 and the axiom e-PSAROEU give the precise form that the downward sloping demand property takes in order to characterize OEU, and provides a non-parametric justification to the practice of analyzing the correlation of prices and quantities.

As mentioned, 0-PSAROEU is equivalent to SAROEU. When $e = \infty$, the e-PSAROEU always holds because $(1+e)^{m(\sigma)} = \infty$.

Given a dataset, we shall calculate the *smallest* e for which the dataset satisfies e-PSAROEU. It is easy to see that such a minimal level of e exists. [6] We explain in Appendices B and C how it is calculated in practice.

[6] In Appendix B, we show that e_* can be obtained as a solution of minimization of a continuous function on a compact space. So the minimum exists.

Definition 8. Minimal e, *denoted* e_*, *is the smallest* $e' \geq 0$ *for which the data satisfies* e'*-PSAROEU.*

The number e_* is a crucial component of our empirical analysis. Importantly, it is the basis of a statistical procedure for testing the null hypothesis of OEU rationality.

As mentioned above, e_* is a bound that has to hold across all observations, and therefore may be sensitive to extreme outliers. It is, however, easy to check the sensitivity of the calculated e_* to an extreme observation. One can re-calculate e_* after dropping one or two observations, and look for large changes (Appendix D.2).

Finally, e_* depends on the prices and the objective probability which a decision maker faces. In particular, it is clear from e-PSAROEU that $1 + e$ is bounded by the maximum ratio of risk-neutral prices (i.e., $\max_{k,k' \in K, s,s' \in S} \rho_s^k / \rho_{s'}^{k'}$).

4 Testing (Objective) Expected Utility

We use our methods to test for perturbed OEU on datasets from three experiments implemented through large-scale online surveys. The datasets are taken from Choi et al. (2014), hereafter CKMS, Carvalho et al. (2016), hereafter CMW, and Carvalho and Silverman (2017), hereafter CS. All of these experiments followed the experimental structure introduced originally by Choi et al. (2007). [7]

It is worth mentioning here that all three papers, CKMS, CMW, and CS, focus on CCEI as a measure of violation of basic rationality. We shall instead look at the more narrow model of OEU, and use e_* as our measure of violations of the model. Our procedure for calculating e_* is explained in Appendices B and C.

4.1 Datasets

CKMS experiment Choi et al. (2014) used the CentERpanel, a stratified online weekly survey of a sample of over 2,000 households and 5,000 individual members in the

[7]We focused on CKMS, CMW, and CS because they have much larger samples than Choi et al. (2007), and collect sociodemographic variables. Choi et al. (2007) estimate a two-parameter utility function based on Gul's (1991) model of disappointment aversion. We report an analysis of Choi et al.'s (2007) dataset in Appendix D.

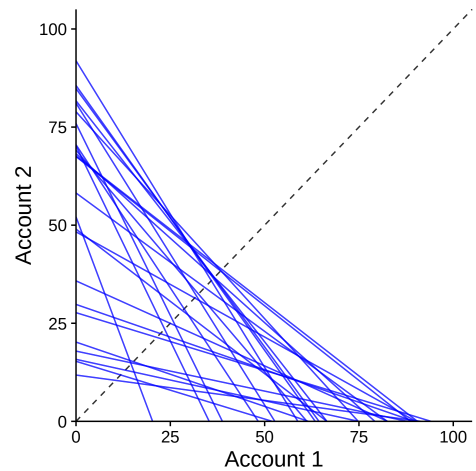

Figure 3: Sample budget lines. A set of 25 budgets from one real subject in Choi et al. (2014).

Netherlands. They implemented experimental tasks using the panel's survey instrument, randomly recruiting subjects from the entire CentERpanel sample. Their experiment was conducted with 1,182 CentERpanel adult members.

The instrument allowed them to collect a wide variety of individual demographic and economic information from the subjects. The main sociodemographic information they obtained include gender, age, education level, household monthly income, occupation, and household composition. [8]

In the experiment, subjects were presented with a sequence of decision problems under risk in a graphical illustration of a two-dimensional budget line on the (x, y)-plane. They were then asked to select a point, an "allocation," by clicking on the budget line. The coordinates of the selected point represent an allocation of points between accounts x and y. They received the points allocated to one of the accounts, x or y, determined at random with equal chance. They were presented a total of 25 budgets, which were selected randomly from the set of budget lines (see Figure 3). The selection of budget lines was independent across subjects, meaning that the subjects were give different sets of budget lines.

We note some interpretations of the design that matter for our discussion later. First, the points that lie on the 45-degree line correspond to equal allocations between the two

[8]Summary statistics of those key individual characteristics are reported in Table 1 in Choi et al. (2014).

Table 1: Sample size for each experiment.

Dataset	CKMS	CMW	CS
Number of subjects	1,182	1,119	1,423
Number of budgets	25	25	25

accounts, and therefore involve no risk. The 45-degree line is the "full insurance" line. Second, we can interpret the slope of a budget line as a price, in the usual sense: if the y-intercept is larger than the x-intercept, points in the y account are "cheaper" than those in the x account.

CMW experiment Carvalho et al. (2016) studied the effect of financial resources on decision making using two internet panel surveys. In their study 2, they administered a portfolio choice task and Choi et al. (2014). They fixed the set of 25 budgets, i.e., all subjects in the survey faced the same set of budgets. A total of 1,119 subjects participated in this study.

CS experiment Carvalho and Silverman (2017) studied the effects of the complexity of financial decision making using the University of Southern California's Understanding America Study (UAS) panel. A portfolio choice task with 25 budgets was induced in their baseline survey. A total of 1,423 subjects participated in this study.

4.2 Results

Summary statistics. We exclude five subjects whose e_* is 0 (i.e., "exact" OEU rational). We calculate e_* for the rest of the 3,719 subjects in the three experiments. The distributions of e_* are displayed in panel A of Figure 4.

The CKMS sample has a mean e_* of 1.289, and a median of 1.316. The CMW subjects have a mean of 1.189 and a median of 1.262, while the CS sample has a mean of 1.143 and a median of 1.128. [9]

Recall that the smaller a subject's e_* is, the closer her choice data to OEU rationality. Of course it is hard to exactly interpret the magnitude of e_*, a problem that we turn to

[9] Since e_* depends on the design of set(s) of budgets, comparing e_* across studies requires caution.

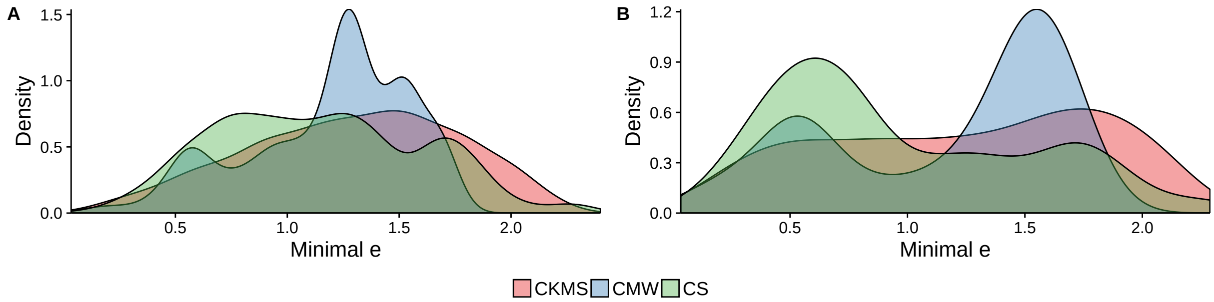

Figure 4: Kernel density estimations of e_* for all subjects (panel A) and for the subsample of subjects whose CCEI = 1 (panel B).

in Section 4.3.

Downward sloping demand and e_* Perturbations in beliefs, prices, or utility, seek to accommodate a dataset so that it is OEU rationalizable. The accommodation can be seen as correcting a mismatch of relative prices and marginal rates of substitution: recall our discussion in the Introduction. Another way to see the accommodation is through the relation between prices and quantities. Our revealed preference axiom, e-PSAROEU, bounds certain deviations from downward sloping demand. The minimal e is therefore a measure of the kinds of deviations from downward sloping demand that are crucial to OEU rationality.

Figure 5 displays "typical" patterns of choices from subjects with large and small values of e_*. The figure represents two selected subjects from our data. Panels A and C plot the observed choices from the different budget lines, and panels B and D plot the relation between $\log(x_2/x_1)$ and $\log(p_2/p_1)$. The idea in the latter plots is that, if a subject properly responds to price changes, then as $\log(p_2/p_1)$ becomes higher, $\log(x_2/x_1)$ should become lower. This relation is also the idea in e-PSAROEU. Therefore, panels B and D in Figure 5 should have a negative slope for the subjects to be OEU rational.

Observe that both subjects in Figure 5 have CCEI = 1, and are therefore consistent with utility maximization. The figure illustrates that the nature of OEU violations has little to do with CCEI.

The subject's choices in panel C are close to the 45-degree line. At first glance, such choices might seem to be rationalizable by a very risk-averse expected utility function. However, as panel D shows, the subject's choices deviates from downward sloping de-

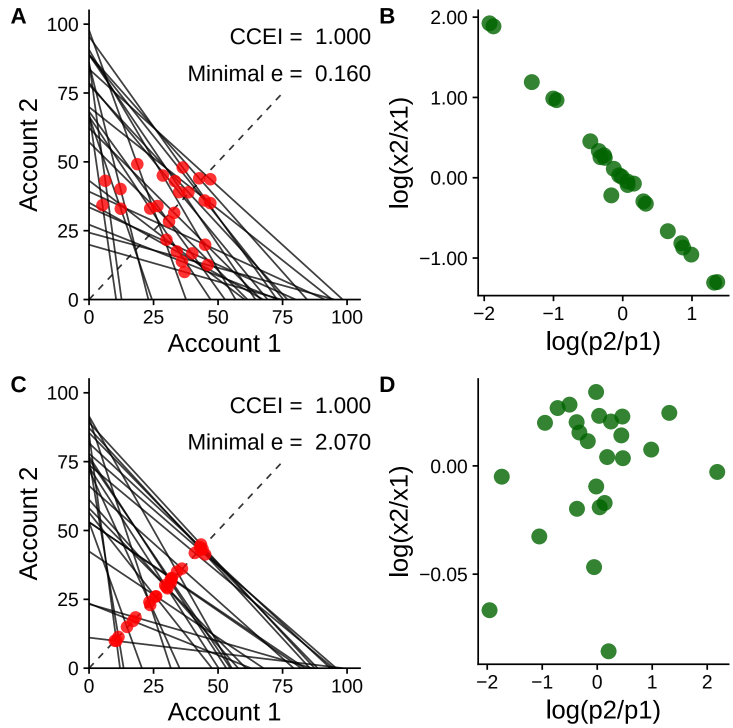

Figure 5: Dataset with CCEI = 1 and low e_* (panels AB) and high e_* (panels CD).

mand, hence cannot be rationalized by any expected utility function. One might be able to rationalize the choices made in panel C with certain symmetric models of errors in choices (like, possibly, "trembling hand" errors), but not with the types of errors captured by our model.

The observation in Figure 5 generalizes this idea. We calculate Pearson's correlation coefficient between $\log(x_2/x_1)$ and $\log(p_2/p_1)$ for each subject in the datasets. [10] Roughly speaking, the correlation coefficient is negative if subjects exhibit downward sloping demand. The correlation coefficient is close to zero if subjects' are not responding to price changes. Figure 6 illustrates the results. The top row of the figure confirms that e_* and the correlation between price and quantity, are positively related. This means that as e_* becomes small, subjects tend to exhibit downward sloping demand. As e_* becomes large, subjects become insensitive to price changes. Across all datasets, CKMS, CMW and CS, e_* and downward sloping demand are positively related.

We should mention the practice by some authors, notably Friedman et al. (2018),

[10] Note that $\log(x_2/x_1)$ is not defined at the corners. We thus adjust corner choices by small constant, 0.1% of the budget in each choice, in calculation of the correlation coefficient.

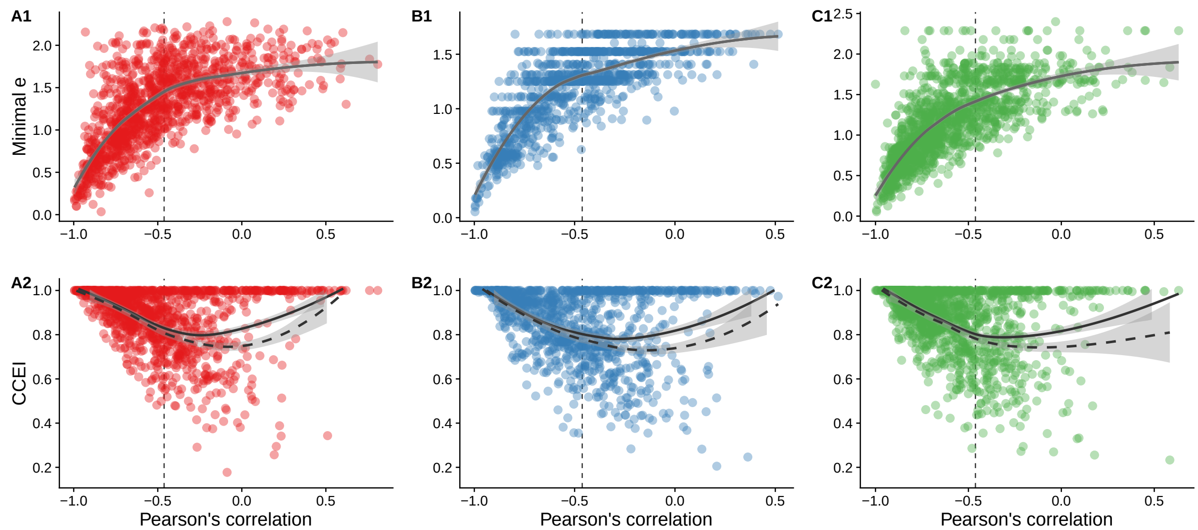

Figure 6: x-axis is Pearson's correlation between $\log(x_2/x_1)$ and $\log(p_2/p_1)$. The vertical dashed line indicates the critical value below which Pearson's correlation is significantly negative (one-tailed test, at 1% level). Solid curves represent LOESS smoothing. Dashed lines in the second row represent LOESS smoothing excluding subjects with CCEI = 1. Panels: (A) CKMS, (B) CMW, (C) CS.

to evaluate compliance with OEU by looking at the correlation between risk-neutral prices and quantities. Our measure is clearly related to that idea, and the empirical results presented in this section can be read as a validation of the correlational approach. Friedman et al. (2018) use their approach to estimate a parametric functional form, using experimental data in which they vary objective probabilities, not just prices. [11] Our approach is non-parametric, and focused on testing OEU, not estimating any particular utility specification.

The bottom row of Figure 6 illustrates the relation between CCEI and the correlation between price and quantity. The relation is not monotonic. Agents who are closer to complying with utility maximization do not display a stronger correlation between prices and quantities. The finding is consistent with our comment about CCEI and OEU rationality: CCEI measures the distance from utility maximization, which is related to parallel shifts in budget lines, while e_* and OEU are about the slope of the budget lines, and about a negative relation between quantities and prices. Hence, e_* reflects better

[11]For the datasets we use, where probabilities are always fixed, the results we report in Figure 5 are analogous to what Friedman et al. (2018) report in their Figure 6. The regression coefficients in their Table 2 are proportional to our estimated correlation coefficients (since beliefs are constant).

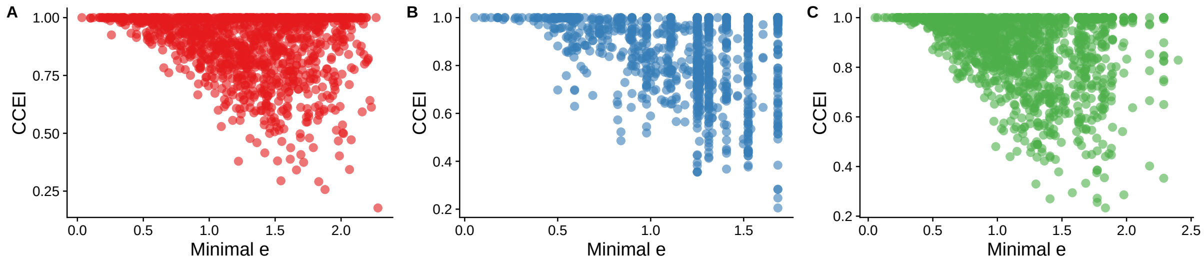

Figure 7: Correlation between e_* and CCEI from (A) CKMS, (B) CMW, and (C) CS.

than CCEI the characterizing properties of OEU.

We should mention that the non-monotonic relation between CCEI and the correlation coefficient seems to be partially driven by subjects who have CCEI = 1. There are 270 (22.8%) subjects whose CCEI scores equal to 1 in CKMS sample, 210 (18.5%) in CMW sample, and 315 (22.0%) in CS sample, respectively. Omitting such subjects weakens the non-monotone relationship. The dotted curves in the bottom row of Figure 6 look at the relation between CCEI and the correlation coefficient excluding subjects with CCEI = 1. These curves also have non-monotonic relation, but they (i) exhibit negative relation on a wider range of the x-axis, and (ii) have wider confidence bands when the correlation coefficient is positive (fewer observations).

We next turn to a direct comparison of e_* and CCEI in our data.

Relationship between e_* and CCEI. Comparing e_* and CCEI, we find that CCEI is not a good indication of the distance to OEU rationality. To reiterate a point we have already made, this should not be surprising as CCEI is meant to test general utility maximization, and not OEU. Nevertheless, it is interesting to see and quantify the relation between these measures in the data.

In panel B of Figure 4, we show the distribution of e_* among subjects whose CCEI is equal to one, which varies as much as in panel A. Many subjects have CCEI equal to one, but their e_*'s are far from zero. This means that consistency with general utility maximization is not necessarily a good indication of consistency with OEU.

That said, the measures are clearly correlated. Figure 7 plots the relation between CCEI and e_*. As we expect from their definitions (*larger* CCEI and *smaller* e_* correspond to higher consistency), there is a negative and significant relation between them (Pearson's correlation coefficient: $r = -0.2573$, $p < 10^{-15}$ for CKMS; $r = -0.2419$, $p < 10^{-15}$ for CMW; $r = -0.3458$, $p < 10^{-15}$ for CS).

Notice that the variability of the CCEI scores widens as the e_* becomes larger. Obviously, subjects with a small e_* are close to being consistent with general utility maximization, and therefore have a CCEI that is close to 1. However, subjects with large e_* seem to have disperse values of CCEI.

Correlation with demographic variables. We investigate the correlation between our measure of consistency with expected utility, e_*, and various demographic variables available in the data. The exercise is analogous to CKMS's findings using CCEI.

We find that younger subjects, those who have high cognitive abilities, and those who are working, are closer to being consistent with OEU than older, low ability, or passive, subjects. For some of the three experiments we also find that highly educated, high-income subjects, and males, are closer to OEU. Figure 8 summarizes the mean e_* along with 95% confidence intervals across several socioeconomic categories. [12] We use the same categorization as in Choi et al. (2014) to compare our results with their Figure 3.

We observe statistically significant (at a 5% level) gender differences in CMW (two-sample t-test, $t(1114) = -2.2074$, $p = 0.0275$) and CS (two-sample t-test, $t(1418) = -4.4620$, $p = 8.76 \times 10^{-6}$), but not in CKMS (two-sample t-test, $t(1180) = -0.8703$, $p = 0.3843$). Male subjects were on average closer to OEU rationality than female subjects in the CMW and CS samples (panel A).

We find significant age effects as well. Panel B shows that younger subjects are on average closer to OEU rationality than older subjects (the comparison between age groups 16-34 and 65+ reveals statistically significant difference in all three datasets; all two-sample t-tests give $p < 10^{-5}$).

We observe weak effects of education on e_* (panel C). [13] Subjects with higher education are on average closer to OEU rationality than those with lower education in CKMS (two-sample t-test, $t(829) = 4.1989$, $p < 10^{-4}$), but the difference is not significant in

[12] Figure D.13 in Appendix D shows correlation between CCEI and demographic variables.

[13] The low, medium, and high education levels correspond to primary or prevocational secondary education, pre-university secondary education or senior vocational training, and vocational college or university education, respectively. It is possible that we observe significant difference depending on how we categorize education levels, but we used the present categorization for comparability across studies.

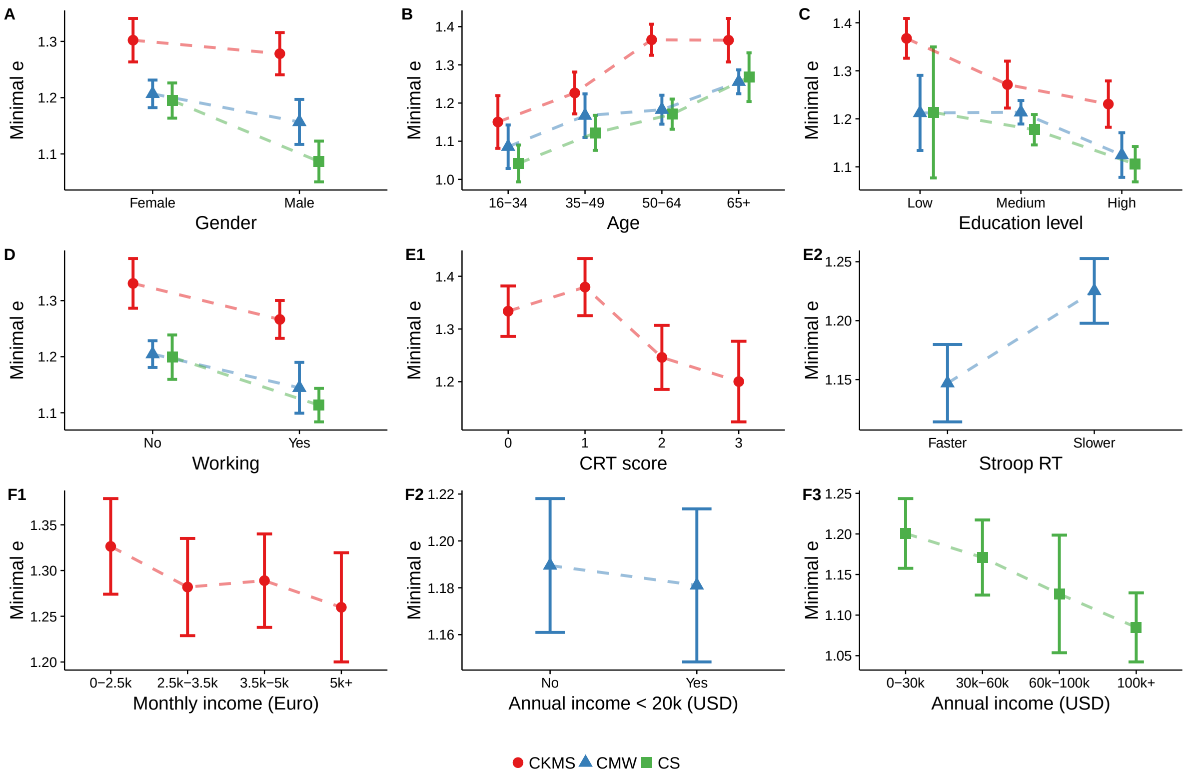

Figure 8: e_* and demographic variables.

the CMW and CS ($t(374) = 1.6787$, $p = 0.0940$ in CMW; $t(739) = 1.4113$, $p = 0.1586$ in CS).

Panel D shows that subjects who were working at the time of the survey are on average closer to OEU rationality than those who were not ($t(1180) = 2.2431$, $p = 0.0251$ in CKMS; $t(1114) = 2.4302$, $p = 0.0153$ in CMW; $t(1419) = 3.3470$, $p = 0.0008$ in CS).

In panels E1 and E2, we classify subjects according to their Cognitive Reflection Test score (CRT; Frederick, 2005) or average log reaction times in numerical Stroop task. CRT consists of three questions, all of which have an intuitive and spontaneous, but incorrect, answer, and a deliberative and correct answer. Frederick (2005) finds that CRT scores (number of questions answered correctly) are correlated with other measures of cognitive ability. In the numerical Stroop task, subjects are presented with a number, such as 888, and are asked to identify the number of times the digit is repeated (in this example the answer is 3, while more "intuitive" response is 8). It has been shown that response times in this task capture the subject's cognitive control ability.

The average e_* for those who correctly answered two questions or more of the CRT is lower than the average for those who answered at most one question. Subjects with lower response times in the numerical Stroop task have significantly lower e_* (two-sample t-test, $t(1114) = -3.345$, $p = 0.0009$).

One of the key findings in Choi et al. (2014) is that consistency with utility maximization measured by CCEI was related with household wealth. When we look at the relation between e_* and household income, there is a negative trend but the differences across income brackets are not statistically significant (bracket "0-2.5k" vs. "5k+" two-sample t-test, $t(533) = 1.6540$, $p = 0.0987$; panel F1). Panel F2 presents similar non-significance between subjects who earned more than 20 thousand USD annually or not in CMW sample (two-sample t-test, $t(1114) = -0.2301$, $p = 0.8180$). When we compare poor households (annual income less than 20 thousand USD) and wealthy households (annual income more than 100 thousand USD) from the CS sample, average e_* is significantly smaller for the latter sample (two-sample t-test, $t(887) = -3.5657$, $p = 0.0004$).

4.3 Minimum Perturbation Test

Our discussion so far has sidestepped one issue. How are we to interpret the absolute magnitude of e_*? When can we say that e_* is large enough to reject consistency with OEU rationality?

To answer this question, we present a statistical test of the hypothesis that an agent is OEU rational. The test needs some assumptions, but it gives us a threshold level (a critical value) for e_*. Any value of e_* that exceeds the threshold indicates inconsistency with OEU at some statistical significance level.

Our approach follows, roughly, the methodology laid out in Echenique et al. (2011) and Echenique et al. (2016). First, we adopt the price perturbation interpretation of e in Section 3.2. The advantage of doing so is that we can use the observed variability in price to get a handle on the assumptions we need to make on perturbed prices. To this end, let $D_{\text{true}} = (p^k, x^k)_{k=1}^K$ denote a dataset and $D_{\text{pert}} = (\tilde{p}^k, x^k)_{k=1}^K$ denote an "perturbed" dataset. Prices $\tilde{p}^k$ are prices p^k measured with error, or misperceived:

$$\tilde{p}_s^k = p_s^k \varepsilon_s^k \text{ for all } s \in S \text{ and } k \in K$$

where $\varepsilon_s^k > 0$ is a random variable.

If the *variance* of ε is large, it will be easy to accommodate a dataset as OEU rational. The larger is the variance of ε, the larger the magnitudes of e that can be rationalized as consistent with OEU. So, our procedure is sensitive to the assumptions we make about the variance of ε.

Our approach to get a handle on the variance of ε is to think of an agent who mistakes true prices p with perturbed prices $\tilde{p}$. If the variance of ε is too large, the agent should not mistake the distribution of p and $\tilde{p}$. In other words, the distributions of p and $\tilde{p}$ should be similar enough that an agent might plausibly confuse the two. Specifically, we imagine an agent who conducts a statistical test for the variance of prices. If the true variance of p is σ_0^2 and the implied variance of $\tilde{p}$ is $\sigma_1^2 > \sigma_0^2$, then the agent would conduct a test for the null of $\sigma^2 = \sigma_0^2$ against the alternative of $\sigma^2 = \sigma_1^2$. We want the variances to be close enough that the agent might reasonably get inconclusive results from such a test. *Specifically, we assume the sum of type I and type II errors in this test is relatively large.*[14]

The details of how we design our test are below, but we can advance the main results. See Figure 9. Each panel corresponds to our results for each of the datasets. The probability of a type I error is η^I. The probability of a type II error is η^{II}. Recall that we focus on situations when $\eta^I + \eta^{II}$ is relatively large, as we want our consumer to plausibly mistake the distributions of p and $\tilde{p}$. Consider, for example, our results for CKMS. The outermost numbers assume that $\eta^I + \eta^{II} = 0.7$. For such numbers, the rejection rates range from 3% to 41%. For the CS dataset, if we look at the second line of numbers, where $\eta^I + \eta^{II} = 0.65$, we see that rejection rates range from 1% to 19%.

Overall, it is fair to say that rejection rates are modest. Smaller values of $\eta^I + \eta^{II}$ correspond to larger values of $\mathrm{Var}(\varepsilon)$, and therefore smaller rejection rates. The figure also illustrates that the conclusions of the test are very sensitive to what one assumes about $\mathrm{Var}(\varepsilon)$, through the assumptions about η^I and η^{II}. But if we look at the largest rejection rates, for the largest values of $\eta^I + \eta^{II}$, we get 25% for CS, 27% for CMW, and 41% for CKMS. Many subjects in the CS, CMW and CKMS experiments are inconsistent with OEU, but at least according to our statistical test, for most subjects the rejections could be attributed to mistakes.

[14] The problem of variance is pervasive in statistical implementations of revealed preference tests, see Varian (1990), Echenique et al. (2011), and Echenique et al. (2016) for example. The use of the sum of type I and type II errors to calibrate a variance, is new to the present paper.

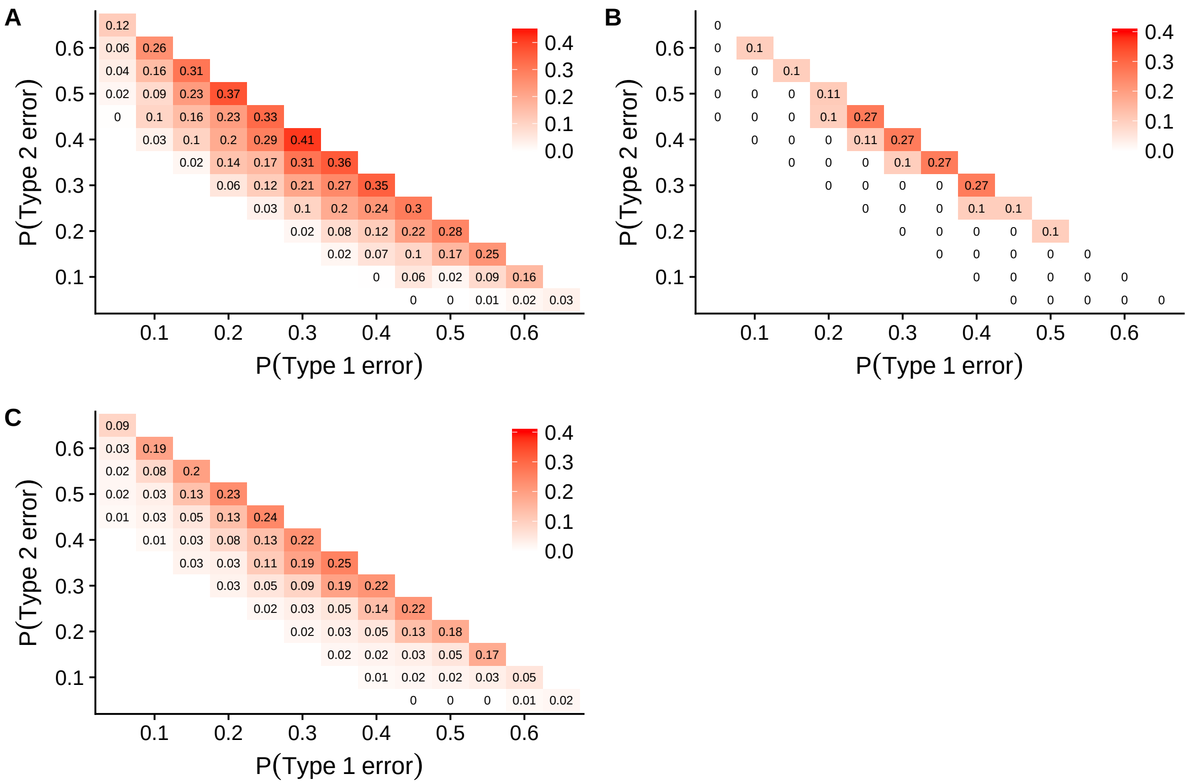

Figure 9: Rejection rates under each combination of type I and type II error probabilities (η^I, η^{II}), from CKMS sample (A), CMW sample (B), and CS sample (C).

Rationale behind the test. We now turn to a more detailed exposition of how we derive our test. Let H_0 and H_1 denote the null hypothesis that the true dataset D_{true} is OEU rational and the alternative hypothesis that D_{true} is not OEU rational. To construct our test, consider a number $\mathcal{E}^*$, which is the result of the following optimization problem. Given a dataset $D_{\text{true}} = (p^k, x^k)_{k=1}^K$:

$$\begin{aligned} \min_{(v_s^k, \lambda^k, \varepsilon_s^k)_{s,k}} \quad & \max_{k \in K, s,t \in S} \frac{\varepsilon_s^k}{\varepsilon_t^k} \\ \text{s.t.} \quad & \log \mu_s^* + \log v_s^k - \log \lambda^k - \log p_s^k - \log \varepsilon_s^k = 0 \\ & x_s^k > x_{s'}^{k'} \implies \log v_s^k \leq \log v_{s'}^{k'}. \end{aligned} \tag{13}$$

Under H_0, the true dataset $D_{\text{true}} = (p^k, x^k)_{k=1}^K$ is OEU rational. A slight modification of Lemma 7 in Echenique and Saito (2015) then implies that there exist strictly positive numbers $\widetilde{v}_s^k$, and $\widetilde{\lambda}^k$ for $s \in S$ and $k \in K$ such that

$$\log \mu_s^* + \log \widetilde{v}_s^k - \log \widetilde{\lambda}^k - \log p_s^k = 0 \quad \text{and} \quad x_s^k > x_{s'}^{k'} \implies \log \widetilde{v}_s^k \leq \log \widetilde{v}_{ts}^{k'}.$$

Substituting the relationship $\tilde{p}_s^k = p_s^k \varepsilon_s^k$ for all $s \in S$ and $k \in K$ yields

$$\log \mu_s^* + \log \widetilde{v}_s^k - \log \widetilde{\lambda}^k - \log \tilde{p}_s^k = \log \varepsilon_s^k \quad \text{and} \quad x_s^k > x_{s'}^{k'} \Longrightarrow \log \widetilde{v}_s^k \leq \log \widetilde{v}_{s'}^{k'},$$

which implies that the tuple $(\widetilde{v}_s^k, \widetilde{\lambda}^k, \varepsilon_s^k)_{s,k}$ satisfies the constraint in problem (13).

Letting $\mathcal{E}^* \left((p^k, x^k)_{k=1}^K \right)$ denote the optimal value of the problem (13), we have

$$\mathcal{E}^* \left((p^k, x^k)_{k=1}^K \right) \leq \max_{k \in K, s, t \in S} \frac{\varepsilon_s^k}{\varepsilon_s^k} = \widehat{\mathcal{E}}$$

under the null hypothesis.

Then, we construct a test as follows:

$$\begin{cases} \text{reject } H_0 & \text{if } \displaystyle\int_{\mathcal{E}^*\left((p^k, x^k)_{k=1}^K\right)}^{\infty} f_{\widehat{\mathcal{E}}}(z) dz < \alpha \\ \text{accept } H_0 & \text{otherwise} \end{cases},$$

where α is the size of the test and $f_{\widehat{\mathcal{E}}}$ is the density function of the distribution of $\widehat{\mathcal{E}} = \max_{k,s,t} \varepsilon_s^k / \varepsilon_t^k$. Given a nominal size α, we can find a critical value C_α satisfying $\Pr[\widehat{\mathcal{E}} > C_\alpha] = \alpha$; we set $C_\alpha = F_{\widehat{\mathcal{E}}}^{-1}(1-\alpha)$, where $F_{\widehat{\mathcal{E}}}$ denotes the cumulative distribution function of $\widehat{\mathcal{E}}$. However, because $\mathcal{E}^* \left((p^k, x^k)_{k=1}^K \right) \leq \widehat{\mathcal{E}}$, the true size of the test is better than α. Concretely,

$$\text{size} = \Pr[\mathcal{E}^* > C_\alpha] \leq \Pr[\widehat{\mathcal{E}} > C_\alpha] = \alpha.$$

Parameter tuning. In order to perform the test, we need to obtain the distribution of $\widehat{\mathcal{E}}$ and its critical value C_α given a significance level α. We obtain the distribution of $\widehat{\mathcal{E}}$ by assuming that ε follows a log-normal distribution $\varepsilon \sim \Lambda(\nu, \xi^2)$. [15]

The crucial step in our approach is the selection of parameters (ν, ξ^2). It is natural to choose these parameters so that there is no price perturbation on average (i.e., $\mathbf{E}[\varepsilon] = 1$). However, as we discussed above, there is no objective guide to choosing an appropriate level of $\text{Var}(\varepsilon)$. Therefore, we use variation in (relative) prices observed in the data.

[15] Note that parameters (ν, ξ^2) correspond to the mean and the variance of the random variable in the log-scale. In other words, $\log \varepsilon \sim N(\nu, \xi^2)$. The moments of the log-normal distribution $\varepsilon \sim \Lambda(\nu, \xi^2)$ are then calculated by $\mathbf{E}[\varepsilon] = \exp(\nu + \xi^2/2)$ and $\text{Var}(\varepsilon) = \exp(2\nu + \xi^2)(\exp(\xi^2) - 1)$.

We have assumed that $\tilde{p}_s^k = p_s^k \varepsilon_s^k$ for all $s \in S$, $k \in K$, and the noise term ε is independent of the random selection of budgets $(p_s^k)_{k,s}$. Hence,

$$\begin{aligned} &\mathrm{Var}(\tilde{p}) = \mathrm{Var}(p) \cdot \mathrm{Var}(\varepsilon) + \mathrm{Var}(p) \cdot \mathbf{E}[\varepsilon]^2 + \mathbf{E}[p]^2 \cdot \mathrm{Var}(\varepsilon) \\ \iff \quad &\frac{\mathrm{Var}(\tilde{p})}{\mathrm{Var}(p)} = \mathbf{E}[\varepsilon]^2 + \left(1 + \frac{\mathbf{E}[p]^2}{\mathrm{Var}(p)}\right) \mathrm{Var}(\varepsilon). \end{aligned}$$

Given the observed variation in $(p_s^k)_{k,s}$, $\mathrm{Var}(\varepsilon)$ determines how much larger (or smaller, in ratio) the variation of perturbed prices $(\tilde{p}_s^k)_{k,s}$ is relative to actual prices.

Our agent has trouble telling the two variances apart. More generally, the agent has trouble telling the distributions of prices apart, that is why she is confusing actual and perceived prices, but the distribution depends only on the variance; so we focus on variance. Consider a hypothesis test for the null hypothesis that the variance of a normal random variable with known mean has variance σ_0^2 against the alternative that $\sigma^2 \geq \sigma_0^2$. Let $\hat{\sigma}_n^2$ be the sample variance.

The agent performs an upper-tailed chi-squared test defined as

$$\begin{aligned} \mathrm{H}_0 &: \ \sigma^2 = \sigma_0^2 \\ \mathrm{H}_1 &: \ \sigma^2 > \sigma_0^2 \end{aligned}$$

The test statistic is:

$$T_n = \frac{(n-1)\hat{\sigma}_n^2}{\sigma_0^2}$$

where n is the sample size (i.e., the number of budget sets). The sampling distribution of the test statistic T_n under the null hypothesis follows a chi-squared distribution with $n-1$ degrees of freedom.

We consider the probability η^I of rejecting the null hypothesis when it is true, a type I error; and the probability η^{II} of failing to reject the null hypothesis when the alternative $\sigma^2 = \sigma_1^2 > \sigma_0^2$ is true, a type II error. The test rejects the null hypothesis that the variance is σ_0^2 if

$$T_n > \chi^2_{1-\alpha,n-1}$$

where $\chi^2_{1-\alpha,n-1}$ is the critical value of a chi-squared distribution with $n-1$ degree of freedom at the significance level α, defined by $\Pr[\chi^2 < \chi^2_{1-\alpha,n-1}] = 1 - \eta^I$. [16]

[16] An alternative approach, without assuming that a distribution for T_n, and based on a large sample approximation to the distribution of T_n, yields very similar results. Calculations and empirical findings are available from the authors upon request.

Under the alternative hypothesis that $\sigma^2 = \sigma_1^2 > \sigma_0^2$, the statistic $(\sigma_0^2/\sigma_1^2) \cdot T_n$ follows a chi-squared distribution (with $n-1$ degrees of freedom). Then, the probability η^{II} of making a type II error is given by

$$\begin{aligned}\eta^{II} &= \Pr[T_n < \chi^2_{1-\alpha,n-1} \mid \mathrm{H}_1 : \sigma_1^2 > \sigma_0^2 \text{ is true}] \\ &= \Pr\left[\frac{\sigma_0^2}{\sigma_1^2} \cdot T_n < \frac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{1-\alpha,n-1}\right] \\ &= \Pr\left[\chi^2 < \frac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{1-\alpha,n-1}\right].\end{aligned}$$

Let $\chi^2_{\beta,n-1}$ be the value that satisfies $\Pr[\chi^2 < \chi^2_{\beta,n-1}] = \eta^{II}$. Then, given η^I and η^{II}, we obtain

$$\Pr\left[\chi^2 < \frac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{1-\alpha,n-1}\right] = \eta^{II} \iff \frac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{1-\alpha,n-1} = \chi^2_{\beta,n-1} \iff \frac{\sigma_1^2}{\sigma_0^2} = \frac{\chi^2_{1-\alpha,n-1}}{\chi^2_{\beta,n-1}}.$$

As a consequence, given a measured variance σ_0^2, calculated from observed prices, and assumed values for η^I and η^{II}, we can back out the minimum "detectable" value of the variance σ_1^2. From this variance of prices, we obtain $\mathrm{Var}(\varepsilon)$.

5 Perturbed Subjective Expected Utility

We now turn to the model of subjective expected utility, in which beliefs are not known. Instead, beliefs are subjective and unobservable. The analysis will be analogous to what we did for OEU, and therefore proceed at a faster pace. In particular, all the definitions and results parallel those of the section on OEU. The proof of the main result (the axiomatic characterization) is substantially more challenging here because both beliefs and utilities are unknown: there is a classical problem in disentangling beliefs from utility. The technique for solving this problem was introduced in Echenique and Saito (2015).

Definition 9. *Let $e \in \mathbf{R}_+$. A dataset $(x^k, p^k)_{k=1}^K$ is* e-belief-perturbed SEU rational *if there exist $\mu^k \in \Delta_{++}$ for each $k \in K$ and a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$ such that, for all k,*

$$y \in B(p^k, p^k \cdot x^k) \implies \sum_{s \in S} \mu_s^k u(y_s) \le \sum_{s \in S} \mu_s^k u(x_s^k) \tag{14}$$

and for each $k, l \in K$ and $s, t \in S$

$$\frac{\mu_s^k/\mu_t^k}{\mu_s^l/\mu_t^l} \le 1 + e. \tag{15}$$

Note that the definition of e-belief-perturbed SEU rational differs from the definition of belief-perturbed OEU rationality, only in condition (15); establishing bounds the perturbations. Here there is no objective probability from which we can evaluate the deviation of the set $\{\mu^k\}$ of beliefs. Thus we evaluate perturbations *among* beliefs, as in (15).

Remark 1. *The constraint on the perturbation applies for each $k, l \in K$ and $s, t \in S$, so it implies for each $k, l \in K$ and $s, t \in S$*

$$\frac{1}{1+e} \leq \frac{\mu_s^k/\mu_t^k}{\mu_s^l/\mu_t^l} \leq 1+e.$$

Hence, when $e = 0$, it must be that $\mu_s^k/\mu_t^k = \mu_s^l/\mu_t^l$. This implies that $\mu^k = \mu^l$ for a dataset that is 0*-belief perturbed SEU rational.*

Next, we propose perturbed SEU rationality with respect to prices.

Definition 10. *Let $e \in \mathbf{R}_+$. A dataset $(x^k, p^k)_{k=1}^K$ is* e-price-perturbed SEU rational *if there exist $\mu \in \Delta_{++}$ and a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$ and $\varepsilon^k \in \mathbf{R}_+^S$ for each $k \in K$ such that, for all k,*

$$y \in B(\tilde{p}^k, \tilde{p}^k \cdot x^k) \implies \sum_{s \in S} \mu_s u(y_s) \leq \sum_{s \in S} \mu_s u(x_s^k), \tag{16}$$

where for each $k \in K$ and $s \in S$

$$\tilde{p}_s^k = p_s^k \varepsilon_s^k, \tag{17}$$

and for each $k, l \in K$ and $s, t \in S$

$$\frac{\varepsilon_s^k/\varepsilon_t^k}{\varepsilon_s^l/\varepsilon_t^l} \leq 1+e. \tag{18}$$

Again, the definition differs from the corresponding definition of price-perturbed OEU rationality only in condition (18), establishing bounds on perturbations. In condition (18), we measure the size of the perturbations by

$$\frac{\varepsilon_s^k/\varepsilon_t^k}{\varepsilon_s^l/\varepsilon_t^l},$$

not $\varepsilon_s^k/\varepsilon_t^k$ as in (9). This change is necessary to accommodate the existence of subjective beliefs. By choosing subjective beliefs appropriately, one can neutralize the perturbation in prices if $\varepsilon_s^k/\varepsilon_t^k = \varepsilon_s^l/\varepsilon_t^l$ for all $k, l \in K$. That is, as long as $\varepsilon_s^k/\varepsilon_t^k = \varepsilon_s^l/\varepsilon_t^l$ for all $k, l \in K$, if we can rationalize the dataset by introducing the noise with some subjective belief μ, then without using the noise, we can rationalize the dataset with another subjective belief μ' such that $\varepsilon_s^k \mu_s'/\varepsilon_t^k \mu_t' = \mu_s/\mu_t$.

Finally, we define utility-perturbed SEU rationality.

Definition 11. *Let $e \in \mathbf{R}_+$. A dataset $(x^k, p^k)_{k=1}^K$ is* e-utility-perturbed SEU rational *if there exist $\mu \in \Delta_{++}$, a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$, and $\varepsilon^k \in \mathbf{R}_+^S$ for each $k \in K$ such that, for all k,*

$$y \in B(p^k, p^k \cdot x^k) \implies \sum_{s \in S} \mu_s \varepsilon_s^k u(y_s) \leq \sum_{s \in S} \mu_s \varepsilon_s^k u(x_s^k), \tag{19}$$

and for each $k \in K$ and $s, t \in S$

$$\frac{1}{1+e} \leq \frac{\varepsilon_s^k / \varepsilon_t^k}{\varepsilon_s^l / \varepsilon_t^l} \leq 1 + e. \tag{20}$$

As in the previous section, given e, we can show that these three concepts of rationality are equivalent.

Theorem 3. *Let $e \in \mathbf{R}_+$ and D be a dataset. The following are equivalent:*

- *D is e-belief-perturbed SEU rational;*
- *D is e-price-perturbed SEU rational;*
- *D is e-utility-perturbed SEU rational.*

In light of Theorem 3, we shall speak simply of e-perturbed SEU rationality to refer to any of the above notions of perturbed SEU rationality.

Echenique and Saito (2015) prove that a dataset is SEU rational if and only if it satisfies a revealed-preference axiom termed the Strong Axiom for Revealed Subjective Expected Utility (SARSEU). SARSEU states that, for any test sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$, if each s appears as s_i (on the left of the pair) the same number of times it appears as s_i' (on the right), then

$$\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq 1.$$

SARSEU is no longer necessary for perturbed SEU-rationality. This is easy to see, as we allow the decision maker to have a different belief μ^k for each choice k, and reason as in our discussion of SAROEU. Analogous to our analysis of OEU, we introduce a perturbed version of SARSEU to capture perturbed SEU rationality. Let $e \in \mathbf{R}_+$.

Axiom 2 (e-Perturbed SARSEU (e-PSARSEU))**.** *For any test sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n \equiv \sigma$, if each s appears as s_i (on the left of the pair) the same number of times it appears as s_i' (on the right), then*

$$\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq (1+e)^{m(\sigma)}.$$

We can easily see the necessity of e-PSARSEU by reasoning from the first order conditions, as in our discussion of e-PSAROEU. The main result of this section shows that e-PSARSEU is not only necessary for e-perturbed SEU rationality, but also sufficient.

Theorem 4. *Let $e \in \mathbf{R}_+$ and D be a dataset. The following are equivalent:*

- *D is e-perturbed SEU rational;*
- *D satisfies e-PSARSEU.*

It is easy to see that 0-PSARSEU is equivalent to SARSEU, and that by choosing e to be arbitrarily large it is possible to rationalize any dataset. As a consequence, we shall be interested in finding a minimal value of e that rationalizes a dataset: such "minimal e" is also denoted by e_*.

We should mention, as in the case of OEU, that e_* depends on the prices which a decision maker faces. It is clear from e-PSARSEU that $1+e$ is bounded by the maximum ratio of prices (i.e., $\max_{k,k' \in K, s,s' \in S} p^k_s / p^{k'}_{s'}$).

6 Proofs

6.1 Proof of Theorems 1 and 2

First, we prove a lemma which shows Theorem 1 and is useful for the sufficiency part of Theorem 2.

Lemma 1. *Given $e \in \mathbf{R}_+$, let $(x^k, p^k)_{k=1}^K$ be a dataset. The following statements are equivalent:*

1. *$(x^k, p^k)_{k=1}^K$ is e-belief-perturbed OEU rational.*
2. *There are strictly positive numbers v^k_s, λ^k, μ^k_s, for $s \in S$ and $k \in K$, such that*

$$\mu^k_s v^k_s = \lambda^k p^k_s, \quad x^k_s > x^{k'}_{s'} \implies v^k_s \le v^{k'}_{s'}, \tag{21}$$

and for all $k \in K$ and $s, t \in S$

$$\frac{1}{1+e} \le \frac{\mu^k_s / \mu^k_t}{\mu^*_s / \mu^*_t} \le 1 + e. \tag{22}$$

3. *$(x^k, p^k)_{k=1}^K$ is e-price-perturbed OEU rational.*

4. *There are strictly positive numbers* $\hat{v}_s^k$, $\hat{\lambda}^k$, *and* ε_s^k *for* $s \in S$ *and* $k \in K$, *such that*

$$\mu_s^* \hat{v}_s^k = \hat{\lambda}^k \varepsilon_s^k p_s^k, \quad x_s^k > x_{s'}^{k'} \implies \hat{v}_s^k \leq \hat{v}_{s'}^{k'},$$

and for all $k \in K$ *and* $s, t \in S$

$$\frac{1}{1+e} \leq \frac{\varepsilon_s^k}{\varepsilon_t^k} \leq 1 + e.$$

5. $(x^k, p^k)_{k=1}^K$ *is e-utility-perturbed OEU rational.*

6. *There are strictly positive numbers* $\hat{v}_s^k$, $\hat{\lambda}^k$, *and* $\hat{\varepsilon}_s^k$ *for* $s \in S$ *and* $k \in K$, *such that*

$$\mu_s^* \hat{\varepsilon}_s^k \hat{v}_s^k = \hat{\lambda}^k p_s^k, \quad x_s^k > x_{s'}^{k'} \implies \hat{v}_s^k \leq \hat{v}_{s'}^{k'},$$

and for all $k \in K$ *and* $s, t \in S$

$$\frac{1}{1+e} \leq \frac{\hat{\varepsilon}_s^k}{\hat{\varepsilon}_t^k} \leq 1 + e.$$

Proof. By the standard way, the equivalence between 1 and 2, the equivalence between 3 and 4, and the equivalence between 5 and 6 hold. Moreover, it is easy to see the equivalence between 4 and 6 with $\varepsilon_s^k = 1/\hat{\varepsilon}_s^k$ for each $k \in K$ and $s \in S$. So to show the result, it suffices to show that 2 and 4 are equivalent.

To show 4 implies 2, define $v = \hat{v}$ and

$$\mu_s^k = \frac{\mu_s^*}{\varepsilon_s^k} \Bigg/ \left(\sum_{s \in S} \frac{\mu_s^*}{\varepsilon_s^k} \right)$$

for each $k \in K$ and $s \in S$ and

$$\lambda^k = \hat{\lambda}^k \Bigg/ \left(\sum_{s \in S} \frac{\mu_s^*}{\varepsilon_s^k} \right)$$

for each $k \in K$. Then, $\mu^k \in \Delta_{++}(S)$. Since $\mu_s^* \hat{v}_s^k = \hat{\lambda}^k \varepsilon_s^k p_s^k$, we have

$$\mu_s^k v_s^k = \lambda^k p_s^k.$$

Moreover, for each $k \in K$ and $s, t \in S$

$$\frac{\varepsilon_s^k}{\varepsilon_t^k} = \frac{\mu_s^k / \mu_t^k}{\mu_s^* / \mu_t^*}.$$

Hence,

$$\frac{1}{1+e} \leq \frac{\varepsilon_s^k}{\varepsilon_t^k} \leq 1+e.$$

To show that 2 implies 4, for all $s \in S$ define $\hat{v} = v$ and for all $k \in K$, $\hat{\lambda}^k = \lambda^k$. For all $k \in K$ and $s \in S$, define

$$\varepsilon_s^k = \frac{\mu_s^*}{\mu_s^k}.$$

For each $k \in K$ and $s \in S$, since $\mu_s^k u_s^k = \lambda^k p_s^k$,

$$\mu_s^* v_s^k = \hat{\lambda}^k \varepsilon_s^k p_s^k.$$

Finally, for each $k \in K$ and $s, t \in S$,

$$\frac{\varepsilon_s^k}{\varepsilon_t^k} = \frac{\mu_s^*/\mu_s^k}{\mu_t^*/\mu_t^k} = \frac{\mu_t^k/\mu_s^k}{\mu_t^*/\mu_s^*}.$$

Therefore, we obtain

$$\frac{1}{1+e} \leq \frac{\varepsilon_s^k}{\varepsilon_t^k} \leq 1+e.$$

□

6.1.1 Necessity of Theorem 2

Lemma 2. *Given $e \in \mathbf{R}_+$, if a data set is e-belief-perturbed OEU rational, then the data set satisfies e-PSAROEU.*

Proof. Fix any sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n \equiv \sigma$ of pairs satisfies conditions (1) and (2). Assuming differentiability of u and interior solution for simplicity, we have for each $k \in K$ and $s \in S$, $\mu_s^k u'(x_s^k) = \lambda^k p_s^k$, or

$$\frac{\mu_s^k}{\mu_s^*} u'(x_s^k) = \lambda^k \rho_s^k.$$

Then,

$$\prod_{i=1}^n \frac{\rho_{s_i}^{k_i}}{\rho_{s_i'}^{k_i'}} = \prod_{i=1}^n \frac{\lambda^{k_i'}(\mu_{s_i}^{k_i}/\mu_{s_i}^*)u'(x_{s_i}^{k_i})}{\lambda^{k_i}(\mu_{s_i'}^{k_i'}//\mu_{s_i'}^*)u'(x_{s_i'}^{k_i'})} = \prod_{i=1}^n \frac{u'(x_{s_i}^{k_i})}{u'(x_{s_i'}^{k_i'})} \prod_{i=1}^n \frac{\mu_{s_i}^{k_i}/\mu_{s_i}^*}{\mu_{s_i'}^{k_i'}/\mu_{s_i'}^*}.$$

The second equality holds by condition (2). By condition (1), the first term is less than one because of the concavity of u. In the following, we evaluate the second term. First, for each (k, s) cancel out the same μ_s^k as much as possible both from the denominator and the numerator. Then, the number of μ_s^k remained in the numerator is $d(\sigma, k, s)$.

Since the number of numerator and the denominator must be the same. The number of remaining fraction is $m(\sigma) \equiv \sum_{s\in S}\sum_{k\in K:d(\sigma,k,s)>0} d(\sigma,k,s)$. So by relabeling the index i to j if necessary, we obtain

$$\prod_{i=1}^{n} \frac{\mu_{s_i}^{k_i}/\mu_{s_i}^*}{\mu_{s_i'}^{k_i'}/\mu_{s_i'}^*} = \prod_{j=1}^{m(\sigma)} \frac{\mu_{s_j}^{k_j}/\mu_{s_j}^*}{\mu_{s_j'}^{k_j'}/\mu_{s_j'}^*}.$$

Consider the corresponding sequence $(x_{s_j}^{k_j}, x_{s_j'}^{k_j'})_{j=1}^{m(\sigma)}$. Since the sequence is obtained by canceling out x_s^k from the first element and the second element of the pairs the same number of times; and since the original sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n}$ satisfies condition (2), it follows that $(x_{s_j}^{k_j}, x_{s_j'}^{k_j'})_{j=1}^{m(\sigma)}$ satisfies condition (2).

By condition (2), we can assume without loss of generality that $k_j = k_j'$ for each j. Therefore, by the condition on the perturbation,

$$\prod_{j=1}^{m(\sigma)} \frac{\mu_{s_j}^{k_j}/\mu_{s_j}^*}{\mu_{s_j'}^{k_j'}/\mu_{s_j'}^*} \leq (1+e)^{m(\sigma)}.$$

Hence,

$$\prod_{i=1}^{n} \frac{\rho_{s_i}^{k_i}}{\rho_{s_i'}^{k_i'}} \leq (1+e)^{m(\sigma)}.$$

□

6.1.2 Sufficiency of Theorem 2

We need three more lemmas to prove the sufficiency.

Lemma 3. *Given $e \in \mathbf{R}_+$, let a dataset $(x^k, p^k)_{k=1}^k$ satisfy e-PSAROEU. Suppose that $\log(p_s^k) \in \mathbf{Q}$ for all $k \in K$ and $s \in S$, $\log(\mu_s^*) \in \mathbf{Q}$ for all $s \in S$, and $\log(1+e) \in \mathbf{Q}$. Then there are numbers v_s^k, λ^k, μ_s^k, for $s \in S$ and $k \in K$ satisfying (21) and (22) in Lemma 1.*

Proof of Lemma 3 The proof is similar to the case in which $e = 0$. By log-linearizing conditions (21) and (22) in Lemma 1, we have for all $s \in S$ and $k \in K$, such that

$$\log \mu_s^k + \log v_s^k = \log \lambda^k + \log p_s^k, \tag{23}$$

$$x_s^k > x_{s'}^{k'} \implies \log v_s^k \leq \log v_{s'}^{k'}, \tag{24}$$

and for all $k \in K$ and $s, t \in S$

$$-\log(1+e) + \log \mu_s^* - \log \mu_t^* \leq \log \mu_s^k - \log \mu_t^k \leq \log(1+e) + \log \mu_s^* - \log \mu_t^*. \quad (25)$$

Matrix A looks as follows:

$$\begin{array}{c} \\ \\ (k,s) \\ (k,t) \\ (l,s) \\ (l,t) \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc|cccccc|ccccc|c} \cdots & v_s^k & v_t^k & v_s^l & v_t^l & \cdots & \cdots & \mu_s^k & \mu_t^k & \mu_s^l & \mu_t^l & \cdots & \cdots & \lambda^k & \lambda^l & \cdots & & p \end{array} \\ \left[\begin{array}{cccccc|cccccc|cccc|c} & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ \cdots & 1 & 0 & 0 & 0 & \cdots & \cdots & 1 & 0 & 0 & 0 & \cdots & \cdots & -1 & 0 & \cdots & -\log p_s^k \\ \cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & -1 & 0 & \cdots & -\log p_s^k \\ \cdots & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & -1 & \cdots & -\log p_s^l \\ \cdots & 0 & 0 & 0 & 1 & \cdots & \cdots & 0 & 0 & 0 & 1 & \cdots & \cdots & 0 & -1 & \cdots & -\log p_s^l \\ & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \end{array}\right] \end{array}.$$

Matrix B has additional rows as follows in addition to the rows in Echenique and Saito (2015):

$$\begin{array}{c} \begin{array}{cccccc|cccccc|cccc|c} \cdots & v_s^k & v_t^k & v_s^l & v_t^l & \cdots & \cdots & \mu_s^k & \mu_t^k & \mu_s^l & \mu_t^l & \cdots & \cdots & \lambda^k & \lambda^l & \cdots & p \end{array} \\ \left[\begin{array}{cccccc|cccccc|cccc|c} & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & -1 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \log(1+e) - \log \mu_s^* + \log \mu_t^* \\ \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 1 & -1 & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \log(1+e) + \log \mu_s^* - \log \mu_t^* \\ \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & -1 & \cdots & \cdots & 0 & 0 & \cdots & \log(1+e) + \log \mu_s^* - \log \mu_t^* \\ \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & 1 & \cdots & \cdots & 0 & 0 & \cdots & \log(1+e) - \log \mu_s^* + \log \mu_t^* \\ & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \end{array}\right] \end{array}.$$

Matrix E is the same as in Echenique and Saito (2015).

The entries of A, B, and E are either 0, 1 or -1, with the exception of the last column of A. Under the hypotheses of the lemma we are proving, the last column consists of rational numbers. By Motzkin's theorem, then, there is such a solution u to $S1$ if and only if there is no rational vector (θ, η, π) that solves the system of equations and linear inequalities

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

Claim There exists a sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*} \equiv \sigma$ of pairs that satisfies conditions (1) and (2) in e-PSAROEU.

Proof. Denote the weight on the rows capturing $\log \mu_s^k - \log \mu_t^k \leq \log(1+e) + \log \mu_s^* - \log \mu_t^*$ by $\theta(k, s, t)$. Then, notice that the corresponding constraint $-\log(1+e) + \log \mu_s^* - \log \mu_t^* \leq$

$\log \mu_s^k - \log \mu_t^k$ is denoted by $\theta(k,t,s)$. So for each $k \in K$ and $s \in S$,

$$n(x_s^k) - n'(x_s^k) + \sum_{t \neq s} \Big[-\theta(k,s,t) + \theta(k,t,s) \Big] = 0$$

Hence

$$\sum_{s \in S} \Big[n(x_s^k) - n'(x_s^k) \Big] = \sum_{s \in S} \sum_{t \neq s} \Big[\theta(k,s,t) - \theta(k,t,s) \Big] = 0$$

□

Claim $\prod_{i=1}^{n^*} \frac{\rho_{s_i}^{k_i}}{\rho_{s_i'}^{k_i'}} > (1+e)^{m(\sigma^*)}$.

Proof. By the fact that the last column must sum up to zero and E has one at the last column, we have

$$\sum_{i=1}^{n^*} \log \frac{p_{s_i'}^{k_i'}}{p_{s_i}^{k_i}} + \log(1+e) \sum_{k \in K} \sum_{s \in S} \sum_{t \neq s} \theta(k,s,t) + \sum_{k \in K} \sum_{s \in S} \sum_{t \neq s} (\theta(k,s,t) - \theta(k,t,s)) \log \mu_s^* = -\pi < 0.$$

Remember that for all $k \in K$ and $s \in S$,

$$n(x_s^k) - n'(x_s^k) = \sum_{t \neq s} \Big[\theta(k,s,t) - \theta(k,t,s) \Big].$$

So for each $s \in S$

$$\sum_{k \in K} \sum_{s \in S} \sum_{t \neq s} \Big[\theta(k,s,t) - \theta(k,t,s) \Big] \log \mu_s^* = \sum_{i=1}^{n^*} \log \frac{\mu_{s_i}^*}{\mu_{s_i'}^*}.$$

Hence,

$$\begin{aligned} 0 &> -\pi \\ &= \sum_{i=1}^{n^*} \log \frac{p_{s_i'}^{k_i'}}{p_{s_i}^{k_i}} - \sum_{i=1}^{n^*} \log \frac{\mu_{s_i}^*}{\mu_{s_i'}^*} + \log(1+e) \sum_{k \in K} \sum_{s \in S} \sum_{t \neq s} \theta(k,s,t) \\ &= \sum_{i=1}^{n^*} \log \frac{\rho_{s_i'}^{k_i'}}{\rho_{s_i}^{k_i}} + \log(1+e) \sum_{k \in K} \sum_{s \in S} \sum_{t \neq s} \theta(k,s,t). \end{aligned}$$

Since $d(\sigma^*, k, s) = n(x_s^k) - n'(x_s^k) = \sum_{t \neq s} \Big[\theta(k,s,t) - \theta(k,t,s) \Big] \leq \sum_{t \neq s} \theta(k,s,t)$, we have

$$m(\sigma^*) \equiv \sum_{s \in S} \sum_{k \in K : d(\sigma^*, k, s) > 0} d(\sigma^*, k, s) = \sum_{s \in S} \sum_{k \in K} \min\{n(x_s^k) - n'(x_s^k), 0\} \leq \sum_{s \in S} \sum_{k \in K} \sum_{t \neq s} \theta(k,s,t).$$

Therefore

$$0 > \sum_{i=1}^{n^*} \log \frac{\rho_{s'_i}^{k'_i}}{\rho_{s_i}^{k_i}} + \log(1+e) \sum_{k\in K} \sum_{s\in S} \sum_{t\neq s} \theta(k,s,t) \geq \sum_{i=1}^{n^*} \log \frac{\rho_{s'_i}^{k'_i}}{\rho_{s_i}^{k_i}} + \log(1+e) m(\sigma^*).$$

That is, $\sum_{i=1}^{n^*} \log \frac{\rho_{s_i}^{k_i}}{\rho_{s'_i}^{k'_i}} > m(\sigma^*) \log(1+e)$. This is a contradiction. □

Lemma 4. *Given $e \in \mathbf{R}_+$, let a dataset $(x^k, p^k)_{k=1}^k$ satisfy e-PSAROEU with respect to μ^*. Then for all positive numbers $\overline{\varepsilon}$, there exist a positive real numbers $e' \in [e, e+\overline{\varepsilon}]$, $\mu'_s \in [\mu^*_s - \overline{\varepsilon}, \mu^*_s + \overline{\varepsilon}]$, and $q_s^k \in [p_s^k - \overline{\varepsilon}, p_s^k]$ for all $s \in S$ and $k \in K$ such that $\log q_s^k \in \mathbf{Q}$ for all $s \in S$ and $k \in K$, $\log(\mu'_s) \in \mathbf{Q}$ for all $s \in S$, and $\log(1+e') \in \mathbf{Q}$, $\mu' \in \Delta_{++}(S)$, and the dataset $(x^k, q^k)_{k=1}^k$ satisfy e'-PSAROEU with respect to μ'.*

Proof of Lemma 4 Consider the set of sequences that satisfy Conditions (1) and (2) in PSAROEU(e):

$$\Sigma = \left\{ (x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^n \subset \mathcal{X}^2 \;\middle|\; \begin{array}{l} (x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^n \text{ satisfies conditions (1) and (2)} \\ \text{in } e\text{-PSAROEU for some } n \end{array} \right\}.$$

For each sequence $\sigma \in \Sigma$, we define a vector $t_\sigma \in \mathbf{N}^{K^2 S^2}$ as in Lemma 9.

Define δ as in Lemma 9. Then, δ is a K^2S^2-dimensional real-valued vector. If $\sigma = (x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^n$, then

$$\delta \cdot t_\sigma = \sum_{((k,s),(k',s'))\in(KS)^2} \delta((k,s),(k',s')) t_\sigma((k,s),(k',s')) = \log\left(\prod_{i=1}^n \frac{\rho_{s_i}^{k_i}}{\rho_{s'_i}^{k'_i}} \right).$$

So the dataset satisfies e-PSAROEU with respect to μ if and only if $\delta \cdot t_\sigma \leq m(\sigma)\log(1+e)$ for all $\sigma \in \Sigma$.

Enumerate the elements in $\mathcal{X}$ in increasing order: $y_1 < y_2 < \cdots < y_N$. And fix an arbitrary $\underline{\xi} \in (0,1)$. We shall construct by induction a sequence $\{(\varepsilon_s^k(n))\}_{n=1}^N$, where $\varepsilon_s^k(n)$ is defined for all (k,s) with $x_s^k = y_n$.

By the denseness of the rational numbers, and the continuity of the exponential function, for each (k,s) such that $x_s^k = y_1$, there exists a positive number $\varepsilon_s^k(1)$ such that $\log(\rho_s^k \varepsilon_s^k(1)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon_s^k(1) < 1$. Let $\varepsilon(1) = \min\{\varepsilon_s^k(1) \mid x_s^k = y_1\}$.

In second place, for each (k,s) such that $x_s^k = y_2$, there exists a positive $\varepsilon_s^k(2)$ such that $\log(\rho_s^k \varepsilon_s^k(2)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon_s^k(2) < \varepsilon(1)$. Let $\varepsilon(2) = \min\{\varepsilon_s^k(2) \mid x_s^k = y_2\}$.

In third place, and reasoning by induction, suppose that $\varepsilon(n)$ has been defined and that $\underline{\xi} < \varepsilon(n)$. For each (k,s) such that $x^k_s = y_{n+1}$, let $\varepsilon^k_s(n+1) > 0$ be such that $\log(\rho^k_s \varepsilon^k_s(n+1)) \in \mathbf{Q}$, and $\underline{\xi} < \varepsilon^k_s(n+1) < \varepsilon(n)$. Let $\varepsilon(n+1) = \min\{\varepsilon^k_s(n+1) \mid x^k_s = y_n\}$.

This defines the sequence $(\varepsilon^k_s(n))$ by induction. Note that $\varepsilon^k_s(n+1)/\varepsilon(n) < 1$ for all n. Let $\bar{\xi} < 1$ be such that $\varepsilon^k_s(n+1)/\varepsilon(n) < \bar{\xi}$.

For each $k \in K$ and $s \in S$, let $\hat{\rho}^k_s = \rho^k_s \varepsilon^k_s(n)$, where n is such that $x^k_s = y_n$. Choose $\mu' \in \Delta_{++}(S)$ such that for all $s \in S$ $\log \mu'_s \in \mathbf{Q}$ and $\mu'_s \in [\bar{\xi}\mu_s, \mu_s/\bar{\xi}]$ for all $s \in S$. Such μ' exists by the denseness of the rational numbers. Now for each $k \in K$ and $s \in S$, define

$$q^k_s = \frac{\hat{\rho}^k_s}{\mu'_s}. \tag{26}$$

Then, $\log q^k_s = \log \hat{\rho}^k_s - \log \mu'_s \in \mathbf{Q}$.

We claim that the dataset $(x^k, q^k)^K_{k=1}$ satisfies e'-PSAROEU with respect to μ'. Let δ^* be defined from $(q^k)^K_{k=1}$ in the same manner as δ was defined from $(\rho^k)^K_{k=1}$.

For each pair $((k,s),(k',s'))$ with $x^k_s > x^{k'}_{s'}$, if n and m are such that $x^k_s = y_n$ and $x^{k'}_{s'} = y_m$, then $n > m$. By definition of ε,

$$\frac{\varepsilon^k_s(n)}{\varepsilon^{k'}_{s'}(m)} < \frac{\varepsilon^k_s(n)}{\varepsilon(m)} < \bar{\xi} < 1.$$

Hence,

$$\delta^*((k,s),(k',s')) = \log \frac{\rho^k_s \varepsilon^k_s(n)}{\rho^{k'}_{s'} \varepsilon^{k'}_{s'}(m)} < \log \frac{\rho^k_s}{\rho^{k'}_{s'}} + \log \bar{\xi} < \log \frac{\rho^k_s}{\rho^{k'}_{s'}} = \delta((k,s),(k',s')).$$

Now, we choose e' such that $e' \geq e$ and $\log(1+e') \in \mathbf{Q}$.

Thus, for all $\sigma \in \Sigma$, $\delta^* \cdot t_\sigma \leq \delta \cdot t_\sigma \leq m(\sigma)\log(1+e) \leq m(\sigma)\log(1+e')$ as $t_\cdot \geq 0$ and the dataset $(x^k, p^k)^K_{k=1}$ satisfies e-PSAROEU with respect to μ.

Thus the dataset $(x^k, q^k)^K_{k=1}$ satisfies e'-PSAROEU with respect to μ'. Finally, note that $\underline{\xi} < \varepsilon^k_s(n) < 1$ for all n and each $k \in K, s \in S$. So that by choosing $\underline{\xi}$ close enough to 1, we can take $\hat{\rho}$ to be as close to ρ as desired. By the definition, we also can take μ' to be as close to μ as desired. Consequently, by (26), we can take (q^k) to be as close to (p^k) as desired. We also can take e' to be as close to e as desired. ■

Lemma 5. *Given $e \in \mathbf{R}_+$, let a dataset $(x^k, p^k)_{k=1}^k$ satisfy e-PSAROEU with respect to μ. Then there are numbers v_s^k, λ^k, μ_s^k, for $s \in S$ and $k \in K$ satisfying* (21) *and* (22) *in Lemma 1.*

Proof of Lemma 5 Consider the system comprised by (23), (24), and (25) in the proof of Lemma 3. Let A, B, and E be constructed from the dataset as in the proof of Lemma 3. The difference with respect to Lemma 3 is that now the entries of A_4 may not be rational. Note that the entries of E, B, and A_i, $i = 1, 2, 3$ are rational.

Suppose, towards a contradiction, that there is no solution to the system comprised by (23), (24), and (25). Then, by the argument in the proof of Lemma 3 there is no solution to System $S1$. Lemma 1 with $\mathbf{F} = \mathbf{R}$ implies that there is a real vector (θ, η, π) such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$ and $\eta \geq 0, \pi > 0$. Recall that $B_4 = 0$ and $E_4 = 1$, so we obtain that $\theta \cdot A_4 + \pi = 0$.

Consider $(q^k)_{k=1}^K$, μ', and e' be such that the dataset $(x^k, q^k)_{k=1}^K$ satisfies e'-PSAROEU with respect to μ', and $\log q_s^k \in \mathbf{Q}$ for all k and s, $\log \mu'_s$ for all $s \in S$, and $\log(1+e') \in \mathbf{Q}$. (Such $(q^k)_{k=1}^K$, μ', and e' exists by Lemma 4.) Construct matrices A', B', and E' from this dataset in the same way as A, B, and E is constructed in the proof of Lemma 3. Note that only the prices, the objective probabilities, and the bounds are different. So $E' = E$, $B' = B$ and $A'_i = A_i$ for $i = 1, 2, 3$. Only A'_4 may be different from A_4.

By Lemma 4, we can choose q^k, μ', and e' such that $|\theta \cdot A'_4 - \theta \cdot A_4| < \pi/2$. We have shown that $\theta \cdot A_4 = -\pi$, so the choice of q^k, μ', and e' guarantees that $\theta \cdot A'_4 < 0$. Let $\pi' = -\theta \cdot A'_4 > 0$.

Note that $\theta \cdot A'_i + \eta \cdot B'_i + \pi' E_i = 0$ for $i = 1, 2, 3$, as (θ, η, π) solves system $S2$ for matrices A, B and E, and $A'_i = A_i$, $B'_i = B_i$ and $E_i = 0$ for $i = 1, 2, 3$. Finally, $B_4 = 0$ so $\theta \cdot A'_4 + \eta \cdot B'_4 + \pi' E_4 = \theta \cdot A'_4 + \pi' = 0$. We also have that $\eta \geq 0$ and $\pi' > 0$. Therefore θ, η, and π' constitute a solution to $S2$ for matrices A', B', and E'.

Lemma 1 then implies that there is no solution to $S1$ for matrices A', B', and E'. So there is no solution to the system comprised by (23), (24), and (25) in the proof of Lemma 3. However, this contradicts Lemma 3 because the dataset (x^k, q^k) satisfies e'-PSAROEU with μ', $\log(1 + e) \in \mathbf{Q}$, $\log \mu'_s \in \mathbf{Q}$ for all $s \in S$, and $\log q_s^k \in \mathbf{Q}$ for all $k \in K$ and $s \in S$. ■

6.2 Proof of Theorems 3 and 4

First, we prove a lemma which proves Theorem 3 and is useful for the sufficiency part of Theorem 4.

Lemma 6. *Given $e \in \mathbf{R}_+$, let $(x^k, p^k)_{k=1}^K$ be a dataset. The following statements are equivalent:*

1. *$(x^k, p^k)_{k=1}^K$ is e-belief-perturbed SEU rational.*

2. *There are strictly positive numbers v_s^k, λ^k, μ_s^k, for $s \in S$ and $k \in K$, such that*

$$\mu_s^k v_s^k = \lambda^k p_s^k, \quad x_s^k > x_{s'}^{k'} \implies v_s^k \leq v_{s'}^{k'}, \tag{27}$$

and for each $k, l \in K$ and $s, t \in S$

$$\frac{\mu_s^k / \mu_t^k}{\mu_s^l / \mu_t^l} \leq 1 + e. \tag{28}$$

3. *$(x^k, p^k)_{k=1}^K$ is e-price-perturbed SEU rational.*

4. *There are strictly positive numbers $\hat{v}_s^k$, $\hat{\lambda}^k$, μ_s, and ε_s^k for $s \in S$ and $k \in K$, such that*

$$\mu_s \hat{v}_s^k = \hat{\lambda}^k \varepsilon_s^k p_s^k, \quad x_s^k > x_{s'}^{k'} \implies \hat{v}_s^k \leq \hat{v}_{s'}^{k'},$$

and for all $k, l \in K$ and $s, t \in S$

$$\frac{\varepsilon_s^k / \varepsilon_t^k}{\varepsilon_s^l / \varepsilon_t^l} \leq 1 + e.$$

5. *$(x^k, p^k)_{k=1}^K$ is e-utility-perturbed SEU rational.*

6. *There are strictly positive numbers $\hat{v}_s^k$, $\hat{\lambda}^k$, μ_s, and $\hat{\varepsilon}_s^k$ for $s \in S$ and $k \in K$, such that*

$$\mu_s \hat{\varepsilon}_s^k \hat{v}_s^k = \hat{\lambda}^k p_s^k, \quad x_s^k > x_{s'}^{k'} \implies \hat{v}_s^k \leq \hat{v}_{s'}^{k'},$$

and for all $k, l \in K$ and $s, t \in S$

$$\frac{\hat{\varepsilon}_s^k / \hat{\varepsilon}_t^k}{\hat{\varepsilon}_s^l / \hat{\varepsilon}_t^l} \leq 1 + e.$$

Proof. By the standard way, the equivalence between 1 and 2, the equivalence between 3 and 4, and the equivalence between 5 and 6 hold. Moreover, it is easy to see the equivalence between 4 and 6 with $\varepsilon_s^k = 1/\hat{\varepsilon}_s^k$ for each $k \in K$ and $s \in S$. So to show the result, it suffices to show that 2 and 4 are equivalent.

To show 4 implies 2, define $v = \hat{v}$ and

$$\mu_s^k = \frac{\mu_s}{\varepsilon_s^k} \bigg/ \left(\sum_{s \in S} \frac{\mu_s}{\varepsilon_s^k} \right)$$

for each $k \in K$ and $s \in S$ and

$$\lambda^k = \hat{\lambda}^k \bigg/ \left(\sum_{s \in S} \frac{\mu_s}{\varepsilon_s^k} \right)$$

for each $k \in K$. Then, $\mu^k \in \Delta_{++}(S)$. Since $\mu_s \hat{v}_s^k = \hat{\lambda}^k \varepsilon_s^k p_s^k$, we have

$$\mu_s^k v_s^k = \lambda^k p_s^k.$$

Moreover, for each $k, l \in K$ and $s, t \in S$

$$\frac{\mu_s^k / \mu_t^k}{\mu_s^l / \mu_t^l} = \frac{\varepsilon_t^k / \varepsilon_s^k}{\varepsilon_t^l / \varepsilon_s^l} \leq 1 + e.$$

To show 2 implies 4, for all $s \in S$ define $\hat{v} = v$ and

$$\mu_s = \sum_{k \in K} \frac{\mu_s^k}{|K|}.$$

Then, $\mu \in \Delta_{++}(S)$. For all $k \in K$, $\hat{\lambda}^k = \lambda^k$. For all $k \in K$ and $s \in S$, define

$$\varepsilon_s^k = \frac{\mu_s}{\mu_s^k}.$$

For each $k \in K$ and $s \in S$, since $\mu_s^k v_s^k = \lambda^k p_s^k$,

$$\mu_s v_s^k = \hat{\lambda}^k \varepsilon_s^k p_s^k.$$

Finally, for each $k, l \in K$ and $s, t \in S$,

$$\frac{\varepsilon_s^k / \varepsilon_t^k}{\varepsilon_s^l / \varepsilon_t^l} = \frac{\mu_t^k / \mu_s^k}{\mu_t^l / \mu_s^l} \leq 1 + e.$$

□

6.2.1 Necessity of Theorem 4

Lemma 7. *Given $e \in \mathbf{R}_+$, if a data set is e-belief-perturbed SEU rational then the data set satisfies e-PSARSEU.*

Proof. Fix any sequence $(x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^n \equiv \sigma$ of pairs satisfies conditions (1)–(3). Assuming differentiability of u and interior solution for simplicity, we have for each $k \in K$ and $s \in S$

$$\mu_s^k u'(x_s^k) = \lambda^k p_s^k.$$

Then,

$$\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s'_i}^{k'_i}} = \prod_{i=1}^n \frac{\lambda^{k'_i} \mu_{s_i}^{k_i} u'(x_{s_i}^{k_i})}{\lambda^{k_i} \mu_{s'_i}^{k'_i} u'(x_{s'_i}^{k'_i})} = \prod_{i=1}^n \frac{u'(x_{s_i}^{k_i})}{u'(x_{s'_i}^{k'_i})} \prod_{i=1}^n \frac{\mu_{s_i}^{k_i}}{\mu_{s'_i}^{k'_i}}.$$

The second equality holds by condition (3). By condition (1), the first term is less than one because of the concavity of u. In the following, we evaluate the second term. First, for each (k, s) cancel out the same μ_s^k as much as possible both from the denominator and the numerator. Then, the number of μ_s^k remained in the numerator is $d(\sigma, k, s)$. Since the number of numerator and the denominator must be the same, the number of remaining fraction is $m(\sigma) \equiv \sum_{s \in S} \sum_{k \in K : d(\sigma,k,s)>0} d(\sigma, k, s)$. So by relabeling the index i to j if necessary, we obtain

$$\prod_{i=1}^n \frac{\mu_{s_i}^{k_i}}{\mu_{s'_i}^{k'_i}} = \prod_{j=1}^{m(\sigma)} \frac{\mu_{s_j}^{k_j}}{\mu_{s'_j}^{k'_j}}.$$

Consider the corresponding sequence $(x_{s_j}^{k_j}, x_{s'_j}^{k'_j})_{j=1}^{m(\sigma)}$. Since the sequence is obtained by canceling out x_s^k from the first element and the second element of the pairs the same number of times; and since the original sequence $(x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^n$ satisfies condition (2) and (3), it follows that $(x_{s_j}^{k_j}, x_{s'_j}^{k'_j})_{j=1}^{m(\sigma)}$ satisfies condition (2) and (3).

By condition (2), we can assume without loss of generality that $s_j = s'_j$ for each j. Fix $s^* \in S$. Then by the robustness condition, for each $j \in \{1, \ldots, m(\sigma)\}$,

$$\frac{\mu_{s_j}^{k_j}}{\mu_{s'_j}^{k'_j}} = \frac{\mu_{s_j}^{k_j}}{\mu_{s_j}^{k'_j}} \leq (1+e) \frac{\mu_{s^*}^{k'_j}}{\mu_{s^*}^{k_j}}.$$

Moreover by condition (3),

$$\prod_{j=1}^{m(\sigma)} \frac{\mu_{s^*}^{k'_j}}{\mu_{s^*}^{k_j}} = 1.$$

Therefore,

$$\prod_{j=1}^{m(\sigma)} \frac{\mu_{s_i}^{k_j}}{\mu_{s'_j}^{k'_j}} \leq (1+e)^{m(\sigma)} \prod_{j=1}^{n} \frac{\mu_{s^*}^{k'_j}}{\mu_{s^*}^{k_j}} = (1+e)^{m(\sigma)},$$

and hence,

$$\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s'_i}^{k'_i}} \leq (1+e)^{m(\sigma)}.$$

□

Remark 2. *We need to show the lemma because in the proof of sufficiency we weaken the dual of the rationality condition.*

6.2.2 Sufficiency of Theorem 4

We need three more lemmas to prove the theorem.

Lemma 8. *Given $e \in \mathbf{R}_+$, let a dataset $(x^k, p^k)_{k=1}^k$ satisfy e-PSARSEU. Suppose that* $\log(p_s^k) \in \mathbf{Q}$ *for all k and s and* $\log(1+e) \in \mathbf{Q}$. *Then there are numbers v_s^k, λ^k, μ_s^k, for $s \in S$ and $k \in K$ satisfying* (27) *and* (28) *in Lemma 6.*

Proof of Lemma 8 The proof is similar to the case in which $e = 0$. By log-linearizing conditions (27) and (28) in Lemma 6, we have for all $s \in S$ and $k \in K$, such that

$$\log \mu_s^k + \log v_s^k = \log \lambda^k + \log p_s^k, \tag{29}$$

$$x_s^k > x_{s'}^{k'} \implies \log v_s^k \le \log v_{s'}^{k'}, \tag{30}$$

and for all $k, l \in K$ and $s, t \in S$

$$\log \mu_s^k - \log \mu_t^k - \log \mu_s^l + \log \mu_t^l \le \log(1+e). \tag{31}$$

Matrix A looks as follows:

$$\begin{array}{c} \\ \\ (k,s) \\ (k,t) \\ (l,s) \\ (l,t) \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccccccccccccccccc} \cdots & v_s^k & v_t^k & v_s^l & v_t^l & \cdots & \cdots & \mu_s^k & \mu_t^k & \mu_s^l & \mu_t^l & \cdots & \cdots & \lambda^k & \lambda^l & \cdots & p \end{array} \\ \left[\begin{array}{cccccc|cccccc|cccc|c} & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ \cdots & 1 & 0 & 0 & 0 & \cdots & \cdots & 1 & 0 & 0 & 0 & \cdots & \cdots & -1 & 0 & \cdots & -\log p_s^k \\ \cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & -1 & 0 & \cdots & -\log p_s^k \\ \cdots & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & -1 & \cdots & -\log p_s^l \\ \cdots & 0 & 0 & 0 & 1 & \cdots & \cdots & 0 & 0 & 0 & 1 & \cdots & \cdots & 0 & -1 & \cdots & -\log p_s^l \\ & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \end{array}\right] \end{array}.$$

Matrix B has additional rows as follows in addition to the rows in Echenique and Saito (2015).

$$\begin{array}{c} \begin{array}{cccccccccccccccccccccc} \cdots & v_s^k & v_t^k & v_s^l & v_t^l & \cdots & \cdots & \mu_s^k & \mu_t^k & \mu_s^l & \mu_t^l & \cdots & \cdots & \lambda^k & \lambda^l & \cdots & p \end{array} \\ \left[\begin{array}{cccccc|cccccc|cccc|c} & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & -1 & 1 & 1 & -1 & \cdots & \cdots & 0 & 0 & \cdots & \log(1+e) \\ \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 1 & -1 & -1 & 1 & \cdots & \cdots & 0 & 0 & \cdots & \log(1+e) \\ & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & & \vdots \end{array}\right] \end{array}.$$

Matrix E is the same as in Echenique and Saito (2015).

The entries of A, B, and E are either 0, 1 or -1, with the exception of the last column of A. Under the hypotheses of the lemma we are proving, the last column consists of rational numbers. By Motzkin's theorem, then, there is such a solution u to $S1$ if and only if there is no rational vector (θ, η, π) that solves the system of equations and linear inequalities

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

Claim There exists a sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^{n^*}_{i=1}$ of pairs that satisfies conditions (1) and (3) in e-PSARSEU.

Proof. The same as the case in which $e = 0$. From matrix B, we obtain a chain $z > \cdots > z'$. Define $x^{k_1}_{s_1} = z$ and $x^{k'_1}_{s'_1} = z'$. By (30), we have -1 in the column of $v^{k_1}_{s_1}$ and 1 in the column $v^{k'_1}_{s'_1}$. So these -1 and 1 are canceled out in A_1. By repeating this, we obtain a sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^{n^*}_{i=1}$ of pairs that satisfies Condition (1). □

Claim The sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^{n^*}_{i=1} \equiv \sigma^*$ satisfies condition (2) in e-PSARSEU.

Proof. Denote the weight on the rows capturing $\frac{\mu^k_s/\mu^k_t}{\mu^l_s/\mu^l_t} \leq 1 + e$ by $\theta(k, l, s, t)$. Note that $\frac{\mu^l_t/\mu^l_s}{\mu^k_t/\mu^k_s} = \frac{\mu^k_s/\mu^k_t}{\mu^l_s/\mu^l_t}$, so we only have the constraint $\frac{\mu^k_s/\mu^k_t}{\mu^l_s/\mu^l_t} \leq 1 + e$ but not $\frac{\mu^l_t/\mu^l_s}{\mu^k_t/\mu^k_s} \leq 1 + e$; hence we will not have $\theta(l, k, t, s)$. On the other hand, we need to have the constraint $\frac{\mu^l_s/\mu^l_t}{\mu^k_s/\mu^k_t} \leq 1 + e$ which is equivalent to $\frac{\mu^k_s/\mu^k_t}{\mu^l_s/\mu^l_t} \geq 1/(1 + e)$. This constraint corresponds to $\theta(l, k, s, t)$.

Let $n(x^k_s) \equiv \#\{i \mid x^k_s = x^{k_i}_{s_i}\}$ and $n'(x^k_s) \equiv \#\{i \mid x^k_s = x^{k'_i}_{s'_i}\}$.

For each $k \in K$ and $s \in S$, in the column corresponding to μ^k_s, remember that we have 1 if we have $x^k_s = x^{k_i}_{s_i}$ for some i and -1 if we have $x^k_s = x^{k'_i}_{s'_i}$ for some i. This is because a row in A must have 1 (-1) in the column v^k_s if and only if it has 1 (-1, respectively) in the column μ^k_s. So in the column in matrix A, we have $n(x^k_s) - n'(x^k_s)$.

Now we consider matrix B. In the column of μ_s^k, we have -1 in the row multiplied by $\theta(k,l,s,t)$ and 1 in the row multiplied by $\theta(l,k,s,t)$. So we also have $-\sum_{l\neq k}\sum_{t\neq s}\theta(k,l,s,t)+\sum_{l\neq k}\sum_{t\neq s}\theta(l,k,s,t)$.

For each $k \in K$ and $s \in S$, the column corresponding to μ_s^k of matrices A and B must sum up to zero; so we have

$$n(x_s^k) - n'(x_s^k) - \sum_{l\neq k}\sum_{t\neq s}\theta(k,l,s,t) + \sum_{l\neq k}\sum_{t\neq s}\theta(l,k,s,t) = 0.$$

Therefore, for each s,

$$\sum_{k\in K}\Big(n(x_s^k) - n'(x_s^k)\Big) = \sum_{k\in K}\left[\sum_{l\neq k}\sum_{t\neq s}\theta(k,l,s,t) - \sum_{l\neq k}\sum_{t\neq s}\theta(l,k,s,t)\right] = 0.$$

□

Claim $\prod_{i=1}^{n^*}\frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} > (1+e)^{m(\sigma^*)}$.

Proof. By the fact that the last column must sum up to zero and E has one at the last column, we have

$$\sum_{i=1}^{n^*}\log\frac{p_{s_i'}^{k_i'}}{p_{s_i}^{k_i}} + \left(\sum_{k\in K}\sum_{l\neq k}\sum_{s\in S}\sum_{t\neq s}\theta(k,l,s,t)\right)\log(1+e) = -\pi < 0.$$

Hence, by multiplying -1, we have

$$\sum_{i=1}^{n^*}\log\frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} - \left(\sum_{k\in K}\sum_{l\neq k}\sum_{s\in S}\sum_{t\neq s}\theta(k,l,s,t)\right)\log(1+e) > 0.$$

Remember that for all $k \in K$ and $s \in S$,

$$n(x_s^k) - n'(x_s^k) = +\sum_{l\neq k}\sum_{t\neq s}\theta(k,l,s,t) - \sum_{l\neq k}\sum_{t\neq s}\theta(l,k,s,t) \leq \sum_{l\neq k}\sum_{t\neq s}\theta(k,l,s,t).$$

Since $d(\sigma^*,k,s) = n(x_s^k) - n'(x_s^k)$, we have

$$\begin{aligned} m(\sigma^*) &\equiv \sum_{s\in S}\sum_{k\in K: d(\sigma^*,k,s)>0} d(\sigma^*,k,s) \\ &= \sum_{s\in S}\sum_{k\in K}\max\{n(x_s^k) - n'(x_s^k), 0\} \\ &\leq \sum_{s\in S}\sum_{k\in K}\sum_{l\neq k}\sum_{t\neq s}\theta(k,l,s,t). \end{aligned}$$

Therefore

$$\begin{aligned}\sum_{i=1}^{n^*} \log \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} &> \left(\sum_{k\in K} \sum_{l\neq k} \sum_{s\in S} \sum_{t\neq s} \theta(k,l,s,t) \right) \log(1+e) \\ &\geq m(\sigma^*) \log(1+e).\end{aligned}$$

This is a contradiction. □

Let $\mathcal{X} = \{x^k_s \mid k \in K, s \in S\}$.

Lemma 9. *Given $e \in \mathbf{R}_+$, let a dataset $(x^k, p^k)^k_{k=1}$ satisfy e-PSARSEU. Then for all positive numbers $\overline{\varepsilon}$, there exist a positive real number $e' \in [e, e+\overline{\varepsilon}]$ and $q^k_s \in [p^k_s - \overline{\varepsilon}, p^k_s]$ for all $s \in S$ and $k \in K$ such that $\log q^k_s \in \mathbf{Q}$ and the dataset $(x^k, q^k)^k_{k=1}$ satisfy e'-PSARSEU.*

Proof of Lemma 9 Consider the set of sequences that satisfy Conditions (1), (2), and (3) in e-PSARSEU:

$$\Sigma = \left\{ (x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1} \subset \mathcal{X}^2 \;\middle|\; \begin{array}{l} (x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1} \text{ satisfies conditions (1), (2), and (3)} \\ \text{in } e\text{-PSARSEU for some } n \end{array} \right\}.$$

For each sequence $\sigma \in \Sigma$, we define a vector $t_\sigma \in \mathbf{N}^{K^2S^2}$. For each pair $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})$, we shall identify the pair with $((k_i, s_i), (k'_i, s'_i))$. Let $t_\sigma((k,s),(k',s'))$ be the number of times that the pair $(x^k_s, x^{k'}_{s'})$ appears in the sequence σ. One can then describe the satisfaction of e-PSARSEU by means of the vectors t_σ. Observe that t depends only on $(x^k)^K_{k=1}$ in the dataset $(x^k, p^k)^K_{k=1}$. It does not depend on prices.

For each $((k,s),(k',s'))$ such that $x^k_s > x^{k'}_{s'}$, define $\delta((k,s),(k',s')) = \log(p^k_s / p^{k'}_{s'})$. And define $\delta((k,s),(k',s')) = 0$ when $x^k_s \leq x^{k'}_{s'}$. Then, δ is a K^2S^2-dimensional real-valued vector. If $\sigma = (x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$, then

$$\delta \cdot t_\sigma = \sum_{((k,s),(k',s')) \in (KS)^2} \delta((k,s),(k',s')) t_\sigma((k,s),(k',s')) = \log \left(\prod_{i=1}^n \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} \right).$$

So the dataset satisfies e-PSARSEU if and only if $\delta \cdot t_\sigma \leq m(\sigma) \log(1+e)$ for all $\sigma \in \Sigma$.

Enumerate the elements in $\mathcal{X}$ in increasing order: $y_1 < y_2 < \cdots < y_N$. And fix an arbitrary $\underline{\xi} \in (0,1)$. We shall construct by induction a sequence $\{(\varepsilon^k_s(n))\}^N_{n=1}$, where $\varepsilon^k_s(n)$ is defined for all (k,s) with $x^k_s = y_n$.

By the denseness of the rational numbers, and the continuity of the exponential function, for each (k,s) such that $x^k_s = y_1$, there exists a positive number $\varepsilon^k_s(1)$ such that $\log(p^k_s \varepsilon^k_s(1)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon^k_s(1) < 1$. Let $\varepsilon(1) = \min\{\varepsilon^k_s(1) \mid x^k_s = y_1\}$.

In second place, for each (k,s) such that $x^k_s = y_2$, there exists a positive $\varepsilon^k_s(2)$ such that $\log(p^k_s \varepsilon^k_s(2)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon^k_s(2) < \varepsilon(1)$. Let $\varepsilon(2) = \min\{\varepsilon^k_s(2) \mid x^k_s = y_2\}$.

In third place, and reasoning by induction, suppose that $\varepsilon(n)$ has been defined and that $\underline{\xi} < \varepsilon(n)$. For each (k,s) such that $x^k_s = y_{n+1}$, let $\varepsilon^k_s(n+1) > 0$ be such that $\log(p^k_s \varepsilon^k_s(n+1)) \in \mathbf{Q}$, and $\underline{\xi} < \varepsilon^k_s(n+1) < \varepsilon(n)$. Let $\varepsilon(n+1) = \min\{\varepsilon^k_s(n+1) \mid x^k_s = y_n\}$.

This defines the sequence $(\varepsilon^k_s(n))$ by induction. Note that $\varepsilon^k_s(n+1)/\varepsilon(n) < 1$ for all n. Let $\bar{\xi} < 1$ be such that $\varepsilon^k_s(n+1)/\varepsilon(n) < \bar{\xi}$.

For each $k \in K$ and $s \in S$, let $q^k_s = p^k_s \varepsilon^k_s(n)$, where n is such that $x^k_s = y_n$. We claim that the dataset $(x^k, q^k)^K_{k=1}$ satisfies e-PSARSEU. Let δ^* be defined from $(q^k)^K_{k=1}$ in the same manner as δ was defined from $(p^k)^K_{k=1}$.

For each pair $((k,s),(k',s'))$ with $x^k_s > x^{k'}_{s'}$, if n and m are such that $x^k_s = y_n$ and $x^{k'}_{s'} = y_m$, then $n > m$. By definition of ε,

$$\frac{\varepsilon^k_s(n)}{\varepsilon^{k'}_{s'}(m)} < \frac{\varepsilon^k_s(n)}{\varepsilon(m)} < \bar{\xi} < 1.$$

Hence,

$$\delta^*((k,s),(k',s')) = \log \frac{p^k_s \varepsilon^k_s(n)}{p^{k'}_{s'} \varepsilon^{k'}_{s'}(m)} < \log \frac{p^k_s}{p^{k'}_{s'}} + \log \bar{\xi} < \log \frac{p^k_s}{p^{k'}_{s'}} = \delta((k,s),(k',s')).$$

Now we choose e' such that $e' \geq e$ and $\log(1+e') \in \mathbf{Q}$.

Thus, for all $\sigma \in \Sigma$, $\delta^* \cdot t_\sigma \leq \delta \cdot t_\sigma \leq m(\sigma)\log(1+e) \leq m(\sigma)\log(1+e')$ as $t_\cdot \geq 0$ and the dataset $(x^k, p^k)^K_{k=1}$ satisfies e-PSARSEU.

Thus the dataset $(x^k, q^k)^K_{k=1}$ satisfies e'-PSARSEU. Finally, note that $\underline{\xi} < \varepsilon^k_s(n) < 1$ for all n and each $k \in K, s \in S$. So that by choosing $\underline{\xi}$ close enough to 1 we can take (q^k) to be as close to (p^k) as desired. We also can take e' to be as close to e as desired. ■

Lemma 10. *Given $e \in \mathbf{R}_+$, let a dataset $(x^k, p^k)^k_{k=1}$ satisfy e-PSARSEU. Then there are numbers v^k_s, λ^k, μ^k_s, for $s \in S$ and $k \in K$ satisfying* (27) *and* (28) *in Lemma 6.*

Proof of Lemma 10 Consider the system comprised by (29), (30), and (31) in the proof of Lemma 8. Let A, B, and E be constructed from the dataset as in the proof of Lemma 8. The difference with respect to Lemma 8 is that now the entries of A_4 may not be rational. Note that the entries of E, B, and A_i, $i = 1, 2, 3$ are rational.

Suppose, towards a contradiction, that there is no solution to the system comprised by (29), (30), and (31). Then, by the argument in the proof of Lemma 8 there is no solution to System $S1$. Lemma 1 with $\mathbf{F} = \mathbf{R}$ implies that there is a real vector (θ, η, π) such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$ and $\eta \geq 0, \pi > 0$. Recall that $B_4 = 0$ and $E_4 = 1$, so we obtain that $\theta \cdot A_4 + \pi = 0$.

Let $(q^k)_{k=1}^K$ vectors of prices and a positive real number e' be such that the dataset $(x^k, q^k)_{k=1}^K$ satisfies e'-PSARSEU and $\log q_s^k \in \mathbf{Q}$ for all k and s and $\log(1 + e') \in \mathbf{Q}$. (Such $(q^k)_{k=1}^K$ and e' exists by Lemma 9.) Construct matrices A', B', and E' from this dataset in the same way as A, B, and E is constructed in the proof of Lemma 8. Since only prices q^k and the bound e' are different in this dataset, only A'_4 may be different from A_4. So $E' = E$, $B' = B$ and $A'_i = A_i$ for $i = 1, 2, 3$.

By Lemma 9, we can choose prices q^k such that $|\theta \cdot A'_4 - \theta \cdot A_4| < \pi/2$. We have shown that $\theta \cdot A_4 = -\pi$, so the choice of prices q^k guarantees that $\theta \cdot A'_4 < 0$. Let $\pi' = -\theta \cdot A'_4 > 0$.

Note that $\theta \cdot A'_i + \eta \cdot B'_i + \pi' E_i = 0$ for $i = 1, 2, 3$, as (θ, η, π) solves system $S2$ for matrices A, B and E, and $A'_i = A_i$, $B'_i = B_i$ and $E_i = 0$ for $i = 1, 2, 3$. Finally, $B_4 = 0$ so $\theta \cdot A'_4 + \eta \cdot B'_4 + \pi' E_4 = \theta \cdot A'_4 + \pi' = 0$. We also have that $\eta \geq 0$ and $\pi' > 0$. Therefore θ, η, and π' constitute a solution to $S2$ for matrices A', B', and E'.

Lemma 1 then implies that there is no solution to $S1$ for matrices A', B', and E'. So there is no solution to the system comprised by (29), (30), and (31) in the proof of Lemma 8. However, this contradicts Lemma 8 because the dataset (x^k, q^k) satisfies e'-PSARSEU, $\log(1 + e) \in \mathbf{Q}$, and $\log q_s^k \in \mathbf{Q}$ for all $k \in K$ and $s \in S$. ∎

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